Listing of the Summaries of NOSYGS or the

Non-Linear Systems Group Seminar

Portland State University

Spring 2011

Friday, 2pm-3pm, NH 307

Organized by: E. Riley and J.J.P. Veerman**.

Summaries by E. Riley

in collaboration with — and with advice from — the speakers.

This term the seminar is dedicated to students who want to do research. The emphasis is on interaction and generating new ideas. Speakers should prepare clear and simple presentations of basic underlying ideas. Technical prowess is appreciated when alluded at, but less so when expounded and insisted upon. More than the details of any single proof or calculation, we aim to understand the underlying principles.

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Flocking Behavior

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April 15

We may wish to model how a flock of birds or a school of fish appears to move 'as one' in three dimensions by first examining the simpler case of motion in one dimension. Suppose a group of objects (cars, birds, masses) is moving in one dimension according to Newton's laws. We will assume that the only forces acting between objects act between nearest neighbors and can be modeled as either a spring (obeying Hooke's Law $F_1 = k_1 d$) or a dashpot (obeying the force-law: $F_2 = -k_2 v$). Here d would be the distance the spring is extended, v is the relative velocity between the neighbors, and k_1 and k_2 are arbitrary positive constants. We are interested in the effect of these forces on both the global behavior and the stability of the system.

To simplify our analysis, we will take all force constants to be 1 and assign coordinates to each of the objects based on their equilibrium location (i.e. $x_i = 0$ indicates that object i is at its equilibrium position). If we have no friction (no dashpots), then for n unit-mass objects we generate a system of equations:

$$\begin{aligned}
\ddot{x}_1 &= -x_1 + x_2 \\
&\vdots \\
\ddot{x}_i &= x_{i-1} - 2x_i + x_{i+1} \\
&\vdots \\
\ddot{x}_n &= x_{n-1} - x_n
\end{aligned}$$

Which can be rewritten (by setting $\dot{x}_i = x_{i+n}$) as the first-order linear differential equation in $2n$ variables $\dot{\mathbf{x}} = A\mathbf{x}$. Where A is the $2n \times 2n$-matrix whose entries are given by the coefficients of the components of $\mathbf{x}$ in the system of equations described above.

The matrix A describing the time evolution of the system can be broken into Jordan blocks and the system can be solved in the frequency domain. There will be a 2x2 block corresponding to the space translation invariance of the system and the conservation of total momentum, and 1x1 blocks corresponding to eigenmodes which evolve in time by rotating in complex space. Each of these modes is independent, so once the initial conditions of the system are transformed into the eigenmode coordinate system, the time evolution of the system is simple to calculate.

An alternate system can be developed along the same lines by including dampers between the objects. The analysis of this system follows the same method as detailed above. The first system studied consisted of just springs between each of the masses. This was a Hamiltonian system that conserved energy. Adding velocity dependent dampers between the masses led to a system whose motion decayed over time. Notice that this is because the nonzero eigenvalues have negative real part.

The effect of an external driving force on the first car was studied. Numerical simulations in the time and frequency domains appeared to give the same answer (this was not yet checked in detail). Applying a "white noise" random force to the first car allowed us to approximate the transfer function of the system by applying all frequencies simultaneously.

An Introduction to Control Theory

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April 22

By taking a dynamic system and introducing an external driving force, we can use the techniques of control theory to analyze the behavior of the system under any driving force (including no force at all). In this talk, we take a brief

look at controllability, the Kalman Rank Condition, and the calculation and use of the transfer function. To motivate our discussion, we continued with the same two systems (with $n = 3$) that Hartfiel used in last week's discussion.

For our purposes, we are going to take a driven dynamic system as a *control system*, with a *state* given by the vector $\mathbf{x}$ and an *input* (the driving force) given by the function $u(t)$. We may also desire to look, for example, at the behavior of a single vehicle. In this case, we could also describe an *output* y that is a (linear) function of $\mathbf{x}$. With this background, we would describe Hartfiel's system as a single-input single-output time-invariant linear continuous-time control system. One of the first steps in analyzing a problem in control theory is the characterization of the system.

