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SYMBOLIC DYNAMICS AND ROTATION NUMBERS

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In the space of binary sequences, minimal sets, that is: sets invariant under the shift operation, that have no invariant proper subsets, are investigated. In applications, such as a piecewise linear circle map and the Smale horseshoe in a mapping of the annulus, each of these sets is invariant under the mapping. These sets can be assigned a unique rotation number equal to the average of the number of ones in the sequences.

One special minimal set is the optimal set for which the convergence of the running average to the rotation number is faster than for any other minimal set. These sequences are instrumental in analytically solving for the width of the parameter intervals for which members of a one parameter family of piecewise linear critical circle maps have rational rotation number.

1. Introduction

In the study of nonlinear maps and flows, aspects of complicated behavior can often be described in terms of symbolic dynamics. In this approach, the action of the mapping restricted to an appropriate subset of the domain, is shown to be conjugate to a subshift on a set of sequences of finitely many symbols[1]). The simplest case is the full shift on the space $\{0, 1)^{Z}$ of two symbol sequences, which arises, for example, in the study of Smale's horseshoe map[2]).

In this study we shall work in the space S of bi-infinite sequences of ones and zeros. We equip the space with the norm $\Sigma_{-\infty}^{+\infty} i_n/2^{|n|}$ where i_n are the digits of an element s of S. This metric induces one to establish an isometry of S to the set of pairs of real numbers $(x, y) \in [0, 1\rangle \times [0, 2\rangle$:

$$x \equiv \sum_{1}^{\infty} \frac{i_n}{2^n}, \quad x \in [0, 1\rangle ,$$

$$y \equiv \sum_{-\infty}^{0} \frac{i_n}{2^{-n}}, \quad y \in [0, 2\rangle \quad \left(\text{includes } \frac{i_0}{2^0}\right).$$

S with the given metric corresponds to the rectangle $[0, 1\rangle \times [0, 2\rangle$ equipped with the induced "Manhattan"-metric (the sum of the x and y distances rather than the root of the squares of the sums). S is then obviously a connected space because the rectangle is connected. When later on we consider one-sided sequences we shall prove that the sets we are interested in project one-to-one onto the x-axis, so that we can identify bi-infinite sequences with semi-infinite ones.

Let σ denote the shift operation, i.e.

$$\sigma(s) = \sigma(\ldots i_{-2}i_{-1}i_0 \cdot i_1i_2 \ldots) = \ldots i_{-1}i_0i_1 \cdot i_2i_3 \ldots .$$

We propose to study sets minimal with respect to the shift. A minimal set is a closed set which is invariant under the transformation of interest and which contains no proper subset with the same properties[3]). The minimal sets we consider can be assigned a "rotation-number", ρ, defined as follows. Let $\rho_n(s) = (1/n)\,\Sigma_1^n\, i_n$, then we consider sets for which the following limits exist and have the same value:

$$\lim_{n\to\infty} \rho_n(s) = \lim_{n\to-\infty} \rho_n(s) \equiv \rho(s):$$

$\rho(s)$ is simply the average number of ones in the sequence. These minimal sets turn out to be Cantor sets in the space of sequences. (A Cantor set is a closed, perfect and totally disconnected set.)

The reason for calling the quantity ρ a rotation-number is that in applications it corresponds to the usual rotation-number. In Hockett and Holmes[4]) the authors consider a Poincaré section of the Josephson junction equation which defines a mapping on the two-dimensional annulus. They prove that, in a certain parameter-range, there is a transversal interaction of stable and unstable manifolds giving rise to a Smale horseshoe. Restricted to the horseshoe, the mapping is conjugate to the shift. They then show that the average number of ones (if this limit exists) of a sequence s corresponding to point x in the horseshoe is a very simple manner related to the rotation-number of x under the Poincaré map. Minimal sets with rotation-number ρ in S represent minimal sets in the horseshoe under the Poincaré map.

In this paper we extend their use of symbolic dynamics in the study of rotation-numbers and indicate applications to invariant sets of a piecewise linear analogue of a critical circle map.

In section 2 we start by constructing a specific minimal set for each irrational number ρ (the number ρ will denote an irrational number throughout this work, unless otherwise mentioned), called the optimal set $\bar{\Sigma}_\rho$. This is the minimal set for which the running averages ρ_n converge faster to ρ than for any other minimal set. The sequences in $\bar{\Sigma}_\rho$ exhibit great regularity. Although non-periodic, each finite subsequence in $s \in \bar{\Sigma}_\rho$ reappears infinitely often. In this section we develop a geometrical algorithm for generating all the elements of $\bar{\Sigma}_\rho$.

Under certain conditions, these sequences correspond to the "order-preserving orbits" constructed by Hockett and Holmes[4]) (i.e., the points on each orbit have the same relative ordering as the orbit of a point under rigid rotation).

In section 3 we prove that the optimal sets are minimal sets as well as Cantor sets in S. The proofs are quite general and apply also to the other "non-optimal" examples of minimal sets that we give in this section. In one example, we construct for each irrational ρ uncountably many minimal sets, all of which are Cantor sets. This result also carries over to the example of Hockett and Holmes: for each rotation-number ρ there are uncountably many minimal Cantor sets in the horseshoe. The other example we give is a construction very similar to Aubry's construction of minimal energy orbits for area preserving monotone twist maps of the annulus[5]).

In section 4 it is argued that the construction of sequences in $\bar{\Sigma}_\rho$ is intimately related to the theory of continued fractions. It is proven that some of the sequences in $\bar{\Sigma}_\rho$ can be constructed from the continued fraction coefficients β_i of ρ, where

$$\rho = \cfrac{1}{\beta_1 + \cfrac{1}{\beta_2 + \cdots}} .$$

This construction is entirely symbolic, i.e.: it consists of concatenating finite subsequences without doing any calculations. The sequences we obtain this way are self-similar, non-periodic sequences, self-repeating on every length-scale. Some of the sequences constructed in this fashion turn up quite naturally in an irrational decimation procedure of a path-integral[6]).

In section 5 we work out a particular application of optimal strings. Consider the complex map $f: z \to z^2$ restricted to the unit-circle $|z| = 1$ parametrized by the angle $x = (1/2\pi)\arg(z)$ (this is simply the angle-doubling map: $\theta: x \to 2x$, mod 1). Then, writing x as a binary number, θ is just the shift map. Among all the minimal sets under θ, the only ones that are contained in a closed interval $[\beta/2, \beta/2 + 1/2]$ for some $\beta \in [0, 1]$ are the optimal sets $\bar{\Sigma}_\rho$ for irrational ρ.

We then modify the map θ by mapping the complement of this inverval to β (see fig. 1) and call the resulting "cut-off" map ϕ_β. The piecewise linear map ϕ_β is now a degree one map. It is an analogue of a critical map in that it has a piece with derivative zero (fig. 12). For the family ϕ_β, we work out exactly the

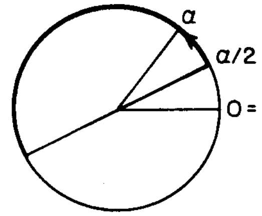

Fig. 1. Angle-doubling on the circle.

length of the locking intervals (the intervals in parameter space where ϕ_β has periodic orbits) using the properties of optimal sequences. This result is then generalized to apply to degree one mappings $\phi_{\tau,\beta}$ obtained in a similar manner from

$$\theta_\tau : x \to \tau x (\mathrm{mod}\ 1)$$

where τ is an arbitrary real number greater than one. In addition to this we find the exact representation of the invariant attracting set $\phi_{\tau,\beta}$ for all rotation numbers. In Kadanoff[7]) and Boyland[8]) a similar "cut-off" map is used to obtain information about orbits in a supercritical map.

Finally in this introduction we give the notation for equivalence-classes that we will use. In $[0, 1\rangle$ an equivalence relation is defined by $\sim$:

$$x_1 \sim x_2 \text{ iff } x_1 = (k\rho + x_2) \bmod 1, \quad k \in \mathbb{Z} .$$

The equivalence-classes

$$\{x | x = (d + k\rho) \bmod 1 |\ k \in \mathbb{Z}\}$$

are denoted by $[d]$, so that for instance $[\rho]$ includes the number 0.

In $\langle 0, 1]$ the equivalence is defined similarly. The only relevant distinction to make is that now 1 is an element of $[\rho]$, whereas 0 is not in the interval. This distinction plays a role in proving some of the results (lemma 3.4).

The number ρ is fixed and irrational throughout the paper except in section 5 where it is being varied.

2. The optimal sets

In this section we address the question of constructing sequences for which the running rotation number $\rho_n(s)$ approaches its limit as fast as possible (optimal sequences). The sequences will in fact be such that, for every k, $\rho_n(\sigma^k s)$ approaches its limit as fast as possible. The construction has a very geometrical flavor to it.

A way of visualizing sequences is the following. For each $n \in \mathbb{Z}$ pick integers p_n that satisfy (fig. 2)

$$i_n \equiv p_n - p_{n-1}\,, \quad i_n \in \{0, 1\}$$

and i_n are the digits of the sequence s. Note that $p_n - p_0$ gives the number of ones in the digits 1 through n (for negative n: $-p_n - p_0$ gives the number of ones in the digits $n+1$ through 0). In what follows p_0 will always be equal to zero. The running rotation number $\rho_n(s)$ is simply the tangent of the line through the origin and (n, p_n) (see fig. 2):

$$\rho_n(s) = \frac{p_n}{n}\,.$$

The question that arises first is: which sequences have converging running averages ρ_n? Denote the limit of ρ_n by ρ. It is then obvious (divide by n) that these sequences are characterized by

$$|n\rho - p_n| = o(n) \quad (n \to \pm\infty)\,.$$

The points (n, p_n) for sequences with a rotation-number ρ lie in an ever-widening band around the line ρx. The width of the band grows slower than n asymptotically.

