TWO DIMENSIONAL GENERALIZATIONS OF HAAR BASES

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Abstract

The subject of this talk (Montevideo, march 1995) are invariant sets of a class of affine iterated function systems in $\mathbb{R}^n$.

The class I consider, roughly speaking, are those systems that have a quotient system living on the torus $\mathbb{R}^n/\mathbb{Z}^n$. This quotient is given by the complete inverse of an expanding matrix with integer entries. The natural question one asks is whether these invariant sets tile $\mathbb{R}^n$. Here, it is proved that in dimension one, and, in some cases, in dimension two, the answer is affirmative. While this was already known in one dimension, our proof is simpler and has a more geometric flavour than other proofs. The principal applications of these ideas is that one can use these tiles to construct wavelets, which, by way of motivation, I explain briefly in the introduction.

1 Introduction

Let $M : \mathbb{R}^n \to \mathbb{R}^n$ be an linear isomorphism with eigenvalues strictly outside the unit circle and preserving $\mathbb{Z}^n$ (that is: has integer entries). Define further $R \subset \mathbb{Z}^n$ a complete set of residues modulo M (that is: R contains precisely one representative in $\mathbb{Z}^n$ of each of the classes $\mathbb{Z}^n/M\mathbb{Z}^n$). By performing a translation we may assume that R contains the origin. Note that R contains $|\det M|$ elements.

We consider the set Λ of expansions on the base M using the set of digits R, or

$$\Lambda(M,R) = \{x \in \mathbb{R}^n \,|x = \sum_{i=1}^{\infty} M^{-i} r_i \text{ with } r_i \in R\} \quad .$$

Definition 1.1 *Let $N : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map preserving $\mathbb{Z}^n$ and $\pi_N : A \to \mathbb{R}^n/N\mathbb{Z}^n$ the canonical projection. A compact set A in $\mathbb{R}^n$ of positive measure is called a tile by $N\mathbb{Z}^n$ if $\pi_N : A \to \mathbb{R}^n/N\mathbb{Z}^n$ is a bijection for Lebesgue almost every point of A.*

When the matrix N is not specified (as in most of this paper), we assume it to be the identity. In this case, we see that a tile is a compact set such that the union of its translates by $\mathbb{Z}^n$ covers $\mathbb{R}^n$, but two translates by distinct elements of $\mathbb{Z}^n$ may intersect in sets of measure zero only.

One of the main results was proved by Gröchenig and Haas [4]. It states that in one dimension with the conditions given here, $\pi : \Lambda(M,R) \to \mathbb{R}^n/\mathbb{Z}^n$ covers exactly q times (almost everywhere) where q is the greatest common divisor of R. The proof of this result was substantially simplified in [6]. In this work, we simplify the proof further (by introducing intersection numbers) and generalize it to include many two-dimensional cases. We also give some counter-examples in the last section.

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In their paper Gröchenig and Haas [4] also give a weak extension of their results to two dimensions. For any 2 by 2 matrix M satisfying our standing assumptions plus another condition, they construct a digit set R such that (M,R) has the tiling property. The extra condition on the matrix M is that if it has two distinct real eigenvalues, these have to be rational.

In higher dimension, there is the result in [6] which states that when $|\det M| = 2$ then, in many cases, Λ is a tile In particular this includes all two- and three-dimensional cases. The method of proof of this latter result is very different from the reasonings we shall use in the present work.

The main results in the present work can be summarized as follows:

Theorem 1.2 *Let M and R as before and suppose they also satisfy*

- *R generates $\mathbb{Z}^n$ under addition and multiplication by $\mathbb{Z}$.*
- *$\phi(x) = \sum_{r\in R} e^{2\pi i r\cdot x}$ has finitely many zeroes.*

Then $\Lambda(M,R)$ is a tile.

The first hypothesis is a clear characterization. Assuming it holds, we will prove that the second hypothesis holds if

- The dimension is one.
- The dimension is two and one of the following statements holds:
  - $|\det M|$ is three.
  - $|\det M|$ is four and in addition the elements of R satisfy a certain "non-resonance" condition (see proposition 3.2).

(In section four these condition are relaxed somewhat by allowing for 'common divisors'.)

After finishing the first draft of this paper, we became aware of several related preprints by Lagarias and Wang. In of these [10], they independently proved results that are essentially equivalent to our theorem 1.2 and proposition 3.2. In another paper [12], they proved the following remarkable result. Let M and R satisfy the standing hypotheses and denote by $\mathbb{Z}[M,R]$ the smallest M-invariant sublattice of $\mathbb{Z}^n$ that contains $D = R - R$ (smallest in the sense that it contains no subset satisfying the same requirements). Then $\Lambda(M,R)$ is a tile by $\mathbb{Z}[M,R]$ unless the following is true. There exists an integer matrix $P \in GL(n,\mathbb{Z})$ such that

- PMP^{-1} is a 'block-triangular' matrix $\begin{pmatrix} A & B \\ \emptyset & C \end{pmatrix}$.
- $P(R)$ is of so-called quasi-product form (for the definition see [12]).

