Listing of the Summaries of NOSYGS or the

Non-Linear Systems Group Seminar

Portland State University

Winter 2010

Friday, 2pm-3pm, NH 346

Organized by: J.J.P. Veerman**) and J. C. Caughman.

Summaries by N. Kraft, A. Keel, G. Swanson, W. Sullivan, R. Benim, E. Riley, J. J. P. Veerman

in collaboration with — and with advice from — the speakers.

This term the seminar is dedicated to students who want to do research, and to faculty who want to jump in and help generate ideas. The emphasis is on interaction and generating new ideas. Speakers should prepare clear and simple presentations of basic underlying ideas. Technical prowess is appreciated when alluded at, but less so when expounded and insisted upon. More than the details of any single proof or calculation, we aim to understand the principles underpinning those.

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Non-Separating Embeddings of Graphs in Closed Surfaces

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January 15, 2pm

Assume all surfaces to be 2-dimensional manifolds which are closed and orientable. A graph is assumed to be a connected, with multiple edges and loops allowed. When a graph G is embedded in a given surface S we consider the thickened ribbon graph $i(G)$ to be a closed, oriented subsection of the surface S. An embedding is considered cellular if each component of $S \setminus i(G)$ is homeomorphic to a disk. These disks will be referred to as the faces of the graph embedding.

The Euler Characteristic, $\chi(S)$, of a surface S is given by:

$$\chi(S) = v - e + f = 2 - 2g(S)$$

This equation is equivalent to:

$$2g(S) = (e - v + 1) - (f - 1)$$

where v is the number of vertices of the surface, e is the number of edges, f is the number of faces, and $g(S)$ denotes the genus of the surface S. Note that $(e-v+1)$ is the first Betti number of S, and $(f-1)$ is called the Betti deficiency.

Clearly, f must be at least one when considering an embedded graph. Let f_- denote the smallest possible value of f for a given graph embedding. Substituting f_- into equation (0.2) makes $g(S)$ become $g_+(G)$, which represents the maximal genus of a surface in which the graph G may be embedded:

$$2g_+(G) = (e - v + 1) - (f_- - 1)$$

From Theorem 1 of the reference, a graph G has a non-separating (not necessarily cellular) embedding in an orientable surface T if and only if:

$$2g(T) \geq (e - v + 1) + (f_- - 1)$$

This theorem can be demonstrated through example, and a rigorous proof is given in the reference below. Notice that, since the embedding $i(G)$ in T need not be cellular, one can add a handle to T in such a way that the the handle does not intersect $i(G)$ and thus increases $g(T)$ by one. This can be repeated any number of times, which causes the left hand side of last inequality to increase, yet has no impact on the right hand side.

Reference: M. Skoviera, *On Non-Separating Embeddings of Graphs in Closed Surfaces*, Acta Math. Univ. Comenianae, Vol. LXI, No.1, 65-68, 1992.

Rotation Systems and Graphs

Nicole Kraft, Mathematics, Portland State

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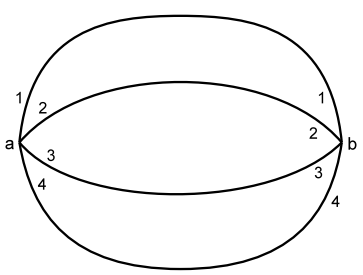
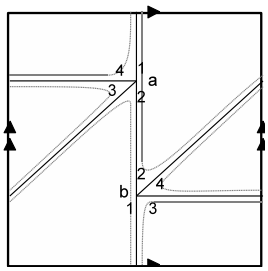
January 22, 2pm

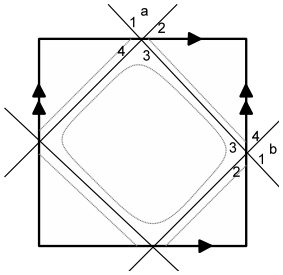
We look at some examples of rotation systems. Suppose we have a graph consisting of two vertices and four edges, each of which connects the two vertices. Consider its "ribbon graph" in a surface (its ϵ neighborhood in the surface) and focus on a neighborhood around each vertex. Call the vertices a and b and label the edges in a clock-wise



(a) Labeling of vertices and edges.