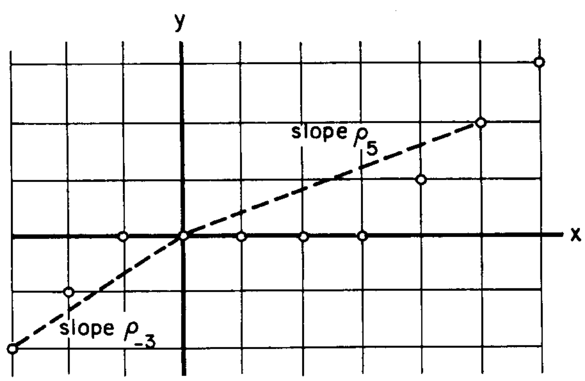

Fig. 2. The dots indicate the pairs $\phi(n, p_n)$. $p_{-2} = -1$ and $p_{-1} = 0$ so $i_{-1} = 1$; $p_5 = 2$ and $p_6 = 3$ so $i_6 = 1$.

Let $I(x)$ denote the integer part of x. We define Σ_ρ (not $\bar{\Sigma}_\rho$), a subset of the sequences with rotation number ρ, as follows:

$$p_n = I(n\rho + d)\,,$$

with the definition of i_n in terms p_n as before. In other words

$$-d \leqslant n\rho - p_n < 1 - d\,, \qquad d \in [0, 1\rangle\,, \tag{2.1}$$

or equivalently

$$0 \leqslant n\rho + d - p_n < 1\,, \qquad d \in [0, 1\rangle\,. \tag{2.2}$$

Sequences constructed this way will have rotation-number ρ (see fig. 3). We will call d the index.

Another subset Σ_ρ^* of sequences with rotation-number ρ is defined in a similar way,

$$-d < n\rho - p_n \leqslant 1 - d\,, \qquad d \in \langle 0, 1]\,, \tag{2.3}$$

or

$$0 \leqslant n\rho + d - p_n \leqslant 1\,, \qquad d \in \langle 0, 1]\,. \tag{2.4}$$

The integers p_n for a sequence s in either Σ_ρ or in Σ_ρ^* are uniquely defined and thus so is the resulting sequence s. If ρ is given with sufficient accuracy many digits of s can be easily calculated with a pocket-calculator using criterion (2.1) or (2.3).

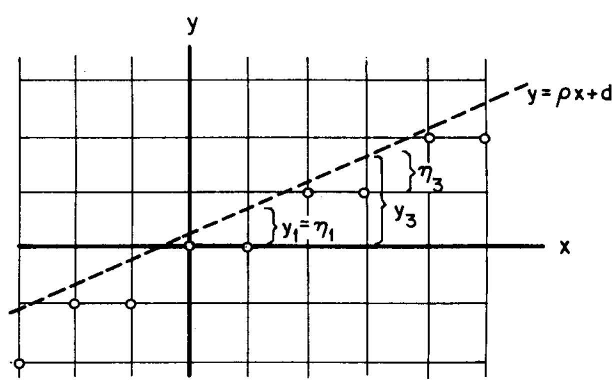

Fig. 3. The dots indicate the pairs (n, p_n).

The optimal set $\bar{\Sigma}_\rho$ (optimal with respect to ρ) is defined to be the union of the elements of Σ_ρ and of Σ^*_ρ. It is easy to see that in (2.2) and (2.4) equality is not attained if $d \not\in [\rho]$ (see section 1 for equivalence-classes), because in this case $d + n\rho$ is irrational for all n. So for every index $d \in [0, 1]\backslash[\rho]$ there is a unique sequence $s \in \bar{\Sigma}_\rho$. If $d \in [\rho]$ equality is attained and there are 2 binary sequences in $\bar{\Sigma}_\rho$ corresponding to that index, one in Σ_ρ and the other in Σ^*_ρ.

Consider the vertical distances η_n in fig. 3 defined by

$$y_n \equiv n\rho + d \quad \text{and} \quad \eta_n \equiv n\rho + d - p_n = y_n \bmod 1 . \tag{2.5}$$

So η_n is generated by a rigid rotation on the circle $[0, 1\rangle$ with initial condition $\eta_0 = d \in [0, 1\rangle$. The sequences in Σ_ρ can also be constructed by partitioning the circle as in fig. 4.

$$\begin{aligned} i_n &= 1 , \quad \text{if} \quad \eta_n \in [0, \rho\rangle , \\ i_n &= 0 , \quad \text{otherwise} . \end{aligned} \tag{2.6}$$

(This is the construction described by Hockett and Holmes[4]).) From definition (2.5) of η_n it follows that $\eta_n \in [0, \rho\rangle$ is equivalent with $p_n - p_{n-1} = 1$, so that we indeed construct the same sequences. The elements of Σ^*_ρ can be gotten by choosing d in $\langle 0, 1]$ and interchanging open and closed endpoints in (2.6). Again for $d \not\in [\rho]$ η_n never assumes the values 0 or ρ.

The map s from $[0, 1]$ to $\bar{\Sigma}_\rho$ that assigns to d the sequence belonging to that index is well-defined on $[0, 1]\backslash[\rho]$. We denote equivalence-classes of sequences with d-equivalent indices by $[s_d]$. Then for

$$s : [0, 1]\backslash[\rho] \to \Sigma_\rho\backslash[s_\rho]$$

we have

Lemma 2.1. The map s restricted to $[0,1]\backslash[\rho]$ is a homeomorphism onto its image.

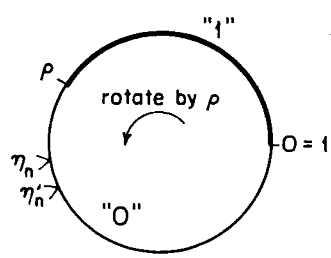

Fig. 4. Rotation by ρ.

Proof. By definition s is well defined and onto. To prove that it is a bijection, let d and d' be two different indices. According to (2.5) we construct two sequences s and s' with

$$\eta'_n = \eta_n + (d' - d) + (p_n - p'_n) \tag{2.7}$$

and η_n and η'_n are rotations on the circle (fig. 4). Their orbits are dense and so for some n, η_n falls in a "1" region while η'_n does not. Thus s is a bijection. To prove continuity let $d \not\in [\rho]$. The numbers η_n, see (2.5), for $-N \leqslant n \leqslant N$ are bounded away from 0 to 1 by a quantity $\delta(N)$ which is decreasing with N. For all d' with $|d' - d| < \delta(N)$, we have in the right-hand side of (2.7)

$$\eta_n + (d' - d) \in [0, 1\rangle, \quad |n| \leqslant N.$$

Therefore $p_n = p'_n$ for $|n| \leqslant N$ and thus

$$d(s, s') < \varepsilon = 2^{-N+2}. \qquad \blacksquare$$

In fact for the bijection part we don't have to restrict s to $[0, 1\rangle \backslash [\rho]$. The map establishes a bijection between $[0, 1\rangle$ and Σ_ρ. This implies that the cardinality of Σ_ρ (and of $\bar{\Sigma}_\rho$) is that of the continuum. (The homeomorphism is needed in a later section.)

The following is a statement of the invariance of $\bar{\Sigma}_\rho$ under the shift.

Lemma 2.2. If $s \in \Sigma_\rho(\Sigma^*_\rho)$ has index d then $s' \equiv \sigma^k s$ is the (unique) element in $\Sigma_\rho(\Sigma^*_\rho)$ with index $d' = (k\rho + d) \bmod 1$.

Proof. The sequence s' can be constructed from the same figure as s by translating the origin over the vector (k, p_k). The line ρx now intersects the new vertical axis at $(k\rho + d) \bmod 1$. See fig. 5. $\blacksquare$

We now give a formal definition of an optimal sequence and in the main theorem below we prove that elements of Σ_ρ satisfy this definition.

Definition. An optimal sequence s (with respect to ρ) is a sequence for which

$$\forall k, n \in \mathbb{Z}, \quad |\rho - \rho_n(\sigma^k s)| < \frac{1}{n}. \tag{2.8}$$

Note that if ρ is rational, say p/q, then $\rho_q(\sigma^k s)$ has to be equal to p/q for all k (this would not be true if we replace $<$ by $\leqslant$ in 2.8 so that $\rho_q(\sigma^k s)$ could also be

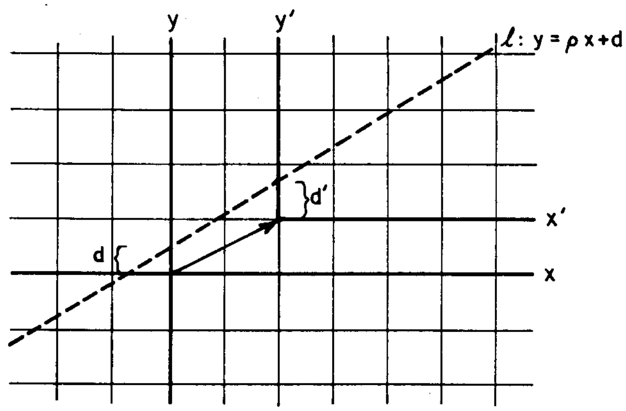

Fig. 5. The shift.

equal to $(p \pm 1)/q$). Now suppose that ρ is irrational and s is optimal. Then in *every* n consecutive digits there are either m or $m+1$ ones where

$$\frac{m}{n} < \rho < \frac{m+1}{n}.$$

For example, if ρ is the golden mean ($=0.6180339887\ldots$) then *every* 3 consecutive digits contain either one or two ones, *every* 4 consecutive digits contain either two or three ones, and so forth. Optimal sequences provide us with the most regular way to make an irrational average with ones and zeros in the sense that there is a minimal amount of accumulation of identical groups of digits: "1101" is possible but "11011101" is not, because $\frac{6}{8}$ is not close enough to ρ (also it contains a subsequence with three ones in three digits which is not allowed).

Our main theorem asserts that all optimal sequence are elements of $\bar{\Sigma}_\rho$ and all elements of $\bar{\Sigma}_\rho$ are optimal sequences (ρ irrational):

Theorem. $s \in \bar{\Sigma}_\rho \Leftrightarrow s$ is optimal.

Proof. $\Rightarrow$: Let $s \in \Sigma_\rho$ so that $\sigma^k s \in \Sigma_\rho$ also.
Using (2.1) and using lemma 2.2 we have

$$-(k\rho + d) \bmod 1 \leq n\rho - n\rho_n(\sigma^k s) < 1 - (k\rho + d) \bmod 1 .$$

Since $(k\rho + d) \bmod 1 \in [0, 1\rangle$, by dividing the above relation by n we see that s is optimal.

For $s \in \Sigma_\rho^*$ $\leqslant$ and $<$ change places and now $(k\rho + d) \bmod 1 \in \langle 0, 1]$ which also proves that s optimal.