In fact, they continue to prove that these exceptional cases also tile $\mathbb{R}^n$ by a lattice, albeit possibly a different one. Curiously, this still leaves one problem unresolved. It is a problem of algebraic nature: Is it true that for any matrix M there is a digit set R of complete residues such that $\mathbb{Z}[M,R] = \mathbb{Z}^n$? Lagarias and Wang seem to have resolved this problem for dimension 2 and 3 (personal communication by Wang, see also [11] for partial results).

The outline of this article is as follows. In the next section we start by deriving an equation very similar to the Perron-Frobenius equation for densities of measures. To prove that Λ is a tile we will have to prove that this equation has only one solution (namely the constant). Most of the discussion of this section can be found in [6], the exception being the notion and subsequent use of intersection numbers to simplify the reasoning. In the third section, we analyze the Perron-Frobenius(-like) equation and prove that in certain cases the only solution is indeed the constant. These cases include the ones mentioned above. In the last section, we sharpen a criterion given in [6] to decide whether $\Lambda(M,R)$ is a tile, and then use that criterion to look at some two-dimensional examples of $\Lambda(M,R)$ that are not tiles.

We end this introduction by indicating a few applications. Let M, R be given such that $\Lambda(M,R)$ is a tile by $\mathbb{Z}^n$ and denote $|\det M|$ by m. Wavelets are widely used in image- and sound-processing as an alternative for Fourier decomposition. They are functions with compact support forming a basis of the square integrable functions on $\mathbb{R}^n$. As additional properties one requires that if $f(x)$ is a basis-function then so is $f(Mx)$ (scaling property). For a detailed account of these notions we refer to [2]. With the help of the above self-similar tiles we can now easily construct such a basis. This construction gives what is called a generalized Haar basis by analogy with a certain one-dimensional construction. Let $\chi_\Lambda(x)$ be the characteristic function on Λ and U an $m \times m$ unitary matrix whose first column consists of the vector with constant entries (namely $m^{-1/2}$). Then the functions

$$\psi_i \stackrel{\text{def}}{\equiv} \sum_{r \in R} U_{ij} \chi_\Lambda(Mx - r)$$

clearly form an orthonormal basis of the function with support on $\cup_{r\in R} M^{-1}(\Lambda + r)$ and for each $r \in R$ constant on $M^{-1}(\Lambda + r)$. Now define

$$f_{ijk} \stackrel{\text{def}}{\equiv} |\det M|^{j/2} \psi_i(M^j x - k)$$

where $j \in \mathbb{Z}^+ i$, $i \in \{0, 1, \cdots |\det M| - 1\}$, and $k \in \mathbb{Z}^n$. Then the set $\{f_{ijk}\}$ forms an orthonormal basis of the square integrable functions on $\mathbb{R}^n$ with the required properties. This was first proved by Gröchenig and Madych [5] (see also [3] and [2] for additional information).

The second application concerns the measure of the compact set obtained in the following way. For every compact set $A \subset \mathbb{R}^n$ define the following affine iterated function system

$$\tau(A) = \cup_{r \in R_t} M^{-1}(A + r) \quad ,$$

where R_t depends on the parameter t. The simplest non-trivial example is when M multiplication by 3 in $\mathbb{R}$ and

$$R_t = \{0, t, 2\}, \text{ with } t \in [0, 1] \quad .$$

As we will see in the following section, there is a unique compact set invariant under τ. Denoting this set by $\Lambda(t)$, one obtains that its Lebesgue measure $\mu(\Lambda(t))$ has the following properties:

$$\begin{aligned} \mu(\Lambda(t)) &= \frac{2}{q} \text{ if } t = \frac{2p}{q} \text{ and } pq \mod 3 = 2 \\ \mu(\Lambda(t)) &= 0 \text{ else} \quad . \end{aligned}$$

The first case is an easy consequence of the theorem by [4] already mentioned, although multiplication by integers greater than 3 introduce some extra problems. The second property follows from a result by [9].

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2 The Perron Frobenius Equation

In this section we define the intersection numbers and show that these satisfy an elementary self-similarity relation. This relation can in turn be translated to an equation of the Perron-Frobenius type. If this equation admits only a constant solution, then $\Lambda(M,R)$ is a tile. Most of the reasoning here has appeared in [6], but the use of intersection numbers simplify that proof.