(b) Graph in torus with ribbon neighborhood.



(c) Another rotation system in the torus.

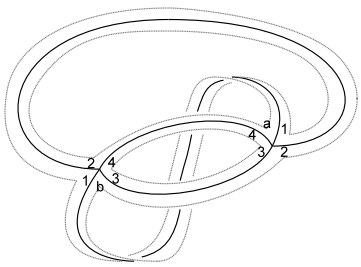
manner listing $a : (1234)$ and $b : (1432)$. This is called a "rotation system" and is invariant under cyclic permutations. For example Figure (1a): $a : (1234)$, $b : (1432)$ is the same rotation system as $a : (3412)$, $b : (4321)$. Since the edges emanating from a can be labeled 1, 2, 3, and 4, without loss of generality, *and* we may start the list (in clock-wise direction) of edges arriving at b beginning with 1, it is easy to see we get at most $3! = 6$ rotation systems. In each case, the edges at a are (1234) (in clock-wise order with respect to the orientation of the surface), while the edges incident at b are:

$$b_1 : (1432) \text{ and } b_2 : (1243) \text{ and } b_3 : (1234) \text{ and } b_4 : (1324) \text{ and } b_5 : (1342) \text{ and } b_6 : (1423) \quad .$$

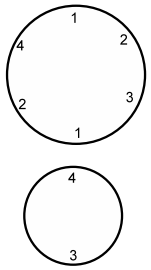
There is a unique correspondence (see the reference) between rotation systems and oriented ribbon graphs — up to homeomorphisms of the ribbon graph. Therefore we can use a rotation system to encode its "cellular embedding" (glue one disk to every boundary component of the ribbon graph). For example Figure (1b) shows the cellular embedding of the rotation system $a : (1234)$, $b : (1243)$. Figure (1c) gives the cellular embedding of a different rotation system associated to the same graph: $a : (1234)$, $b : (1234)$.

Define the set B of half edges at each vertex: $B = \{1_a, 1_b, 2_a, 2_b, 3_a, 3_b, 4_a, 4_b\}$, the involution θ which swaps the half edges of each edge (so that $\theta^2 = \mathrm{Id}$). and σ, the clock-wise, cyclic permutation of half-edges at each vertex. Thus for the rotation system given by $a : (1234)$, $b : (1243)$, see Figure (1b):

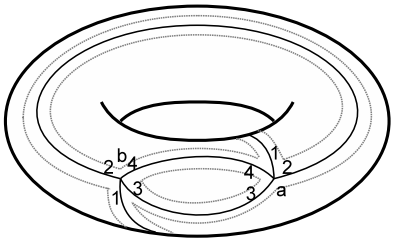
$$\begin{array}{llll} \theta(1_a) = 1_b & \theta(1_b) = 1_a & \sigma(1_a) = 2_a & \sigma(1_b) = 2_b \\ \theta(2_a) = 2_b & \theta(2_b) = 2_a & \sigma(2_a) = 3_a & \sigma(2_b) = 4_b \\ \theta(3_a) = 3_b & \theta(3_b) = 3_a & \sigma(3_a) = 4_a & \sigma(3_b) = 1_b \\ \theta(4_a) = 4_b & \theta(4_b) = 4_a & \sigma(4_a) = 1_a & \sigma(4_b) = 3_b \end{array}$$



(a) Ribbon graph of a: (1234), b: (1243)



(b) Disks corresponding to orbits.



(c) Graph embedded in a torus

From a rotation system we get a unique cellular embedding as follows. The ribbon graph has the same Euler characteristic as the graph, see Figure (2a). The number of boundary components of the ribbon graph equals the

number of orbits of $\langle \theta \circ \sigma \rangle$. Thus the closed (oriented) surface in which the graph is cellularly embedded has Euler characteristic $\chi = v - e + f$, where v and e are the number of vertices and edges of the graph, respectively, and f is the number of orbits of $\theta\sigma$ in the rotation system. In the last example there are two orbits $(1_a \to 2_b \to 3_a \to 1_b \to 2_a \to 4_b)$ and $(4_a \to 3_b)$, see Figure (2b).