$\Leftarrow$: Suppose $s \notin \bar{\Sigma}_\rho$. According to (2.1) and (2.3) there are then $l, m \in \mathbb{Z}$ such that for s:

$$m\rho - p_m = \alpha\,, \quad l\rho - p_l = \beta\,, \quad \text{and} \quad |\alpha - \beta| \geqslant 1\,.$$

Now let p'_n count the ones in $\sigma^m s$, then:

$$|(l-m)\rho - p'_{l-m}| = |\alpha - \beta| \geqslant 1\,;$$

so that s is not optimal. ∎

It can be concluded that the sequences constructed in (2.1) and (2.3) for irrational ρ have a very balanced distribution of ones and zeros. This observation leads us to expect the orbits corresponding to these sequences in the horseshoe mapping of Hockett and Holmes[4]) to have nice properties. If there are any order-preserving-orbits, one would expect these orbits to be candidates.

3. Minimal sets

In this section we prove that $\bar{\Sigma}_\rho$ for each ρ is a minimal set under the shift operation. The existence of this minimal set was already proven by Hedlund[3]), although his approach is slightly different. We attempt to do a more detailed analysis that leads to a picture of $\bar{\Sigma}_\rho$, its elements being interpreted as (pairs of) real numbers. We will prove that $\bar{\Sigma}_\rho$ is a Cantor set. In addition we will give some other examples of minimal Cantor sets. All these results apply to the case that Hockett and Holmes have studied but the results are interesting also in their own right.

The lemma that we want to prove next is most easily understood in the context of fig. 6. For the construction of s in, say, Σ_ρ^* we draw the line $l: y = \rho x + d$. According to (2.4), when l crosses a horizontal integer-axis then p_n increases by one unit. Project the positions x_n of the crossing of $y = n$ down to the x-axis:

$$x_n = \frac{1-d}{\rho} + \frac{n}{\rho}\,. \tag{3.1}$$

The convention is that if x_n for some n falls in the interval $[l-1, l\rangle$ then

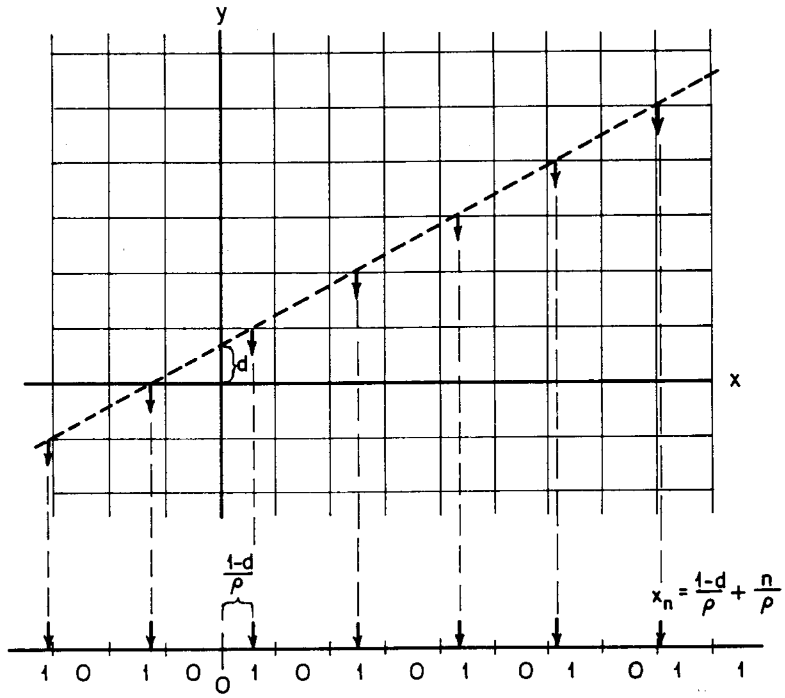

Fig. 6. Projection on the x-axis.

$i_l = 1$. Given the positions x_n one can immediately (by hand) construct the sequence of ones and zeros. (For $s \in \Sigma_\rho$ redefine: $i_l = 1$ if $x_n \in \langle l-1, l]$.)

Lemma 3.1. (a) $\bar{\Sigma}_\rho$ is a closed set. In fact (b) $\bar{\Sigma}_\rho$ is the closure of $\Sigma_\rho \backslash [s_\rho] = \Sigma^*_\rho \backslash [s_\rho]$.

Proof. (a) is proven by showing that $\bar{\Sigma}_\rho$ is equal to the set of its contact-points. We prove now that an arbitrary sequence not in $\bar{\Sigma}_\rho$ is not a contact-point.

So let $r \notin \bar{\Sigma}_\rho$. Just as in the proof of $\Leftarrow$ of the theorem in section 2, there are integers $m, l \in \mathbb{Z}$ such that

$$|(l-m)\rho - p'_{l-m}| = |\alpha - \beta| \geq 1 ,$$

where p'_k counts the ones in $\sigma^m r = r'$. However, every sequence s in $\bar{\Sigma}_\rho$ obeys (theorem in section 2)

$$|(l-m)\rho - p_{l-m}| < 1 ,$$

p_k counting the ones in s. So r' and s have a different number of ones in $l-m$ digits. Then r and s have a different number of ones in l digits.
Or

$$d(s,r) \geqslant 2^{-l}, \quad \forall s \in \bar{\Sigma}_\rho ,$$

and r is not a contact-point.

To prove (b) we have to show that the sequences with index in $[\rho]$ are contact-points of $\Sigma_\rho \backslash [s_\rho]$.

Let $s \in \Sigma_\rho$ have an index $d = -k\rho \bmod 1$, then according to (3.1)

$$x_n = \frac{1+(k\rho \bmod 1)}{\rho} + \frac{n}{\rho} .$$

The equation $x_n = l$ with l integer has a solution $l = k$. So one of the x_n falls on a boundary between intervals: $x_n = +k$. (See fig. 7.) Consider a sequence $s' \in \Sigma_\rho \backslash [s_\rho]$ with index $d' = d + \delta$ $(0 < \delta \ll 1)$. By definition, the kth digits of s and s' are the same; in both cases x_n lands in the kth interval $\langle k-1, k]$. It can be seen from fig. 7 and proven in the manner in which continuity in lemma 2.1 is proven, that for $\delta \downarrow 0$ the distance $d(s, s')$ is decreasing to zero. (In fact this is left-continuity.) This proves that there is an $s \in \Sigma_\rho \backslash [s_\rho]$ in every neighborhood of s. ∎

This proves the hardest lemma in this section. We now have the following:

Lemma 3.2. All orbits in $\bar{\Sigma}_\rho$ are dense in $\bar{\Sigma}_\rho$.

Proof. Orbits $\sigma^k s$ of an element s in $\bar{\Sigma}_\rho \backslash [s_\rho]$ with index d correspond to orbits $(d + k\rho) \bmod 1$ in $[0, 1] \backslash [\rho]$ according to lemma 2.2. The latter orbits are obviously dense in $[0, 1] \backslash [\rho]$. Then, because of the two spaces being homeomorphic, the orbit $\sigma^k s$ is dense in $\bar{\Sigma}_\rho \backslash [s_\rho]$. Now, the elements $[s_\rho]$ are in the closure of $\bar{\Sigma}_\rho \backslash [s_\rho]]$, so $\sigma^k s$ is also dense in all of $\bar{\Sigma}_\rho$.

Consider now the orbit of $r \in [s_\rho]$. We just proved that $\sigma^k s$ is dense. Then

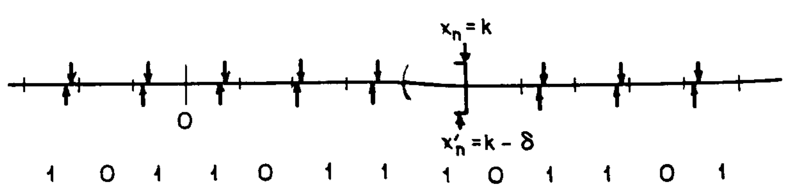

Fig. 7. Upper sequence s with $d \in [\rho]$, lower sequence s' with $d' \notin [\rho]$. s and s' are very close.

for every ε there is an N such that

$$d(\sigma^{-N}s, r) < \frac{\varepsilon}{2^{N+1}}$$

and therefore: $d(s, \sigma^N r) < \varepsilon$ which proves that the orbit of r is dense. ■

Note that also all forward, $k > 0$, orbits are dense everywhere in $\bar{\Sigma}_\rho$, because so are $(d + k\rho) \bmod 1$ for $k > 0$ in $[0, 1]$.

This enables us to state our first main theorem of this section:

Theorem 1. (Hedlund): $\bar{\Sigma}_\rho$ is a minimal set (with respect to σ).

Proof. $\bar{\Sigma}_\rho$ is closed and invariant under σ. Further, every orbit under the shift is everywhere dense, so there is no smaller invariant subset in $\bar{\Sigma}_\rho$. ■

In order to make our proof of the next theorem easier, but also with an eye on applications to be discussed in section 5 we would like to be able to think about sequences as though they were points on the real line. As discussed in the introduction we can identify the space of sequences S with the rectangle $R = [0, 1\rangle \times [0, 2\rangle$ as follows:

$$s \rightarrow (x, y),$$

where

$$s = \cdots i_{-2} i_{-1} i_0 \cdot i_1 i_2 \cdots$$

and

$$x = \sum_1^\infty i_n/2^n, \quad y = \sum_0^\infty i_{-n}/2^n.$$

The norm on the rectangle is just the induced norm

$$\|(x, y)\| = \sum_{-\infty}^{+\infty} i_n/2^{|n|} = |x| + |y|$$

and so R (which is now equipped with the Manhattan metric) and S are isomorphic. Interesting questions, of course, are what do $\bar{\Sigma}_\rho$ and $\cup_{\rho\,\mathrm{irr}} \bar{\Sigma}_\rho$ look like in R? However, in later applications we want to work with semi-infinite sequences and we shall therefore only study the projection of $\bar{\Sigma}_\rho$ on the x-axis (fig. 8): $(x, y) \to (x, 0)$ where x corresponds with a semi-infinite sequence.

Let α denote a real number such that (α, β) corresponds to an element s of $\bar{\Sigma}_\rho$ with index d. The ordering of $\bar{\Sigma}_\rho$ with respect to d is (almost) the same as the ordering with respect to the magnitude of α.