One of the goals of control theory is to make the system behave in the way we want it to. In this case, we would ask whether or not the system is *completely controllable*. By completely controllable, we mean that for any initial condition x_0 and any final condition x_f and any (finite) time T, there exists a control $\mathbf{u}(t)$ such that the solution $x(t)$ to the system with control $\mathbf{u}(t)$ and initial condition x_0 obeys the condition that $x(T) = x_f$.

Fortunately, for linear time-invariant systems (both discrete- and continuous-time) systems, this question has a simple answer.

Let our system Σ be described as:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

for $\mathbf{x} \in \mathbb{R}^n$, A an $n \times n$ matrix, $\mathbf{u}(t) : \mathbb{R}^{\geq 0} \to \mathbb{R}^m$, and B an $n \times m$ matrix. We then construct the $n \times nm$-matrix:

$$U = [B : AB : A^2B : \cdots : A^{n-1}B],$$

whose columns are just the sequence of matrices B, AB, etc. placed next to each other.

Theorem 1 (Kalman Rank Condition)**.** *The system Σ is completely controllable if and only if U is full rank.*

Applying the Kalman Rank Condition to both types of systems described last week shows that they are both completely controllable.

Note that it is important to separate the question of *stability* from that of controllability.

If we look at the solution to Σ with initial condition x_0 and scalar control $u(t)$, we may write it our exactly as:

$$x(t) = e^{At}x_0 + e^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau,$$

and if we introduce an exponential control $u(t) = e^{st}$ (where s is not an eigenvalue of A), we may evaluate the integral and get:

$$x(t) = e^{At}(x_0 - (sI - A)^{-1}B) + (sI - A)^{-1}Be^{st},$$

and our output $y = Cx$ becomes:

$$y(t) = Cx(t) = Ce^{At}(x_0 - (sI - A)^{-1}B) + C(sI - A)^{-1}Be^{st}.$$

The term $C(sI - A)^{-1}Be^{st}$ is the *pure exponential response*, and $C(sI - A)^{-1}B$ is the *transfer function* of the system (in particular when B is such that only one component of Σ feels the control, and C is such that our output is a single component of $\mathbf{x}(t)$ - i.e. when Σ is a single-input-single-output (SISO) system, although the term has been generalized).

In our cases, we can calculate the transfer functions readily using $B = (000100)^T$ and $C = (000001)$ (or whatever C we desire should we wish to look at a different component of $\mathbf{x}$). Henceforth, we will use 'Σ' to refer to the undamped system and '$\tilde{\Sigma}$' to refer to the damped system. The transfer function for Σ is given by:

$$G(s) = \frac{s}{s^6 + 4s^4 + 3s^2}$$

and that for $\tilde{\Sigma}$ by:

$$\tilde{G}(s) = \frac{s^3 + 2s^2 + s}{s^6 + 4s^5 + 7s^4 + 6s^3 + 3s^2}$$

What is particularly useful is to suppose that we are using an exponential input with an imaginary argument: $u(t) = e^{i\omega t}$. If we want to look at the behavior of the system driven by a sine wave with unit amplitude and frequency ω, we can use $u(t) = \sin \omega t = \mathrm{Im}(e^{i\omega t})$. Our output becomes:

$$y(t) = |G(i\omega)| \sin(\omega t + \arg[G(i\omega)])$$

where $|G(i\omega)|$ is easily seen to be the gain as a function of frequency and $\arg[G(i\omega)]$ the phase-shift. Plotting the two versus log-frequency produces a Bode plot, and is a common technique for the analysis of complex linear systems.