Let X be a a closed ball in $\mathbb{R}^n$, or, more generally a complete compact metric space. Define the space $H(X)$ of closed subsets of X. The following construction now defines the so-called distance on $H(X)$. For $A,B \in H(X)$, let $N_\epsilon(A)$ denote the open ϵ n eighborhood of a set A. The Hausdorff distance between A and B is $\mathrm{Hd}\,(A,B)$:

$$\mathrm{Hd}\,(A,B) = \inf\{\epsilon > 0 | A \subset N_\epsilon(B) \text{ and } B \subset N_\epsilon(A)\} \quad .$$

This distance induces a topology on $H(X)$ so that $H(X)$ is a complete compact metric space [7]. Limits in this topology will be denoted by Hlim. In $H(\mathbb{R}^n)$ we define:

$$\tau : H(\mathbb{R}^n) \to H(\mathbb{R}^n)$$

by

$$\tau(A) = \cup_{r\in R} M^{-1}(A+r) \quad .$$

It is easy to prove that τ is a contraction (see [7]) and its unique fixed point is precisely the set Λ as defined before. Hence we obtain that Λ is 'self similar' or:

$$\Lambda = \cup_{r\in R} M^{-1}(\Lambda+r)$$

or, equivalently:

$$M\Lambda = \cup_{r\in R}(\Lambda+r) \quad . \tag{2.1}$$

The (Lebesgue) measure of the set of this last equation is, of course, $|\det M|$ times the measure of Λ. From the righthand side of the equation one then concludes easily that translates of Λ by distinct elements of $\mathbb{R}$ intersect in sets of measure zero.

Denote $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and let $\pi : H(\mathbb{R}^n) \to H(\mathbb{T}^n)$ be induced by the usual canonical projection. It is easy to verify that π is continuous. Let

$$W : H(\mathbb{T}^n) \to H(\mathbb{T}^n)$$

be induced by the usual complete inverse of M on the torus. Clearly, W has $|\det M|$ branches.

Lemma 2.1 $\pi\Lambda = \mathbb{T}^n$.

Proof: (see also [6].) The following diagram commutes:

$$\begin{array}{ccc} H(\mathbb{R}^n) & \xrightarrow{\tau} & H(\mathbb{R}^n) \\ \downarrow \pi & & \downarrow \pi \\ H(\mathbb{T}^n) & \xrightarrow{W} & H(\mathbb{T}^n) \end{array}$$

Noting that for any compact K we have

$$\pi\Lambda = \pi\ \mathrm{Hlim}_{k\to\infty}\tau^k K = \ \mathrm{Hlim}_{k\to\infty} W^k K = \mathbb{T}^n \quad ,$$

the lemma is easily implied. ■

Thus $\cup_{v\in\mathbb{Z}^n}(\Lambda+v) = \mathbb{R}^n$ and it follows from Baire's theorem that Λ has non-empty interior.

Self-similarity (2.1) implies that for each $y \in \Lambda$ there is at least one $r \in \mathbb{R}$ such that $My - r \in \Lambda$. So define $t : \Lambda \to \Lambda$ as

$$t(y) = [\cup_{r\in R}\{My - r\}] \cap \Lambda \quad .$$

On the other hand each point $x \in \Lambda$ has $|\det M|$ preimages $M^{-1}(x + r)$. Thus t preserves Leabesgue measure:

$$\sum_{ty=x} |dy| = |dx| \quad . \tag{2.2}$$

In fact, since t is expanding, we have the following result.

Proposition 2.2 *t is ergodic with respect to the Lebesgue measure.*

Proof: For a detailed proof see [6]. ■

This last fact has important consequences. Define for $k \geq 1$:

$$\Lambda^{(k)} = \{x \in \Lambda|\{x + \mathbb{Z}^n\} \cap \Lambda \text{ has at least } k \text{ points }\} \quad .$$

Then $\Lambda^{(k)}$ is t-invariant. So, by the ergodic theorem $\Lambda^{(k)}$ has either full measure or measure zero. Denoting Lebesgue measure by μ, we have

$$\mu(\Lambda) = \max\{k \in \mathbb{N}|\mu(\Lambda^{(k)}) > 0\} \quad .$$

Call this number ℓ. Thus the canonical projection of Λ to the torus is ℓ to 1 in almost every point of the torus.

Definition 2.3 (Intersection Numbers) *The intersection numbers are the values of the function $\nu : \mathbb{Z}^n \to \mathbb{R}^+$, defined as follows:*

$$\nu(k) = \mu((\Lambda + k) \cap \Lambda) \quad .$$

This function ν satisfies a simple property due to self-similarity (2.1). First define the difference set <u>with multiplicity</u>:

$$D = R - R \stackrel{\text{def}}{\equiv} \{d \in \mathbb{Z}^n | \exists r_1, r_2 \in R \ \text{ such that } \ d = r_1 - r_2\} \quad . \tag{2.3}$$

(For instance, the element 0 occurs at least $|\det M|$ times in D, namely $0 = r - r$ for all elements r in R.)