Lemma 3.3. For elements s in $\Sigma_\rho: d > d' \Leftrightarrow \alpha > \alpha'$. Similarly for Σ_ρ^*.

Proof. $\Rightarrow$: Consider two sequences s and s' with indices $d > d'$ (fig. 9). There is a smallest positive integer k such that x_k and x_k' fall in different intervals. According to (3.1) we have $x_k < x_k'$. This implies that $\alpha > a'$.

$\Leftarrow$: The inverse is obtained by noting that $\alpha > \alpha'$ implies $x_k < x_k'$ for some k and thus $d > d'$. ■

To understand the next lemma, recall from section 2 that there are two different sequences for $d = k\rho \bmod 1$, one in Σ_ρ and one in Σ_ρ^*. We will call these sequences s and s*, corresponding to the real numbers α and α^*.

Lemma 3.4. Consider the sequences in $\bar{\Sigma}_\rho$ then (a) $d > d' \Leftrightarrow \alpha > \alpha'$ and $d \neq d'$; (b) Let $d = +k\rho \bmod 1$, there are two sequences with index d, s and s^* which differ only in two digits and if $k < 0$ then $\alpha = \alpha^* + 2^{k-1}$. If $k > 0$ then $\alpha = \alpha^*$. (See also fig. 8.)

Proof. The first statement (a) is equivalent to lemma 3.3. For the proof of (b), note that according to (3.1) the positions x_n (figs. 6 and 7) in the construction of

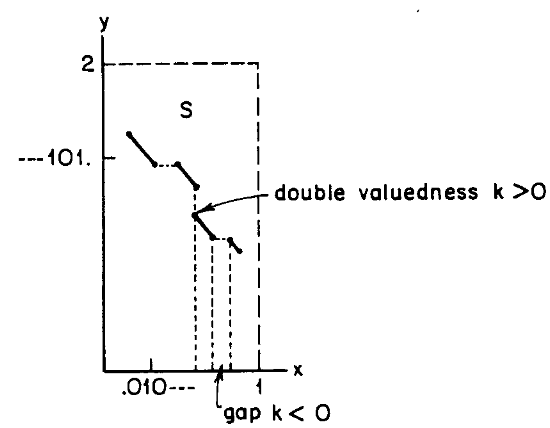

Fig. 8. The set $\bar{\Sigma}_\rho$ for irrational ρ.

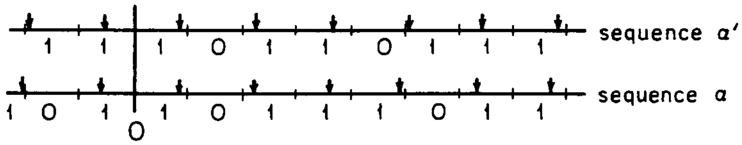

Fig. 9. Two sequences.

s and of s^* are the same. In both cases, there is an n such that $x_n = -k$ and no other x_n fall on a boundary. In constructing s and s^* the only difference is:

	i_{-k}	i_{-k+1}
s	1	0
s^*	0	1

and all the other digits are the same. In particular

$$\alpha = \alpha^* + 2^{k+1}, \quad \text{for } k < 0 . \tag{3.2}$$

For $k = 0$, the indices are different, $d = 0$ and $d^* = 1$ and (a) applies (compare (2.1) and (2.3)). For $k > 0$, in projecting down on the real axis the digits that are different are deleted and α and α^* are identical. ∎

These lemmas are somewhat subtle. Their consequences, however, are far-reaching. First of all, different elements s and s' of $\bar{\Sigma}_\rho$ correspond to different α and α' except when s and s' both have index $k\rho$ mod 1, k > 0. Since both sets, $\bar{\Sigma}_\rho$ and its projection, are so closely related we will denote them both (an abuse of notation) by $\bar{\Sigma}_\rho$. The statements we make below hold for both sets (unless otherwise mentioned). In other words, $\bar{\Sigma}_\rho$ stands for those real numbers in $[0, 1\rangle$ whose binary expansion yields the right half of a sequence in $\bar{\Sigma}_\rho$. The only sequences that map to the same number are:

$$s(d) \text{ in } \Sigma_\rho, \quad s(d) \text{ in } \Sigma^*_\rho \to \alpha(d) ,$$

where

$$d = k\rho \bmod 1 , \quad \text{for } k > 0 . \tag{3.3}$$

Another consequence of the lemmas is that we can conclude the ordering of α

is the same as the ordering with respect to d, except that at $d = k\rho \bmod 1$ $k \leqslant 1$ there are "gaps" in $\bar{\Sigma}_\rho$ on the real axis given by (3.2). Note that in (3.2) there are no $\alpha' \in \bar{\Sigma}_\rho$ such that

$$\alpha < \alpha' < \alpha^* .$$

α and α^* have the same index so $\alpha' \in \bar{\Sigma}_\rho$ is greater than both α and α^* or smaller. Since the gaps occur at all values $d = k\rho \bmod 1$, they are dense in $\bar{\Sigma}_\rho$ and also in the projection of $\bar{\Sigma}_\rho$.

Notice that the ordering of one orbit in $\bar{\Sigma}_\rho$ is exactly the same as the ordering of $(d + k\rho) \bmod 1$. That means restricted to $\bar{\Sigma}_\rho$ the shift is order preserving[7]. The main conclusion that we want to draw is:

Theorem 2. The real numbers in $\bar{\Sigma}_\rho$ form a Cantor set which is contained in a closed interval of length $\frac{1}{2}$ and not in any smaller interval and which has Lebesgue measure zero.

Proof. $\bar{\Sigma}_\rho$ is closed. The orbits of x under iteration are dense everywhere, so every point is a limit point. The "gaps" are dense, i.e. between every two points in $\bar{\Sigma}_\rho$ there is a gap. Thus $\bar{\Sigma}_\rho$ is totally disconnected. The smallest number α in $\bar{\Sigma}_\rho$ has index 0, the greatest has index 1. Both 0 and 1 are equal to $0 \cdot \rho \bmod 1$ and therefore the numbers differ in the first digit only. So, if one has numerical value α, then the other has the value $\alpha + \frac{1}{2}$.

The gaps have length 2^{-k-1} $k \geqslant 1$. Since $\Sigma_1^\infty 1/(2^{k+1}) = \frac{1}{2}$ they form a set of full measure in $[\alpha, \alpha + \frac{1}{2}]$. The remainder has measure zero. ■

Note that when we project $\bar{\Sigma}_\rho$ on the y-axis, none of the reasoning changes except that we have one digit (the zeroth) more to worry about. This projection would yield a Cantor set contained in an interval of length one.

As we have said before, these results carry over to the study of the dynamics on a horseshoe (shift on bi-infinite sequences). In Hockett and Holmes[4]) the horseshoe for a time 1 map of a particular flow is constructed. The domain of the time 1 map is an annulus. It is proven there that the orbits of this map, corresponding to orbits in $\bar{\Sigma}_\rho$, can be assigned rotation numbers in the way we defined them. $\bar{\Sigma}_\rho$ corresponds to a minimal Cantor set in the horseshoe. Under certain additional conditions orbits in $\bar{\Sigma}_\rho$ are indeed order-preserving. We shall show below that there are in fact uncountably many minimal Cantor sets for each irrational ρ. These other sets are not optimal.

Let ρ be a fixed irrational number. We will now construct a continuum of minimal Cantor sets $\bar{\Sigma}_\rho(e)$[4]). They will be continuously parametrized by the parameter e and have rotation number ρ. The construction is as in fig. 10. We draw two lines, each with points on it separated by a distance $2/\rho$ (associated with rotation-number $\rho/2$). The parameter e gives the distance between the two first points on each line. The sequences are again defined by projecting down the coordinates of these points and assigning a "1" to an interval that contains one of these points.

Obviously, if one interval is to contain at most one of the projected points then e must be contained in $[1, 2/\rho - 1]$. Note, however, that in the figure we can interchange the upper with the lower line and still obtain the same sequence. To eliminate such duplications, e is restricted to

$$e \in [1, 1/\rho]\,.$$

For $e = 1/\rho$ we have the optimal set $\bar{\Sigma}_\rho$ back. For e and ρ fixed and d varying in $[0, 2]$, one proves by an analysis similar to the one in this section and the two lemmas in section 2, that $\bar{\Sigma}_\rho(e)$ is also a minimal Cantor-set (for each e). More complicated sets, parametrized by n parameters $e_1 \ldots e_n$, can be constructed by drawing $n+1$ lines each with points on it separated distance $(n+1)/\rho$.

The last example of a minimal Cantor set is constructed in a way very similar to the construction of the optimal set in section 2. Instead of a straight line $y = \rho x + d$, we draw a periodically oscillating line (see fig. 11). Let $g(x)$ be a function with period 2 and

$$|g'(x)| < 1\,.$$

Eqs. (2.2) and (2.4) are replaced by

$$S_\rho\colon\ 0 \leqslant n\rho + d + g(n\rho + d) - p_n < 1\,,\quad d \in [0, 1\rangle\,,$$

$$S^*_\rho\colon\ 0 < n\rho + d + g(n\rho + d) - p_n \leqslant 1\,,\quad d \in \langle 0, 1]\,.$$

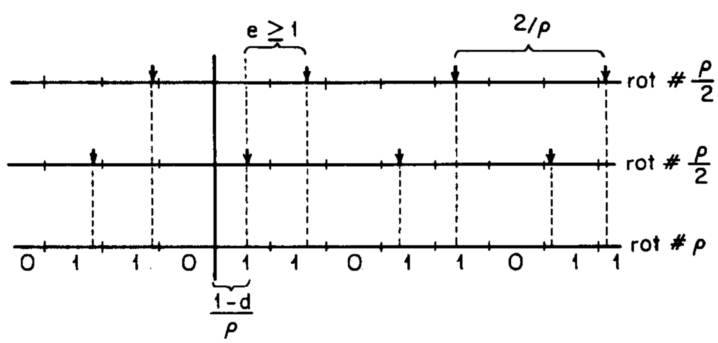

Fig. 10. A sequence belonging to $\bar{\Sigma}_\rho(e)$.