An Introduction to Numerical Analysis

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April 29

If we wish to approximate the behavior of a system with computation, our choice of numerical method can affect the accuracy of our solution for a given step size. The goal of this talk was to show the tradeoffs in choosing one MATLAB package over another. Given a differential equation with an initial condition:

$$y' = f(t, y), \qquad y(0) = y_0 \tag{1}$$

we wish to approximate the solution $y(t)$ numerically. In doing so, there are some costs we may wish to consider:

- Human cost: How much work do *we* have to do?
- Computational cost: How many computations and how much memory is needed?
- Accuracy cost: How large an error are we willing to tolerate?

In this talk, we will only consider the first and last of these questions and the tradeoff inherent between the two.

If f is Lipschitz, then we are guaranteed a unique solution. We may then wonder about the *stability* of that solution under perturbation. That is, if $\gamma(t)$ is our numerical approximation to Equation (1), then the solution $\tilde{\gamma}(t)$ to the perturbed system:

$$y' = f(t, y) + \delta(t), \qquad y(0) = y_0 + \epsilon$$

where both $\delta(t)$ and ϵ are 'small', is not so different from $\gamma(t)$. In some cases, there is a sensitive dependence on initial conditions that prevents such stability (i.e. for even small ϵ, the difference between $\gamma(t)$ and $\tilde{\gamma}(t)$ grows very quickly).

Let us suppose that our system is of the form:

$$y' = \lambda y + g(t),$$

with the $\mathrm{Re}(\lambda) \ll 0$, then we refer to the system as 'stiff'. This can be a consideration when examining numerical methods (and MATLAB has methods optimized for stiff differential equations).

We begin by looking at Euler's method and noting that we are effectively modeling each time step linearly, and measuring the error with the error from the corresponding Taylor polynomial. We could gain some accuracy by using instead a second-order Taylor approximation for $y(t)$, but at the cost of having to compute $g'(t)$ (or $\partial f/\partial t$). For Euler's method, the local error (error from a single step) is $O(h^2)$, where h is the step size, and the global error is $O(h)$. Extending this method to a Taylor approximation of order m gives us a better error ($O(h^m)$ locally), but the cost is in computing higher derivatives and then being required to work with an $m \times m$ system.

Instead, we can reduce the human cost by looking instead at the Runge-Kutta methods. They are higher order methods, but do not require the computation of derivatives. The cost moves to the evaluation of the function (and integrals of the function), but we can let the machine take care of that. In this case, the 'order' of the method relies upon the method of numerical integration. For example, the trapezoid method yields $O(n^2)$, and is called a 'two-step' or 'RK2' method. If we use Simpson's Rule for evaluating the integral, we get a four-step method with $O(n^4)$ error.

In MATLAB, ode45 is based on Runge-Kutta(4,5). This means that it is using Simpson's Rule for the integration. The '5' has to do with another aspect of numerical analysis: how many points are being used to predict the solution one time step away. In this case, five of them are used. The documentation for MATLAB tells us the other algorithms that are used.

Integral Equations with Delay and Their Applications to Biology and Economics

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May 6

Dr. Hritonenko kindly agreed to address NOSYGS in addition to her colloquium talk at the department. She chose to give us an overview of some of her previous line of research in integral models.

The primary model of interest is called the *Vintage Capital Model*, and the main problem is to maximize the integral:

$$I = \int_{t_0}^{T} \rho(t)[u(y(t)) - \lambda(t)m(t)]dt$$

with the *state equations*:

$$y(t) = \int_{a(t)}^{t} \beta(\tau, t)m(\tau)d\tau,$$
$$R(t) = \int_{a(t)}^{t} m(\tau)d\tau,$$

and a set of constraints and initial conditions.

To find the structure of optimal trajectories, we look at $T = \infty$ and attempt to make the integrals as simple as possible by making $u(c)$ linear in c, $\beta(\tau, t) = b + 0\tau + b_1$ and $\lambda(t)$ constant. This gives us a square wave for $m(t)$ indicating the 'replacement echoes'.

We next leave the ideal world of going to $T = \infty$ and have a finite time problem with $T < \infty$, which changes the shape of $m(t)$ creating 'anticipation echoes' of the end of the period.