Proposition 2.4 *The function ν satisfies the following equation:*

$$\frac{1}{|\det M|} \sum_{d\in D} \nu(Mj - d) = \nu(j) \quad .$$

Proof: We have:

$$\begin{aligned} \sum_{r_1,r_2\in R} \mu((\Lambda + r_1 - r_2 + Mj) \cap \Lambda) &= \sum_{r_1,r_2\in R} \mu((\Lambda + r_1 + Mj) \cap (\Lambda + r_2)) = \\ \mu(M(\Lambda + j) \cap M\Lambda) &= |\det M|\ \mu((\Lambda + j) \cap \Lambda) \quad . \end{aligned}$$

■

Let Ω be the space

$$\{\sum a(k)z^k | k \in \mathbb{Z}^n,\ a(k) \in \mathbb{C}\} \quad .$$

Define the 'transition operator' $T : \Omega \to \Omega$:

$$T(\sum a(k)z^k) = \frac{1}{|\det M|} \sum_{j\in\mathbb{Z}^n} \sum_{d\in D} a(Mj - d)z^j \quad . \tag{2.4}$$

Note that T operates on functions defined on the torus $\mathbb{T}^n$. By definition, the functions $f = z^0$ (the constant function) and $f = \sum \nu(k) z^k$ are eigenfunctions of the transition operator both with eigenvalues 1. It is not clear whether there are other eigenfunctions associated with the eigenvalue 1.

Let us now return to equation (2.2) for the existence of an invariant measure. Let $\lambda(x)$ be a probability measure with a continuous density $h(x)$:

$$|d\lambda(x)| = h(x)|dx|$$

The lefthand side of (2.2) now becomes the Perron-Frobenius operator:

$$\sum_{ty=x} |dy| = \sum_{r\in R} h(M^{-1}(x+r)) \cdot \frac{dy}{|\det M|} \quad .$$

The following theorem reinterprets the operator T just defined as a (generalized) Perron-Frobenius operator. Before stating the theorem, we need some notation. Define

$$\phi(x) = \sum_{r\in R} e^{2\pi i r\cdot x} \quad , \tag{2.5}$$

Define the real, non-negative weight function

$$w(x) = \frac{|\phi(x)|^2}{|\det M|^2} \quad . \tag{2.6}$$

Theorem 2.5 *The transition operator (2.4) can be extended to continuous functions of the torus and is given by:*

$$Tf(x) = \sum_{j\in J} f((M^\dagger)^{-1}(x+j))\, w((M^\dagger)^{-1}(x+j)) \quad ,$$

where $(M^\dagger)$ denotes the transpose of M and J is any complete set of residues modulo $(M^\dagger)$. Moreover,

$$\sum_{j\in J} w((M^\dagger)^{-1}(x+j)) = 1 \quad ,$$

and

$$T(\sum_{k\in\mathbb{Z}} \nu(k) e^{2\pi i k\cdot x}) = \sum_{k\in\mathbb{Z}} \nu(k) e^{2\pi i k\cdot x} \quad .$$

Proof: Define

$$z^k \quad = \quad e^{2\pi i k\cdot x}$$

where the $\cdot$ denotes the usual innerproduct of $\mathbb{R}^n$. Now it is an unpleasant, although straightforward, exercise to write 2.4 in the correct form, but the details are in [4]. The extension to the continuous functions is immediately clear by the Stone-Weierstrass theorem.

For the second part, note that by construction $T1 = 1$. Substituting 1 for f yields the relation. The operator has the specified eigenfunction by construction. ∎

In passing we remark that since $\nu(k) = \nu(-k)$, the eigenfunction can be written as:

$$f(x) = \sum_{k\in\mathbb{Z}^n} \nu(k) \cos(2\pi k\cdot x) \quad ,$$

and is a function from the torus ($\mathbb{T}^n$) to the reals. We wish to establish conditions under which $Tf = f$ only admits the constant solution. If those conditions are satisfied then all intersection numbers except $\nu(0)$ must be zero and $\Lambda(M, R)$ is a tile.

The following seems to be a folklore result and was brought to our attention by Wiesław Szlenk.

Proposition 2.6 *If $w(x) > 0$, then the only continuous solution of $Tf = f$ is the constant solution.*

Proof: A non-constant continuous function on the torus has at least a maximum and minimum. Let x be an absolute extremum. From theorem 2.5 we conclude that the value of f at x is the weighted average average of the values of f at the inverse images of x under $M^{\dagger}$. Then these inverse images must also be absolute extrema. Going backward indefinitely, it is well-known that these preimages form a dense set. ■

Unfortunately, this proposition doesn't get us very far. Since $w(0) = 1$, we must have that $w(M^{\dagger -1}(j)) = 0$ when $j \in J - \{0\}$. In [1] a more general result is stated in which zeroes of $w(x)$ are allowed. However, they do require other conditions on $w(x)$ which are not satisfied here. Indeed, by only allowing $w(x)$ to have zeroes the otherwise simple problem of proposition 2.6 becomes highly complicated. In fact, in general one does not have uniqueness as we shall see.

3 Solving the Perron-Frobenius Equation

In this section we prove our main result in three steps. We will use the notation established in the previous sections. In the first of these, we establish that in a number of interesting cases the zeroes of the weight $w(x)$ of the Perron-Frobenius operator are isolated. We then derive the condition on M and R for which $Tf = f$ has a unique solution in the class of trigonometric polynomials. Finally, we verify these conditions in a number of cases in dimension one and two.