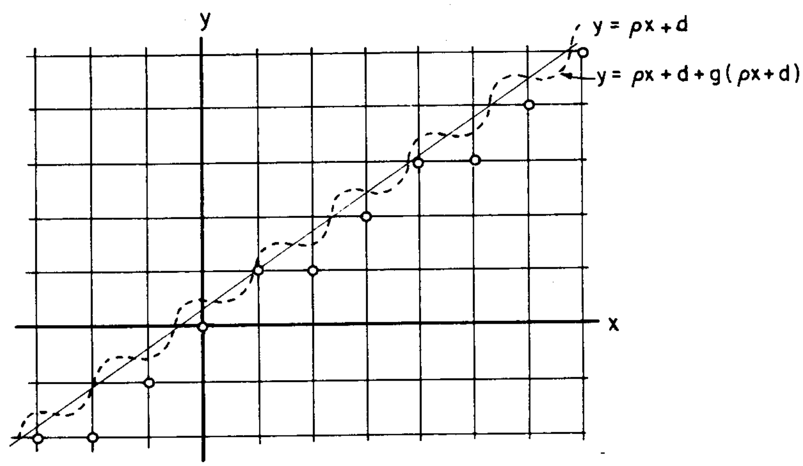

Fig. 11. Construction with a periodic function g of sequence.

Note that if $g(x)$ has period 1 then $g(x\rho)$ has period $1/\rho$. Similar to (2.5) we define y_n to be

$$y_n = n\rho + d + g(n\rho + d)\,. \tag{3.4}$$

Again the same analysis as before will show that the union of S_ρ and S_ρ^* is a minimal Cantor set. We mention this example specifically because it plays an important role in the study of ground state configurations in the Frenkel–Kontorova model[5]). In fact, (3.4) has exactly the same form as the expression for ground state configurations with rotation-number ρ in the work of Aubry and Le Daeron (except that in their work g can possibly have countably many jumps per period).

Finally we mention without proof that if $\rho = p/q$ where p and q relative prime then the set $\bar{\Sigma}_\rho$ is finite. It actually consists solely of one sequence and its iterates of which there are q, since the sequence is q-periodic. For example:

$$\bar{\Sigma}_{1/3} = \{(100)^\infty, (010)^\infty, (001)^\infty\}\,.$$

4. Continued fractions

In this section we consider again the optimal sequences as defined in section 3. We will restrict ourselves to semi-infinite sequences. First we will (for any sequence optimal to ρ) consider the sequence of numbers $\{p_n/n\}_0^\infty$ and prove that the continued fraction expansion of the number ρ is a subsequence of these numbers. On the other hand we can also construct a semi-infinite sequence from the knowledge of the continued fraction expansion of ρ. The construction of this sequence is symbolic; it involves no calculations, only symbolic manipulations of ones and zeros. This sequence is then shown to be a member of $\bar{\Sigma}_\rho$.

The continued fraction expansion $\{q_n/Q_n\}_0^\infty$ with continued fraction coefficients $\{\alpha_i\}_1^\infty$ of the number ρ is defined as follows[9,10]:

$$\begin{aligned} q_{i+1} &= \alpha_{i+1} q_i + q_{i-1}\,, \quad \text{with } q_0 = 0\,,\ \ q_1 = 1\,, \\ Q_{i+1} &= \alpha_{i+1} Q_i + Q_{i-1}\,, \quad \text{with } Q_0 = 1\,,\ \ Q_1 = \alpha_1\,; \end{aligned} \tag{4.1}$$

so that

$$\frac{q_i}{Q_i} = \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cdots \cfrac{1}{\alpha_i}}}\,,$$

and

$$\rho = \lim_{n\to\infty} \frac{q_n}{Q_n}\,,$$

or

$$\rho = [\alpha_1, \alpha_2, \alpha_3, \ldots]\,.$$

Proposition. Let $s \in \bar{\Sigma}_\rho$ have an index $d \in [0, 1]$. Then there is an $N > 0$ such that $\{q_n/Q_n\}_N^\infty$ is a subsequence of the running averages $\{\rho_n(s)\}_1^\infty = \{p_n/n\}_1^\infty$.

Proof. The continued fractions q_n/Q_n have the following basic property[9]. The numbers q_n and Q_n are those numbers for which

$$|Q_n\rho - q_n| < |Q\rho - q|\,, \quad \forall q, Q \in \mathbb{N} \,|\, Q < Q_n \tag{4.2}$$

and

$$|Q_n\rho - q_n| < 1/Q_n\,.$$

According to the definition of p_n in (2.1) and (2.3) (figs. 2 and 3), if we choose N

so that

$$Q_N > 1/d \quad \text{and} \quad Q_N > 1/(1-d)$$

$\{q_n/Q_n\}_N^\infty$ will be a subsequence of $\{p_n/n\}_1^\infty$. ■

Note that (4.2) provides us with an excellent and fast criterion for finding q_n/Q_n numerically while constructing s. From that one can deduce the coefficients α_i.

Consider a sequence s with index d and shift this sequence Q_n places. The new sequence $s' = \sigma^{Q_n} s$ has index

$$d' = (d + Q_n \rho) \bmod 1 = d + Q_n \rho - q_n ,$$

which is closer to d than the index of any other $\sigma^Q s$ with $Q < Q_n$. The sequence s' could therefore be very close to s and start out with many digits the same as in s. The greater Q_n, the more digits might be indentical. This insight leads us to construct a semi-infinite sequence by repeating larger and larger pieces and putting them together in one sequence. The sequence resulting from the following construction will be called the scaling sequence.

First we define some operations in order to be able to do symbolic manipulation with sequences. Define "$\cdot$" as repeating and "$\oplus$" as concatenation. So:

$$3 \cdot 010 \oplus 111 = 010010010111 .$$

Note that we can multiply a sequence only by a positive integer and that the operations are not commutative. The truncation $t_N(s)$ is the subsequence of s consisting of its first N digits.

Example. $t_4(3 \cdot 010 \oplus 111) = .0100$

The construction of the scaling sequence draws on a complete analogy with the definition of continued fractions in (4.1), where now q_i stands for the number of ones in the first Q_i digits. The zeroth digit, or $t_{Q_0} s$ (left of the binary point), is defined to be "0" (step 0). The construction of step 1 is: write down a subsequence of length $\alpha_1 (= Q_1)$ with one $(= q_1)$ "1". If $\alpha_1 > 1$ there is nore than one possibility of doing this. The convention ("initial condition") is that the "1" always comes as the last (α_1st) digit. We have constructed the first Q_1 digits of s or $t_{Q_1} s$. The other steps are given by (compare with (4.1))

$$t_{Q_{n+1}} s = \alpha_{n+1} \cdot t_{Q_n} s \oplus t_{Q_{n-1}} s . \tag{4.3}$$

We give three examples; the bars indicate the digit Q_n with the numbers q_n and Q_n written above resp. below it:

$$\begin{array}{ll} & \quad\;\; 1\;\, 1\;\, 2\;\;\; 3\;\;\;\;\; 5\;\;\;\;\;\;\; 8\;\;\;\;\;\;\;\;\;\; 13 \\ \rho = [1, 1, 1, \dots]: & 0\cdot 1|0|1|10|101|10110|10110101|\dots, \\ & \quad\;\; 1\;\, 2\;\, 3\;\;\; 5\;\;\;\;\; 8\;\;\;\;\;\; 13\;\;\;\;\;\;\;\;\; 21 \end{array}$$

$$\begin{array}{ll} & \quad\;\; 1\;\, 1\;\;\;\;\;\; 3\;\;\;\; 4\; 11 \\ \rho = [2, 1, 2, 1, 2, 1, \dots]: & 0\cdot 01|0|01001|010|0100100101001001001|\dots, \\ & \quad\;\; 2\;\, 3\;\;\;\;\;\; 8\;\;\;\; 11\; 30 \end{array}$$

$$\begin{array}{ll} & \quad\;\;\;\;\;\; 1\;\;\;\;\;\;\;\; 2\;\;\;\;\;\;\;\; 3 \\ \rho = [5]: & 0\cdot 00001|00001|00001|\dots. \\ & \quad\;\;\;\;\;\; 5\;\;\;\;\;\;\; 10\;\;\;\;\;\; 15 \end{array}$$

Note that if ρ is rational, there is a finite number of coefficients but to obtain the sequence we keep repeating to finally get a periodic sequence.

The main theorem of this section is

Theorem. The scaling sequence belongs to $\bar{\Sigma}_\rho$, in fact, they have index ρ.

The proof is a generalization of the integer decomposition as described by Kadanoff[7]). The reader may profit from checking the statements in the proof against the examples of scaling sequences given above.

Proof. We will consider the k-truncation $t_k s$ of the scaling sequence s and prove that the number of ones p_k in $t_k s$ satisfies relation (2.1) with α equal to ρ. In order to be able to do this for every $k \in \mathbb{N}$ we have to "decompose" $t_k s$ into terms like $t_{Q_n} s$. First decompose $t_k s$ as follows:

$$t_k s = \delta_n \cdot t_{Q_n} s \oplus \text{"rest"},$$

where Q_n is the greatest continued fraction denominator smaller than or equal to k. By construction $\delta_n \leq \alpha_{n+1}$. Then repeat this step for "rest", and so on:

$$t_k s = \bigoplus_{i=0}^{n-1} \delta_{n-i} \cdot t_{Q_{n-i}} \oplus t_r s,$$

where $\delta_{n-i} \leq \alpha_{n-i+1}$ and $r < \alpha_1$ and if $\delta_{n-i} = \alpha_{n-i+1}$ then $\delta_{n-i-1} = 0$ (Recall that $\alpha_{n-i+1} \cdot t_{Q_{n-i}} s \oplus t_{Q_{n-i-1}} s = t_{Q_{n-i+1}} s$.) We can now evaluate $k\rho - p_k$ as follows:

$$k\rho - p_k = \sum_{i=0}^{n-1} \delta_{n-i}(\rho Q_{n-i} - q_{n-i}) + (\rho r - p_r).$$

The problem reduces to finding an upper- and lower-bound for this last expression. In the rest-term $r < \alpha_1$ and therefore $p_r = 0$ (number of ones). The smallest value of $\rho t - p_r$ is 0 and its greatest value is $\rho(\alpha_1 - 1)$.