The 'delay' shows up in the limits of integration in a natural way - we turn our attention to a harvesting problem: how to maximize revenue from harvesting. This results in the model:

$$X(t) = h\int_{t-a(t)}^{t} M(t-u)X(u)du + (1-h)\int_{t-T}^{t} M(t-u)X(u)du,$$

again with constraints and initial conditions.

Dr. Hritonenko finished with a quick look at the future of these problems as nonlinear integral equations with delay.

The Dynamics of Zombie Outbreaks

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May 13

In this presentation, we examine several models based upon the 'SIR' epidemiological model: Susceptible, Infected, Recovered (or Removed). In the SIR model, individuals move between the three compartments at various rates, and it is assumed that there is a natural birth and death rate, plus a rate of infection and rate of recovery/mortality. The difference in the Zombie (SZR) model is that individuals can move not only from Susceptible to Zombie, but also from Removed to Zombie (since Zombies are the walking dead). This changes the dynamics from the usual SIR model. In either case, the models are nonlinear (including an 'SI' or 'SZ' term), and so equilibrium stability is determined by looking at the Jacobian evaluated at the given equilibrium. In this case, no Zombies (and no Removed) is an instable equilibrium.

After examining the basic (SZR) model, we add a compartment - assuming another class: Infected. That is, the assumption is that we have Susceptible, Infected (with a latent period before becoming zombies), Zombie, and Removed. This adds an additional dimension to the system creating the 'SIZR' model, but the no Zombie (no Infected, No Removed) equilibrium is again unstable.

Next we add a 'Quarantine' compartment (assuming that all I, Z, are moved to Q at various rates and that Quarantined individuals die and move to R at some rate also). In this case the stability of the 'everyone not a zombie' equilibrium could be determined by looking at the characteristic equation of a 5×5 matrix, or more simply by using the next generation method to determine the basic reproductive ratio R_0. If $R_0 > 1$, then the equilibrium is not stable, if $R_0 < 1$, then the equilibrium is stable. The actual value depends on the rate of quarantine, the higher the quarantine rate, the smaller R_0 becomes.

We then return to the SIZR model, but assume there is a treatment (so we can move individuals from Z to S). This allows an equilibrium in which there are zombies, but not everyone is infected. The disease-free equilibrium is still present, and is still unstable. The coexistence equilibrium is stable.

Finally, we look at a model in which the humans organize periodic attacks to wipe out the zombies. If we assume increasing kill ratios (more and more zombies removed), then the zombies can be eradicated if the attacks happen frequently enough.

Ergodic Theory

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May 20

For this talk Baldivieso gave us an overview of the principles of ergodic theory, that is, the statistical properties of a dynamic system with respect to a measure on the underlying space. First we took a moment to unify the terminology between measure theory and probability theory. For the rest of this summary, we will use the standard measure theoretic notation. Next, Baldivieso reminded us of the basic principles of both probability and measure theory.

Given two measure spaces (in both cases, X_i, the whole space, has measure 1): $(X_1, \mathscr{F}_1, \mu_1)$ and $(X_2, \mathscr{F}_2, \mu_2)$ and a transformation $T : X_1 \to X_2$. We say that T is *measurable* if the inverse-image of an element of $\mathscr{F}_2$ is always in $\mathscr{F}_1$. Further, T is *measure-preserving* when the measure of the pre-image of an element of $\mathscr{F}_2$ (using μ_1) is the same as the measure of the element of $\mathscr{F}_2$ (using μ_2).

Two measures used in Balvdivieso's presentation were Lebesgue measure and Gauss measure. The first transformation was the Decimal Map $Tx = 10x \mod 1$, which is measure preserving when we take our space as the unit interval and the measure as the Lebesgue measure on the line. This leads us to our first theorem:

Theorem 2. *For a measure space $(X, \mathscr{F}, \mu$ and transformation $T : X \to X$ the following two statements are equivalent:*

1. *T is measure-preserving.*
2. *For any f in $L_1(X, \mu)$: $\int_X f(x)d\mu = \int_X f(Tx)d\mu$.*

The type of measure that is preserved depends upon T, and there may be many measures preserved by any particular transformation.