It is conceivable that our methods extend to a more general two-dimensional context, but certainly they cannot be generalized to higher dimensions without substantial change.

We will limit ourselves to a special case by requiring that the zeroes of $w(x)$ be isolated. Clearly, since $w(x)$ is a trigonometric function, this requirement is satisfied in dimension 1. In general, we have that (see equations (2.5) and (2.6)

$$w(x) = 0 \Leftrightarrow \phi(x) = 0$$

and $\phi : \mathbb{T}^n \to \mathbb{C}$ is a smooth periodic function. Clearly, we expect ϕ to have isolated zeroes only in non-degenerate one- or two-dimensional cases. So these will be the only cases we deal with.

Proposition 3.1 *Suppose that the dimension $n = 2$ and $|\det M| = 3$. $w(x)$ has only isolated zeroes if and only if R spans $\mathbb{R}^2$.*

Proof: Let $R = \{(0, 0), r_1, r_2\}$ and suppose that r_1 and r_2 are independent. Interpreting these two vectors as column vectors, define the matrix

$$N = (r_1, r_2) \quad ,$$

and the corresponding coordinate change $y = N^\dagger x$, where $N^\dagger$ is the transposed of N. Then

$$\phi(y) = 1 + e^{2\pi i y_1} + e^{2\pi i y_2} \quad .$$

(Note that the $y_i = r_i \cdot x$ are independent.) The image of ϕ in $\mathbb{C}$ can easily be visualized as a 'flattened' torus $\mathcal{T}$ (see figure 3.1). Notice that $y \in N^\dagger[0, 1]^2$. Thus ϕ covers the flattened torus $|\det N^\dagger|$ times. Therefore, the point $0 \in \mathcal{T}$ has exactly $2|\det N \dagger|$ pre-images. ∎

Proposition 3.2 *Suppose that the dimension n is two and m is four. Suppose that $\{r_1, r_2\} \subset R$ spans $\mathbb{R}^2$ and write $R = \{(0, 0), r_1, r_2, a_1r_1 + a_2r_2\}$. Then w has finitely many zeroes if and only if for all $j, k \in \mathbb{Z}$:*

$$(a_1, a_2) \notin \left\{ (1, \frac{2k+1}{2j+1}), (\frac{2k+1}{2j+1}, 1), (\frac{2k+1}{2j+1}, -\frac{2k+1}{2j+1}) \right\} \quad .$$

Proof: Using the conventions and notation of the previous proof and $r_3 = a_1r_1 + a_2r_2$, we get

$$w(y) = 1 + e^{2\pi i y_1} + e^{2\pi i y_1} + e^{2\pi i (a_1 y_1 + a_2 y_2)} \quad .$$

Rewrite this as:

$$\begin{aligned} \phi(y) &= (e^{-2\pi i \frac{y_1}{2}} + e^{2\pi i \frac{y_1}{2}})e^{2\pi i \frac{y_1}{2}} \\ &\quad + (e^{-2\pi i \frac{1}{2}(a_1 y_1 + (a_2 - 1)y_2)} + e^{2\pi i \frac{1}{2}(a_1 y_1 + (a_2-1)y_2)})e^{2\pi i \frac{1}{2}(a_1 y_1 + (a_2+1)y_2)} \quad . \\ &= 2e^{2\pi i \frac{y_1}{2}} \left\{ \cos(2\pi \frac{y_1}{2}) + \cos(2\pi \frac{a_1 y_1 + (a_2 - 1)y_2}{2})e^{2\pi i \frac{1}{2}((a_1-1)y_1 + (a_2+1)y_2)} \right\} \quad . \end{aligned}$$

This gives zero if one of the two following holds

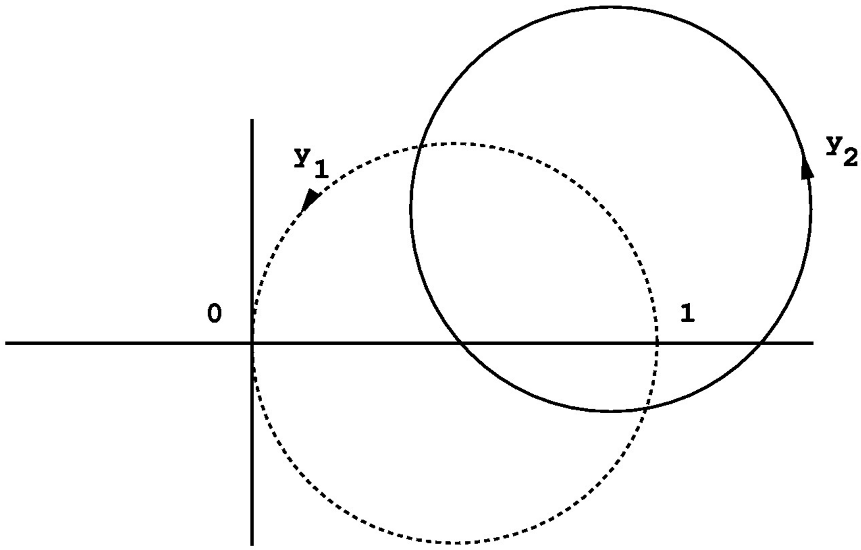

Figure 3.1: *A flattened torus.*

- A. Both cosines yield zero.
- B. The exponential is real and the cosines cancel.