Recall the property of continued fraction[9,10]) that for $n - i$ odd $\rho Q_{n-i} - q_{n-i}$ is greater than zero and for $n - i$ even it is smaller than zero. For $k < Q_{2n+1}$ we can calculate the maximum of $\rho k - p_k$ by simply adding up all the possible positive contributions and likewise we calculate the minimum:

$$\min_{k<Q_{2n+1}} \{\rho k - p_k\} = \sum_{1}^{n} \alpha_{2i+1}(\rho Q_{2i} - q_{2i}) + (\alpha_1 - 1)\rho\,,$$

$$\max_{k<Q_{2n+1}} \{\rho k - p_k\} = \sum_{1}^{n} \alpha_{2i}(\rho Q_{2i-1} - q_{2i-1}) + 0\,.$$

Using definition (4.1) we can rewrite these expressions; for instance:

$$\sum_{1}^{n} \alpha_{2i} q_{2i-1} = \sum_{1}^{n} q_{2i} - q_{2i-2} = q_{2n}\,.$$

This yields:

$$\min_{k<Q_{2n+1}} \{\rho k - p_k\} = \rho(Q_{2n+1} - 1) - (q_{2n+1} - 1)\,,$$

$$\max_{k<Q_{2n+1}} \{\rho k - p_k\} = \rho(Q_{2n} - 1) - q_{2n}\,.$$

This tells us that the minimum occurs at digit $Q_{2n+1} - 1$ and the maximum at digit $Q_{2n} - 1$. Taking the limit for $n \to \infty$ and noting that all contributions have the same sign, we find

$$-\rho < \rho k - p_k < 1 - \rho\,.$$

Thus the scaling sequence satisfies (2.1) with index ρ. (Note that inequality is not attained here because we consider only $k > 0$.) ■

The question now arises how we can change the first steps of the construction of the scaling sequence and still end up in $\bar{\Sigma}_\rho$. Suppose first that α_1 is greater than 1. The first step in assembling the scaling sequence is to place the "1" at digit number α_1. Suppose we put it at digit number $\alpha_1 - 1$. In carrying out the recursive recipe (4.3) to piece the whole sequence together, it is clear that every "1" now moves ahead precisely one digit. This means that we have built σs, the shifted scaling sequence (compare with our second example!). One can continue shifting like this until the "1" is at digit number one. For further shifting if "1" is at digit number one, remove the first bar, permute the first Q_2 digits cyclically (one step) to the left and then pick up the construction as in (4.3). One can see this by explicitly doing this for the examples given!

It should be obvious that if we change the total number of "1's" at any step of the construction the resulting sequence is certainly not in $\bar{\Sigma}_\rho$. (By adding one "1" for instance, we would obtain $(q_n+1)/Q_n$ as continued fractions of ρ.)

To find analogous recursive relations for sequences in $\bar{\Sigma}_\rho$ with $d \not\in [\rho]$ seems a harder problem, although, of course, we can approximate every sequence in $\bar{\Sigma}_\rho$ arbitrarily well by the shifts of the scaling sequence.

In the last part of this section we prove that the scaling sequence for quadratic irrational numbers ρ is an attracting fixed point of a very simple linear operator T defined on the space of semi-infinite sequences.

Consider the subset of quadratic irrational rotation-numbers ρ with $\rho = [(\alpha_1 \ldots \alpha_k)^\infty]$. (All quadratic irrationals have continued fraction coefficients which are eventually periodic[9].) The construction of the scaling sequence with such a rotation number is k-periodic; after performing k steps with (4.3) we have the same coefficients for the construction. If one could perform k steps at one time the coefficients of the construction would again be constant. One would construct $t_{Q_k}s, t_{Q_{2k}}s, \ldots, t_{Q_{nk}}s, \ldots$ etc. An interesting way of doing this is described in Feigenbaum and Hasslacher[6]).

To construct the scaling sequence s define a linear operator T so that: $T\colon t_{Q_k}s \to t_{Q_{2k}}s$.

Proposition. If T satisfies the above equation for the scaling sequence s with ρ quadratic irrational, then for all $n>0$:

$$T\colon t_{Q_{nk}}s \to t_{Q_{(n+1)k}}s$$

This proposition tells us that if T constructs $t_{Q_{2k}}s$ from $t_{Q_k}s$ then we can construct $t_{Q_{3k}}s$ and on. For $\rho = [2,1,2,1,\ldots]$ as in the second example we write down T.

$$T \begin{array}{l} 0 \to 010 \\ 1 \to 01001 \end{array}$$

In the proof we will assume k to be 2 for simplicity. This proof can be generalized easily to $k>2$.

Proof. By induction. Suppose that the proposition holds for $n = 1, 2, 3, \ldots, N$, then for $n = N + 1$ we have by applying (4.3) twice and relying on the linearity of T:

$$\begin{aligned} T(t_{Q_{(N+1)2}}s) &= T[\alpha_{2N+2}\cdot(\alpha_{2N+1}\cdot t_{Q_{2N}}s \oplus t_{Q_{2N-1}}s) \oplus t_{Q_{2N}}s] \\ &= \alpha_{2N+2}\cdot[\alpha_{2N+1}\cdot T(t_{Q_{2N}}s) \oplus T(t_{Q_{2N-1}}s)] \oplus T(t_{Q_{2N}}s) \\ &= \alpha_{2N+2}\cdot(\alpha_{2N+1}\cdot t_{Q_{2N+2}} \oplus t_{Q_{2N+1}}s) \oplus t_{Q_{2N+2}}s\,. \end{aligned}$$

And by the 2-periodicity of α_n this expression equals $t_{Q_{2N+4}}s$. The proposition then holds for $n = N + 1$ as well. By definition of T, the proposition holds for $n = 1$. ■

Remark. The fact the sequence constructed in this fashion is invariant under the inverse of T which decreases the number of digits, makes these sequences apt for the decimation procedure for path integrals[6]).

5. The piecewise linear circle map

The fact that $\bar{\Sigma}_\rho$ is invariant under the shift means that the real numbers α associated with the one-sided version of $\bar{\Sigma}_\rho$ (see section 3) are invariant under

$$\theta : \alpha \to 2\alpha \bmod 1\,.$$

Recall that to the semi-infinite sequences $s = \{O_n\}_{n=1}^{\infty}$ in $\bar{\Sigma}_\rho$ we assign real numbers as follows:

$$\alpha = \sum_1^\infty \frac{i_n}{2^n}\,, \quad \rho = \lim_{N\to\infty} \frac{1}{N} \sum_1^N i_n\,.$$

The set of numbers x corresponding to s in $\bar{\Sigma}_\rho$ is also called $\bar{\Sigma}_\rho$ and ρ assumes one value only, the rotation number associated with $\bar{\Sigma}_\rho$. The results for optimal sequences will lead us to construct a modification ϕ_β of θ. For the family ϕ_β (piecewise linear) we shall calculate exactly the width of the p/q resonance interval. (A p/q-resonance interval is the interval of values for α such that ϕ_α has a stable q-periodic orbit with rotation-number p/q.) In the last part of this section we generalize our result to a slightly wider class of piecewise linear circle-maps.

In section 3 we found that the real numbers $\alpha \in \bar{\Sigma}_\rho$ form a Cantor set contained in $[\alpha_\rho/2, \alpha_\rho + 1/2]$, for some α_ρ, and in no smaller interval. This observation leads to the following construction, similar in spirit to the construction described by Boyland[8]) and Kadanoff[7]). Replace the map θ by ϕ_β (fig. 12) $\beta \in [0, 1]$

$$\phi_\beta \begin{cases} x \to \beta, & x \in \left[0, \dfrac{\beta}{2}\right\rangle, \\ x \to 2x \bmod 1, & x \in \left[\dfrac{\beta}{2}, \dfrac{\beta+1}{2}\right\rangle, \\ x \to \beta, & x \in \left[\dfrac{\beta+1}{2}, 1\right\rangle. \end{cases}$$

For every β ϕ_β is a continuous degree one critical circle map. If $\beta = \alpha_\rho$ then restricted to $\bar{\Sigma}_\rho$ the map ϕ_{α_ρ} is the map $2x \bmod 1$. The invariant set $\bar{\Sigma}_\rho$ can then be constructed on the circle by taking out $\langle(\alpha_\rho + 1)/2, \alpha_\rho/2\rangle$ and all its inverse iterates under ϕ_{α_ρ} (see fig. 13). Note that $\alpha_\rho/2$ and $(\alpha_\rho + 1)/2$ correspond to sequences with index 0 and 1 resp. and the difference between their inverse images correspond exactly to the "gaps" in the Cantor set from lemma 3.4 and theorem 2 in section 3.

We are going to study the rotation-number of ϕ_β as a function of β. The rotation-number is defined in terms of the lift Φ_β of ϕ_β [11]),

$$\text{rotation-number} = \lim_{n\to\infty} \frac{\Phi^n_\beta(x) - x}{n}.$$

We have:

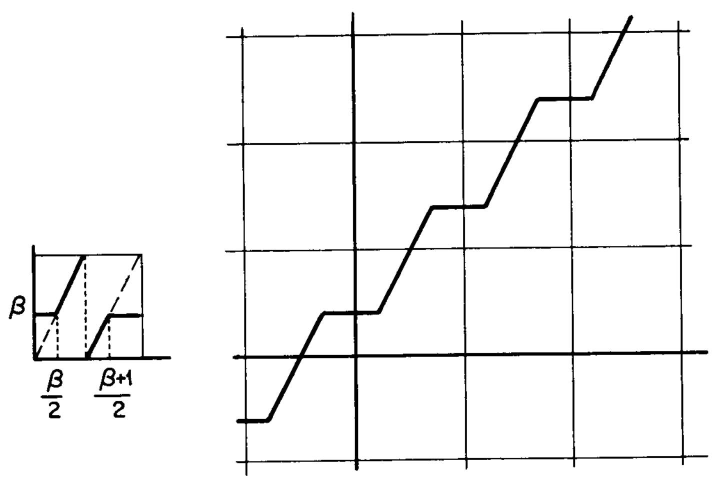

Fig. 12. The map ϕ_B with its lift Φ_B.

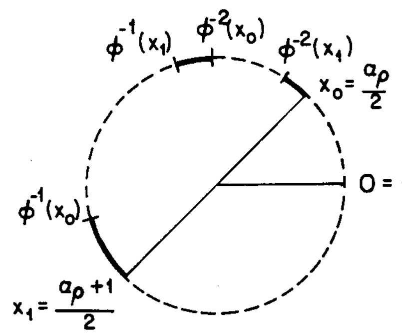

Fig. 13. 3 steps in the construction of the Cantor-set $\bar{\Sigma}_\rho$ by inverse images.