It could also be that for some set $A \in \mathscr{F}$, we have $T^{-1}A = A$, then we say that A is invariant under T. For a probability space (which we may take to mean $\mu(X) = 1$), we say that A is nontrivial so long as $0 < \mu(A) < 1$. We say that T is *ergodic* if for all $A \in \mathscr{F}$, $T^{-1}A = A$ implies that A is trivial. This gives us our second theorem:

Theorem 3. *For a measure space $(X, \mathscr{F}, \mu$ (with $\mu(X) = 1$) and measure-preserving transformation $T : X \to X$ the following three statements are equivalent:*

1. *T is ergodic.*
2. *If $\mu(A) > 0$, then $\bigcup_{n=1}^{\infty} T^n A = X$.*

3. If $\mu(A) > 0$ and $\mu(B) > 0$, then there is an $n \geq 1$ such that $\mu(T^n A \cap B) > 0$.

We follow this immediately with:

Theorem 4 (the Birkhoff Ergodic Theorem)**.** *For a measure space $(X, \mathscr{F}, \mu$ (with $\mu(X) = 1$), if $T : X \to X$ is a measure-preserving transformation and f is invertible, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = f^*(x)$$

for some $f^* \in L^1(X, \mu)$ with $f^*(Tx) = f^*(x)$ for almost every x. Furthermore if T is ergodic, then f^* is constant and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \int_X f d\mu.$$

We finished with some examples and a brief look at *mixing transformations* and their relationship to ergodicity.

Modeling Neuronal Impulses

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June 3

We first look at the features of the Hodgkin-Huxley model of the propagation of electrical impulses along the squid giant axon. This is a complicated system involving four coupled differential equations with two 'fast' and two 'slow' variables and six parameters. If we examine the fast-slow phase plane, we see a cubic nullcline and the existence of a Hopf bifurcation.

The Fitzhugh simplified system uses only two coupled differential equations with three parameters. The Fitzhugh-Nagumo simplification of the Hodgkin-Huxley model holds one of the 'fast' and one of the 'slow' variables constant - the remaining 'fast' variable represents *excitation* and the remaining 'slow' variable the *recovery* of the system. The retains many of the analytic features of the Hodgkin-Huxley model. The Fitzhugh-Nagumo model can be expressed as the following equations.

$$\dot{x} = c\left(y + x - \frac{x^3}{3} + z(t)\right)$$
$$\dot{y} = -\frac{x - a + by}{c}$$

Here, z(t) represents an applied force or stimulus. The x variable is the fast or excitable variable. The y variable is the slow or recovery variable. The parameters a,b,c are bounded.

Specific example: define the parameters as follows: $c = 2, b = \frac{1}{2}, a = \frac{3}{4}$. This changes the Fitzhugh-Nagumo model to:

$$\dot{x} = 2\left(y + x - \frac{x^3}{3} + z(t)\right)$$
$$\dot{y} = -\frac{x}{2} + \frac{3}{8} - \frac{y}{4}$$

Take $z(t) = 0$. There is a fixed point which is representative of the steady/resting state of the neuron, $x_* = 1.08004$. We find that the fixed point is a stable spiral. Note that the stability of a fixed point is not so much a description of

the point as it is the points around it. That is to say, the stability of a fixed point tells us how the system behaves *near* the fixed point. So, any orbit the passes near our fixed point will spiral onto the fixed point.

Now, Let z(t) be a constant function. This function can be thought of as a signal that is trying to go across a neuron. We will consider the fixed point of the resting system to be the initial position of the stimulated system.The system undergoes a Hopf Bifurcation when $x* = \sqrt{\frac{7}{8}}$. The fixed point, or resting state, of the system changes from a stable spiral to an unstable spiral which approached a limit cycle. From here, we can see that the Fitzhugh-Nagumo "simplification" of the Hodgkin-Huxley Model does retain the important qualitative features.