Case A:

$$\begin{cases} \dfrac{y_1}{2} & = \dfrac{n+\frac{1}{2}}{2} \\ \dfrac{a_1 y_1 + (a_2-1) y_2}{2} & = \dfrac{m+\frac{1}{2}}{2} \end{cases} .$$

This is equivalent to

$$\begin{pmatrix} 1 & 0 \\ a_1 & a_2-1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} n+\frac{1}{2} \\ m+\frac{1}{2} \end{pmatrix} .$$

These equations are dependent if and only if $a_2 = 1$ <u>and</u>

$$a_1(2n+1) = 2m+1 \Leftrightarrow a_1 = \frac{2m+1}{2n+1} .$$

Case B:

$$\begin{cases} \dfrac{(a_1-1)y_1 + (a_2+1)y_2}{2} & = n + \dfrac{\delta}{2} \\ \epsilon \dfrac{y_1}{2} & = \dfrac{a_1 y_1 + (a_2-1)y_2}{2} + m + \dfrac{1-\delta}{2} \end{cases} ,$$

where $\delta \in \{0,1\}$ and $\epsilon \in \{-1,1\}$. Or, equivalently:

$$\begin{pmatrix} a_1-1 & a_2+1 \\ -(a_1-\epsilon) & -(a_2-1) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2n+\delta \\ 2m+1-\delta \end{pmatrix} .$$

where $\epsilon \in \{-1,1\}$ and $\delta \in \{0,1\}$. These equations are dependent if and only if

$$\text{determinant} = (1-\epsilon)a_2 + 2a_1 - 1 - \epsilon = 0 .$$

Thus when $\epsilon = 1$ we have dependency when

$$a_1 = 1 \text{ and } a_2 = \frac{2n - 2m + 2\delta - 1}{2n + 2m + 1} = \frac{2k+1}{2j+1} \quad .$$

When $\epsilon = -1$

$$a_1 = -a_2 \text{ and } a_2 = \frac{2m - 2n - 2\delta + 1}{2m + 2n + 1} = \frac{2k+1}{2j+1} \quad .$$

■

Let f be a non-constant eigenfunction of the transition operator. The set of absolute extrema of f will be denoted by $\mathcal{E}_f$. The set of zeroes of the weight function w will be denoted by $\mathcal{Z}_w$.

In the following result we use the fact that the eigenfunction $f(x) = \sum \nu(k) e^{2\pi i k \cdot x}$ we are looking for, has finitely many Fourier components (that is: f is a trigonometric polynomial).

Proposition 3.3 *Let f be a non-constant trigonometric eigenfunction of the transition operator. If $\mathcal{Z}_w$ is finite then $\mathcal{E}_f$ is a finite union of cycles under $M^\dagger$.*

Proof: Suppose that $\mathcal{E}_f$ is not finite, then it is a finite collection of arcs (because f is a trigonometric polynomial). From the proof of proposition 2.6 we conclude that

$$M^{\dagger -1}(\mathcal{E}_f) \subseteq \mathcal{E}_f \cup \mathcal{Z}_w \quad .$$

Define

$$K = \mathcal{E}_f - \cup_{i=0}^{\infty} M^{\dagger i}(\mathcal{Z}_w) \quad .$$

Since $\cup_{i=0}^{\infty} M^{\dagger i}(\mathcal{Z}_w)$ is countable, we see that K is not empty and we have:

$$M^{\dagger -1}(K) \subseteq K \quad .$$

However, since $M^{\dagger -1} : H(\mathbb{T}^2) \to H(\mathbb{T}^2)$ is a contraction K contains a compact set, we have that

$$\text{Hlim}_{\,n \to \infty} M^{\dagger -n}(K) = \mathbb{T}^2 \quad .$$

This gives a contradiction, proving that $\mathcal{E}_f$ is finite.

Notice that the pre-image of each extremum contains at least one extremum and the complete pre-images of distinct points are distinct. It then follows that the pre-image of each extremum contains exactly one extremum. This implies the result. ■

Theorem 3.4 *Let $\mathcal{Z}_w$ is finite. Then there exists $j \in \mathbb{Z}$ such that $x_0 \in \mathcal{E}_f$ implies the following statements:*
i) There are at least two $k \in \mathbb{Z}^n$ such that

$$x_0 = (M^j - I)^{\dagger -1} k \in [0,1]^n \quad .$$

ii) For all $r \in R$, we have

$$r \cdot x_0 \in \mathbb{Z}^n \quad .$$

Remark: Note that the hypothesis of this theorem can only hold if the dimension n takes the value 1 or 2.