Lemma 5.1. Let α correspond to $s \in \bar{\Sigma}_\rho$ and i_j the digits of the binary expansion of α. Then

$$\lim_{n\to\infty} \frac{\Phi^n_{\alpha_\rho}(\alpha) - \alpha}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_1^n i_j \, .$$

Proof. Restricted to $\bar{\Sigma}_\rho$, ϕ_{α_ρ} is just the shift. The integer part of $\Phi_{\alpha_\rho}(\alpha)$ increases by one if $\alpha \bmod 1 \geqslant \frac{1}{2}$, in other words if the first digit is a one. ∎

To proceed we quote some important results for circle-maps. These statements and their proofs can be found in Boyland[8]) or Newhouse, Palis, Takens[11]) and Ito[12]). Let End(S^1) be the set of all continuous degree one circle maps with C^0-topology. Let h and g be maps in End(S^1) such that their lifts H and G are non-decreasing. The following results then hold:

Lemma 5.2. Every map $f \in \mathrm{End}(S^1)$ has a closed rotation-interval $[\rho_1(f), \rho_2(f)]$, i.e.: for every ρ in this interval there is an orbit of f with rotation number ρ and no orbit has rotation-number not in this interval.

Lemma 5.3. The map ρ that assigns to f the endpoints of its rotation-interval:

$$\rho\colon \mathrm{End}(S^1) \to \mathbb{R}^2$$

is continuous.

Lemma 5.4. $\rho(h) = \lim_{n\to\infty} (H^n(x) - x)/n$ exists and is independent of x. (h has a single well-defined rotation-number.)

Lemma 5.5. Let $G(x) < H(x)$ for all x. Then if $\rho(g)$ or $\rho(h)$ irrational $\rho(g) < \rho(h)$. If $\rho(g)$ and $\rho(h)$ are rational, equality is allowed: $\rho(g) \leqslant \rho(h)$.

Lemma 5.6. If $\rho(g)=p/q$, a rational number $\Leftrightarrow$ g has a p/q periodic orbit.

From all this it follows that there is a well-defined continuous mapping that assigns to ϕ_β a single rotation-number. The ϕ_β form a one-parameter family of mappings for which the topology carries nicely over to the parameter space $\beta\in[0,1]$: β_1 and β_2 are close if and only if ϕ_{β_1} and ϕ_{β_2} are C^0-close. So we define a continuous mapping R that assigns to β the rotation number of ϕ_β.

$$R:\ \beta\to\rho(\phi_\beta)\,.$$

According to lemma 5.5 R is non-decreasing and strictly increasing on those values of β where $\rho(\phi_\beta)$ is irrational. It follows that R may have plateaus where $\rho(\phi_\beta)$ is rational, say p/q and where (lemma 5.6) ϕ_β has a p/q periodic orbit. The parameter values for which we find these p/q plateaus are denoted by $I_{p/q}$ (p/q-resonance intervals).

The idea of the construction is that to some extent we already know what the map R is (namely on the points where it assumes *ir*rational values).

Lemma 5.7. Let $\alpha_\rho/2$ be the element in $\bar{\Sigma}_\rho$ with index $d=0$, then ϕ_{α_ρ} has rotation number ρ (ρ irrational).

Proof. ϕ_{α_ρ} restricted to $[\alpha_\rho/2,(\alpha_\rho+1)/2]$ in just $2x \bmod 1$. There is an invariant set $\bar{\Sigma}_\rho$ in $[\alpha_\rho/2,(\alpha_\rho+1)/2]$ and orbits in that set have rotation-number ρ. Therefore (lemma 5.4) all orbits have rotation-number ρ. ■

On the other hand we can construct α_ρ: since it has index ρ (because $\alpha_\rho/2$ has index zero and lemma 2.2), its binary sequence is just the scaling sequence defined in section 4. In addition we know that R is strictly increasing at α_ρ (lemma 5.5). We thus know the map R for all values of β where $R(\beta)$ is irrational.

Lemma 5.8. The set I of values β for which $R(\beta)=\rho(\phi_\beta)$ is irrational, has zero (Lebesque) measure.

Proof. I consists of the numbers α_ρ for ρ irrational and α_ρ corresponds to the scaling sequence in Σ_ρ. Therefore I is a subset of $\cup_{\rho\,\mathrm{irr}}\Sigma_\rho$ on the real line. For each irrational ρ, θ (see beginning of section) maps the set of real numbers Σ_ρ to itself and is a bijection restricted to Σ_ρ (in $\bar{\Sigma}_\rho$ sequences with indices 0 and 1 get mapped to the sequence with index ρ). So θ restricted to $\cup_{\rho\,\mathrm{irr}}\Sigma_\rho$ is a bijection. But θ is expanding (derivative 2) so $\mu(\cup_{\rho\,\mathrm{irr}}\Sigma_\rho)=\mu(\theta(\cup_{\rho\,\mathrm{irr}}\Sigma_\rho))\in\{0,1\}$. No number whose binary expansion begins with .0011 or with .1100 can be an element of $\cup_{\rho\,\mathrm{irr}}\Sigma_\rho$ (compare eq. 2.8)). So we also have:

$$\mu\left(\bigcup_{\rho\,\mathrm{irr}}\Sigma_\rho\right)\leq\tfrac{7}{8}.$$

Therefore $\mu(I)\leq\mu(\cup_{\rho\,\mathrm{irr}}\Sigma_\rho)=0.$ ■

Note that in the same way we can prove that $\cup_{\rho\,\mathrm{irr}}\Sigma^*_\rho$ has measure zero and therefore $\cup_{\rho\,\mathrm{irr}}\bar{\Sigma}_\rho$ has measure zero. It follows from this lemma that the complement of I (or the union of all $I_{p/q}$) has measure 1.

We can think of the interval $I_{p/q}$ as being squeezed in between values α_ρ where $R(\alpha_\rho)$ is irrational (see fig. 14). Because $R(\beta)$ is continuous and increasing we can find the length of $I_{p/q}$ by

$$I_{p/q}\subset[\alpha_{\rho-},\alpha_{\rho+}],$$

$$\rho_\pm=\frac{p}{q}\pm\varepsilon\quad(\varepsilon\text{ irrational}).$$

and taking the limit $\varepsilon\to 0$. The sequences $\alpha_\rho^\pm$ can be easily constructed (being scaling sequences). In this manner we prove:

Theorem 1. The length of $I_{p/q}$ is $1/(2^q-1)$. The reader may profit from comparing the statements in the proof with the table below.

Proof. Let $\rho_\pm=p/q\pm\varepsilon$ with ε irrational. For small enough ε, $\rho_\pm$ will be very well approximated by p/q and so p/q will be a continued fraction of $\rho_\pm$. In fact let $p/q=[\alpha_1,\ldots,\alpha_k]$, the α_i are not unique, then $\varepsilon\to 0$ implies $\alpha_{k+1}\to\infty$ [9]).

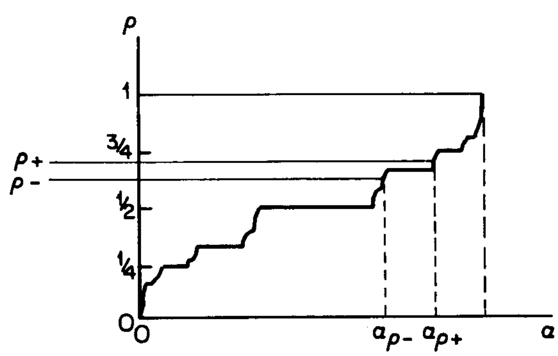

Fig. 14. The map R (a Cantor-function).

Knowing the coefficients we can now construct the sequences $\alpha_{\rho\pm}$. Both will have p ones in the first q digits. This group of q digits will then be repeated α_{k+1} times (see (4.3)) in the next step of the construction:

$$\alpha_{\rho\pm} = \underset{1}{*****} \overset{p}{\underset{q}{|}} \underset{2}{*****} \cdots \underset{a_{k+1}}{*****} \cdots | \cdots .$$

Obviously, for $\varepsilon \to 0$, $\alpha_{\rho\pm}$ will be a periodic sequence as its limit. The rotation-number in the limit sequence will be p/q. The difference between $\alpha_{\rho+}$ and $\alpha_{\rho-}$ depends only on the first q digits.

To construct $\alpha_{\rho+}$ and $\alpha_{\rho-}$ let $0 \leq n < q$. At digit $q-n$ we have p_{q-n} ones where p_{q-n} satisfies (2.2):

$$0 \leq (q-n)\rho - p_{q-n} + d < 1$$

and

$$d = \rho = \frac{p}{q} \pm \varepsilon ,$$

so

$$0 \leq p + \frac{1-np}{q} - p_{q-n} \pm \varepsilon(q-n+1) < 1 .$$

In this last equation $(1-n)p/q$ is fractional for $n \neq 1$. So, for $n \neq 1$ and ε small enough, p_{q-n} does not depend on the sign of the ε-term. However, for $n = 1$ it does. For the digits $i_{q-1} = p_{q-1} - p_{q-2}$ and $i_1 = p_q - p_{q-1}$ we have:

$$\alpha_{\rho-}: \quad i_{q-1} = 0 , \quad i_q = 1 ,$$
$$\alpha_{\rho+}: \quad i_{q-1} = 1 , \quad i_q = 0 .$$

We take the limit as $\varepsilon \to 0$ and the sequences become periodic and

$$\alpha_{\rho+} = \alpha_{\rho-} + 2^{-q} + 2^{-2q} + \cdots = \alpha_{\rho-} + 1/(2^q - 1) .$$

Thus $I_{p/q} = [\alpha_{\rho-}, \alpha_{\rho-} + 1/(2^q - 1)]$. ∎

We note that this proves the relation

$$\sum_{(p,q)\in U} \frac{1}{2^q - 1} = 1 ,$$

where U is the set of pairs (p, q) such that p and q are relative prime, $p<q$ and $q \geqslant 2$. Note that $\alpha_{\rho+}$ can be calulated easily using the fact that $\alpha_{\rho+}$ is the scaling sequence for $\rho = p/q$.