Proof: Let j be the product of the periods of the finitely many cycles referred to in the previous proposition. Then the first part follows from the periodicity of f and the fact that for both the minimum and the maximum we have:

$$M^{\dagger j} x_0 = x_0 + k \quad .$$

The second equation comes because $M^{\dagger -1}x_0$ contains exactly one point x_1 of $\mathcal{E}_f$ by the previous proposition and so $w(x_1) = 1$. The statement for x_0 follows by periodicity. ■

We consider some applications of this theorem. The first is due to [4], but with a different proof. The second is an extension of this result.

Proposition 3.5 *In one dimension $\pi : \Lambda(M, R) \to \mathbb{R}/\mathbb{Z}$ covers exactly q times (almost everywhere) where q is the greatest common divisor of R.*

Proof: First divide R by q. Now assume that $\Lambda(M, R)$ is not a tile. Then the transition operator for the new system still has a non constant eigenfunction f associated with the eigenvalue 1. Note that by the previous theorem, f has one extremum in $(0, 1)$. Call this extremum x_0. Thus

$$x_0 = \frac{k}{M^n - 1} = \frac{p}{q}$$

in smallest terms. By the second part of the theorem:

$$r \cdot \frac{p}{q} \in \mathbb{Z} \quad .$$

Thus all r are divisible by q. This is a contradiction. ■

Notice that the theorem implies that $\Lambda(M, R)$ is a tile if and only if the greatest common divisor of R is 1. In fact, $\Lambda(M, qR)$ is a tile by $q\mathbb{Z}$ (see definition 1.1). A more general version of this last statement is proved in lemma 4.4.

Proposition 3.6 *Suppose M and R are such that w has only isolated zeroes. If two elements of R satisfy $|r_1 \times r_2| = 1$, then $\Lambda(M, R)$ is a tile.*

Proof: As before, if $\Lambda(M, R)$ is not a tile, then the transition operator for the system has a non constant eigenfunction f associated with the eigenvalue 1. It must have an extremum $x_0 \in [0, 1]^2 - \mathbb{Z}^2$. Interpreting the vectors r_1 and r_2 of the proposition as column vectors, define the matrix

$$N = (r_1, r_2) \quad .$$

By the second part of the theorem, we have that $N^\dagger x_0 \in \mathbb{Z}^2$. But by definition $N^\dagger$ restricted to $\mathbb{Z}^2$ is a bijection. So this gives a contradiction. ■

In special cases one can derive stronger results then the previous corollary. For example, when $M = 2I$. For j as in the theorem, define $\Gamma_j = [\frac{1}{2^j - 1} \cdot \mathbb{Z}]^2 \cap [0, 1]^2$. Now we have that

$$\mathcal{E}_f \subseteq \Gamma_j \quad .$$

It is easy to check whether, for a given R, $r \cdot \Gamma_j$ is in $\mathbb{Z}^2$.

4 Examples

In this section we state a sharp criterion for $\Lambda(M,R)$ to be a tile by $\mathbb{Z}^n$. Using that criterion we give some two dimensional examples of sets $\Lambda(M,R)$ that are not tilings by $\mathbb{Z}^2$.

Recall the definition of the difference D of R (equation (2.3)). Define $G \subseteq \mathbb{Z}^n$ as

$$G(M,R) = \{x \in \mathbb{Z}^n \,|x = \sum_{i=1}^{\infty} M^i d_i \text{ with } d_i \in D\} \quad .$$

The first proposition is an easy consequence of results of [6].

Proposition 4.1 *$\Lambda(M,R)$ is a tile by $\mathbb{Z}^n$ if and only if $G = \mathbb{Z}^n$.*

Proof: $\Leftarrow$: See [6].
$\Rightarrow$: Suppose that Λ is a tile and there is a $k \in \mathbb{Z}^n$ such that $k \notin G$. Then by the tiling property, we have that for all $v \in G$, $\mu((\Lambda + k) \cap (\Lambda + v)) = 0$. Thus

$$\cup_{v \in G}\{\Lambda + v\} \subseteq \mathbb{R}^n - \{\Lambda + k\}$$

This contradicts property 1.13 of [6]. ∎

Definition 4.2 *Let M and R be as usual and A a linear isomorphism whose matrix has integer entries. We call A a common divisor of (M,R) if $A^{-1}MA \in GL(n,\mathbb{Z})$ and $A^{-1}R \in \mathbb{Z}^n$.*

In one dimension, the definition reduces to the usual one. Furthermore, the definition is consistent in that the following lemma holds.