Table I which we give below was made as follows. For a given ρ we calculated the scaling sequence by (2.2), setting $d = \rho$. The bars that indicate continued fraction denominators were found numerically by criterion (4.2). The continued fraction coefficient can then easily be deduced through (4.1). One can check that the resonance intervals have length $\alpha_+ - \alpha_- = \frac{1}{31}$.

Consider again the space of sequences S. To generalize the foregoing we

TABLE I

$\rho = .19900\ldots$	1 00001|00001 . . . 00001 5 2 39	0|. . .	$\alpha_- = 1/31$	$\rho = [5, 39, \ldots]$
$\rho = .20100\ldots$	1 0001|0|00010 . . . 00010 5 2 39	0001|. . .	$\alpha_+ = 2/31$	$\rho = [4, 1, 39, \ldots]$
$\rho = .39900\ldots$	2 01|0|01|01001 . . . 01001 5 2 39	010|. . .	$\alpha_- = 9/31$	$\rho = [2, 1, 1, 39, \ldots]$
$\rho = .40100\ldots$	2 01|010|01010 . . . 01010 5 2 39	01|. . .	$\alpha_+ = 10/31$	$\rho = [2, 2, 39, \ldots]$
$\rho = .5990\ldots$	3 1|0|101|10101 . . . 10101 5 2 39	10|. . .	$\alpha_- = 21/31$	$\rho = [1, 1, 2, 39, \ldots]$
$\rho = .60100\ldots$	3 1|0|1|10|10110 . . . 10110 5 2 39	101|. . .	$\alpha_+ = 22/31$	$\rho = [1, 1, 1, 1, 39, \ldots]$
$\rho = .79900\ldots$	4 1|110|1|11101 . . . 11101 5 2 39	11101|. . .	$\alpha_- = 29/31$	$\rho = [1, 3, 1, 39, \ldots]$
$\rho = .80100\ldots$	4 1|1110|11110 . . . 11110 5 2 39	1|. . .	$\alpha_+ = 30/31$	$\rho = [1, 4, 39, \ldots]$

regard them now as numbers on the base τ where $\tau > 1$. As in section 3 we associate numbers x and y with each sequence as follows:

$$s = \cdots i_{-2} i_{-1} i_0 \cdot i_1 i_2 \cdots,$$

$$\tilde{x} = \sum_1^{\infty} i_n / \tau^n, \quad \tilde{y} = \sum_0^{\infty} i_{-n} / \tau^n,$$

but now

$$x = \tilde{x} \bmod 1, \quad y = \tilde{y} \bmod \tau.$$

Note that this mapping from S to the rectangle R a bijection only for $\tau = 2$. It is onto for $1 < \tau \leqslant 2$ and it is one-to-one for $\tau \geqslant 2$.

The set of optimal sequences $\bar{\Sigma}_\rho$ is invariant under the shift. The set of numbers $\tilde{\alpha} \in \bar{\Sigma}_\rho$ corresponding to one-sided sequences in $\bar{\Sigma}_\rho$, is now invariant under multiplication by τ and so is the set $\{\alpha\} = \{\tilde{\alpha} \bmod 1\}$. Furthermore the ordering of the projections $\tilde{\alpha}$ is (almost) the same as the ordering with respect to the index: lemmas 3.3 and 3.4 still hold if α is replaced by $\tilde{\alpha}$. This implies that the smallest number $\tilde{\alpha}$ in $\bar{\Sigma}_\rho$ has index 0 and the biggest has index 1. They differ only in the first digit, i.e.: their difference is $1/\tau$. This enables us to do a similar construction as before.

We construct a degree one continuous circle map which now has two parameters τ and β: $\phi_{\tau,\beta}$. The construction is similar to the previous one, β is a cut-off parameter, τ is the slope of the slanted part of the map (figs. 15 and 16). It can be seen from the figure that for $\beta = 0$ and for $\beta = 1/(\tau - 1)$ the map $\phi_{\tau,\beta}$ has a fixed point. The parameter range of β that we are interested in is

$$\beta \in \left[0, \frac{1}{\tau - 1}\right].$$

We expect the map $\phi_{\tau,\beta}$ to attain all rotation-numbers in $\langle 0, 1\rangle$ for some β in the specified interval.

Let $\alpha \in \bar{\Sigma}_\rho$ have digits i_n and

$$\rho = \lim_{n \to \infty} \frac{1}{n} \sum_1^{\infty} i_j.$$

As before, if β has the value α_ρ where α_ρ is constructed with index $d = \rho$, then restricted to $\bar{\Sigma}_\rho$, the map ϕ_{τ,α_ρ} is just multiplication by τ. The rotation-number of the orbits of the invariant set $\bar{\Sigma}_\rho$ is the average number of ones, ρ. Therefore we

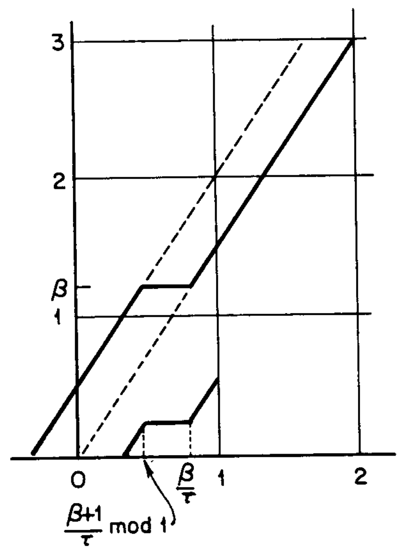

Fig. 15. Construction of the truncated map $\phi_{\tau,\beta}$ for $\tau = 1\frac{1}{2}$. Fixed point when $\beta = 0$ or $\beta = 2 \neq 1/(1\frac{1}{2} - 1)$.

can associate the rotation-number ρ with (every orbit of) the mapping ϕ_{τ,α_ρ}. (Note that α_ρ depends on τ.)

Let $\tau > 1$. Theorem 1 generalizes to

Theorem 2. The resonance interval $I_{p/q}$ of the family $\phi_{\tau,\beta}$, τ fixed, has length $(\tau - 1)/(\tau^q - 1)$. The union of all $I_{p/q}$ has full measure.

Proof. The p/q resonance intervals are constructed exactly as before. The particular we have that in the binary expansions of $\alpha_{\rho-}$ and $\alpha_{\rho+}$, the digits i_{nq-1} and i_{nq} interchange (in the limit $\varepsilon \to 0$ see proof of theorem 1).

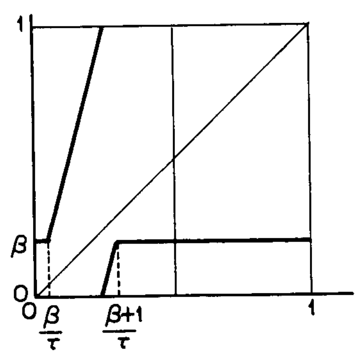

Fig. 16. Construction of $\phi_{\tau,\beta}$ for $\tau = 4$. Fixed point when $\beta = 0$ or $\beta = 1/3 = 1/(4 - 1)$.

Therefore

$$\begin{aligned}\alpha_{p+} &= \alpha_{p-} + (\tau^{-q+1} - \tau^{-q}) + (\tau^{-2q+1} - \tau^{-2q}) + \cdots \\ &= \alpha_{p-} + (\tau - 1)(\tau^{-q} + \tau^{-2q} + \cdots) \\ &= \alpha_{p-} + \frac{\tau - 1}{\tau^q - 1}.\end{aligned}$$

To prove the union of $I_{p/q}$ has the full measure we can try and prove that the complement I of $\cup I_{p/q}$ has zero measure. For $\tau \geqslant 2$ the proof of lemma 5.8 applies. However, for all $\tau > 1$, we can also prove directly the equality:

$$\sum_{(p,q)\in U} \frac{\tau - 1}{\tau^q - 1} = \frac{1}{\tau - 1},$$

where U is the set of pairs (p, q) such that p and q are relative prime, $p < q$ and $q \geqslant 2$. By definition of Euler's ϕ function this can be written as

$$\sum_{q\geqslant 2} \phi(q) \frac{\tau - 1}{\tau^q - 1} = \frac{1}{\tau - 1}$$

(see Hardy and Wright[13]). Expanding the expressions as power-series in $1/\tau$ and using the properties of ϕ [13]) one proves the equality for all $\tau > 1$. ■

We want to stress the importance of lemma 5.3 in connection with this treatment of the function R: $\beta \to \rho(\phi_{\tau,\beta})$. This lemma asserts that if we study maps ψ_n such that

$$\lim_{n\to\infty} \|\psi_n - \phi_{\tau,\beta}\| = 0$$

in the C^0 topology, then ψ_n has rotation interval $[\rho_{1n}, \rho_{2n}]$ such that

$$\lim_{n\to\infty} \rho_{1n} = \lim_{n\to\infty} \rho_{2n} = \rho(\phi_{\tau,\beta}).$$

In other words, we have a tool to estimate the rotation interval of an arbitrary mapping ψ if it is close to $\phi_{\tau,\beta}$ for some τ and some β.

In recent papers the metric properties of the invariant set $\bar{\Sigma}_\rho$ have been studied in a more general context[7,14]). For our piecewise linear mapping, lemma

3.3 and 3.4 assert that the ordering of points of one orbit in $\bar{\Sigma}_\rho$ is exactly the same as the ordering of points under rigid rotation $(d + k\rho) \bmod 1$. The distances between a point and its iterates, however, are not preserved. In particular Kadanoff[7] and Sarkar[14] give expressions for the distance between α_ρ/τ and its Q_{2m}th iterate under the map ϕ_{τ,α_ρ} with irrational rotation number ρ (ρ has continued fractions q_n/Q_n). Since in this paper we developed an explicit representation for the attracting set $\bar{\Sigma}_\rho$, the conclusions of these two authors might be proven in the piecewise linear case.

In fact, for the two-parameter family $\phi_{\tau,\beta}$ and τ given we have constructed the function $R: \beta \to \rho(\phi_{\tau,\beta})$ explicitly. Let $\rho(\phi_{\tau,\beta}) = \rho$ for some given values of τ and β; for $q_{\tau,\beta}$ we have constructed the invariant attracting set: it is the set of optimal sequences with rotation number ρ (as numbers on the base τ and modulo 1).

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