Lemma 4.3 *If R is a complete set of residues modulo $M\mathbb{Z}^n$, then $A^{-1}R$ is a complete set of residues modulo $A^{-1}MA\mathbb{Z}^n$.*

Proof: Recall that D is the difference set of R. A set R of cardinality $|\det M|$ is a complete set of residues if and only if

$$D \cap M\mathbb{Z}^n = 0 \quad .$$

Thus the elements $A^{-1}r_i$ of $A^{-1}R$ satisfy:

$$A^{-1}D \cap A^{-1}M\mathbb{Z}^n = 0 \quad .$$

Since $\mathbb{Z}^n \subset A\mathbb{Z}^n$, the result follows. ∎

Lemma 4.4 *Let (M,R) have common divisor A and suppose that $\Lambda(A^{-1}MA, A^{-1}R)$ is a tile by $\mathbb{Z}^n$. Then $\Lambda(M,R)$ is a tile by $A\mathbb{Z}^n$.*

Proof: One easily verifies that

$$\begin{aligned} \Lambda(M,R) &= A \cdot \Lambda(A^{-1}MA, A^{-1}R) \quad . \\ G(M,R) &= A \cdot G(A^{-1}MA, A^{-1}R) \quad . \end{aligned}$$

By proposition 4.1, $G(A^{-1}MA, A^{-1}R) = \mathbb{Z}^n$, thus $G(M,R) = A\mathbb{Z}^n$. Now our result follows from [6], theorem 1.22. ∎

As examples define

$$
\begin{aligned}
M_n &= \begin{pmatrix} 2 & n \\ 0 & 2 \end{pmatrix} \quad , \\
R_s &= \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} s \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} s \\ 1 \end{pmatrix} \right\} \quad , \\
A_m &= \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \quad .
\end{aligned}
$$

Notice that we cannot use the results of section 3 to prove that $\Lambda(M_n, R_1)$ is a tile, because $w(x) = (1 + e^{2\pi i x})(+e^{2\pi i y})$, whose zeroes are not isolated. However, we will employ the criterion of proposition 4.1 to prove this.

Proposition 4.5 *$\Lambda(M_n, R_1)$ is a tile.*

Proof: The difference set D_1 of R_1 consists of the points $(\delta_1, \delta_2) \in \mathbb{Z}^2$ with $\delta_i \in \{-1, 0, 1\}$. (The multiplicities are of no consequence here.) Denote by G_i all those points in $\mathbb{Z}^2$ whose second coordinate lies in the interval $[-2^i + 1, 2^i - 1]$. To prove the proposition we will prove that for all i, $G_i \subset G$.

We first show that $G_0 \subset G$. G_0 is the set $\mathbb{Z} \times \{0\}$. The matrix M_n acts on this sets as multiplication by 2 (in $\mathbb{Z}$). So representing G_0 on the basis M_n with digits $(\delta_1, 0)$ where $\delta_1 \in \{-1, 0, 1\}$ is thus equivalent to representing all numbers in $\mathbb{Z}$ as

$$
\sum_{i=0}^{\infty} 2^i \mathcal{E} \;\text{ where }\; \mathcal{E} = \{-1, 0, 1\} \quad .
$$

Now we show that if $G_i \subset G$ then $G_{i+1} \subset G$. We are done if for every point $(c, d) \in G_{i+1}$ there is a point $(a, b) \in G_i$ such that

$$
M_n \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} 2a + nb - \delta_1 \\ 2b - \delta_1 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \quad .
$$

These equations can be solved by determining b and δ_2 from the second equation and then a and δ_1 from the first equation.

By induction on i, we obtain that for all $i \in \mathbb{Z}$, $G_i \subset G$. ■

Corollary 4.6 *For $n, s \in \mathbb{Z}$, $\Lambda(M_{ns}, R_s)$ is a tile by $A_s\mathbb{Z}^2$.*

Proof: Apply lemma 4.4 to obtain that

$$
\Lambda(M_{ns}, R_s) = A_s \cdot \Lambda(M_n, R_1) \quad .
$$

Then use proposition 4.5. ■

We remark that only for s odd is R_s a complete set of residues modulo $M_{ns}\mathbb{Z}^2$.

In the above examples G is a group. From proposition 3.5 it easily follows that this is always the case in one dimension. In higher dimensions, the situation is more complicated. For instance, in dimension two, consider the system (M_1, R_3) (see [8]). One easily verifies that this system does not admit a common divisor with determinant of absolute value greater than one. The reason being, of course, that matrix multiplication is not commutative. In this case one also easily verifies that $G(M_1, R_3)$ is not a group: This set contains $(0, 1)$ and $M(0, 1) = (1, 2)$. The cross-product of these two vectors is 1, so the only additive subgroup of $\mathbb{Z}^2$ containing both of them is $\mathbb{Z}^2$. It is easy to verify that $(1, 0)$ and $(2, 0)$ are not in $G(M_1, R_3)$. What happens in this case is that $G + \Lambda$ covers $\mathbb{R}^2$ more than once. One checks easily that $3 + 4\cos(2\pi x) + 2\cos(4\pi x)$ is a non-trivial eigenfunction, with eigenvalue 1, of the operator T associated with this problem as discussed in section 2.

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