Reference: J.L. Gross and T.W. Tucker, *Topological graph theory*, Wiley Interscience, 1987

Stability of Nearest Neighbor Traffic

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Jan 29, 2pm

We model agents and their interaction as a infinite string of linear oscillators in $\mathbb{R}$ of mass 1 and with a preferred velocity u_0: $\forall k \in \mathbb{Z}$

$$\begin{aligned} \ddot{x}_k &= f\{(x_k - h_k) - \rho_-(x_{k-1} - h_{k-1}) - \rho_+(x_{k+1} - h_{k+1})\} \\ &\quad + g\{(\dot{x}_k - u_0) - r_-(\dot{x}_{k-1} - u_0) - r_+(\dot{x}_{k+1} - u_0)\} \end{aligned}$$

In a "truly decentralized" model, agents cannot measure our absolute position without reference to an external computer (ie: GPS); they can only perceive relative distances and speeds. Thus in this case the sum of the coefficients of the positions $(\rho_- + \rho_+)$ must be one in the LHS of the equation, as well as the sum of the coefficients of the velocities $(r_- + r_+ = 1)$. However, on earth we tend to travel in or on a more or less static medium. This allows us to locally measure (from frictional forces) our own *speed* with respect to that medium. Thus for practical applications $r_- + r_+$ need not be 1. The equation given here is the general linear model (in $\mathbb{R}$) where cars try to achieve relative positions $h_i - h_k$ with respect to one another and there is a preferred velocity u_0. After a coordinate change (and rescaling time) we obtain: $\forall k \in \mathbb{Z}$

$$\ddot{z}_k = -\{z_k - \rho_- z_{k-1} - \rho_+ z_{k+1}\} + g\{\dot{z}_k - r_- \dot{z}_{k-1} - r_+ \dot{z}_{k+1}\} \quad . \qquad (*)$$

Stability of large, but finite flocks is a complicated notion. This is due to the fact that even though the system may have a stable equilibrium, perturbations can grow while propagating through the flock. If this is case then obviously very large flocks are impossible. By and large what is at issue is the following. Denote the finite system (add boundary conditions to the previous equation) with N agents (or cars) by a subscript N. Thus for each N we have a linear system, that we write as follows:

$$\dot{z}_N = M_N z_N \quad .$$

The translation invariance of the system can used to show that each M_N must have an eigenvalue 0. Now it is perfectly possible for the matrices M_N to have eigenvalues with non-positive real part, while still the response of the system of size N to a perturbation in the initial condition of a given car grows polynomially in N.

The infinite problem offers some simple insights. In Equation (*), set $r_\pm = \rho_\pm = 1/2$ and write $Z(t) \equiv \sum_{-\infty}^{\infty} z_k(t) e^{-ik\theta}$. Barring summability assumptions, the set of equations can be rewritten as:

$$\frac{d}{dt}\begin{pmatrix} Z \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -(1-\cos\theta) & g(1-\cos\theta) \end{pmatrix}\begin{pmatrix} Z \\ \dot{Z} \end{pmatrix}$$

The eigenvalues an be calculated as functions of θ. They have 0 real part when $\cos\theta = 1$.

Now set $\rho_\pm = \rho$ and multiply Equation (*) by $\dot{z}_k$. Again, assume summability and sum over k. Denote by $(.,.)$ the ℓ_2 inner product, by z the vector whose k-th component equals z_k and by Tz the vector whose k-th component equals z_{k-1}. Summing the equation over k yields:

$$\frac{d}{dt}\left[\frac{1}{2}(\dot{z},\dot{z}) + \frac{1}{2}(z,z) - \rho(z,Tz)\right] = g[(\dot{z},\dot{z}) - (r_- + r_+)(\dot{z},T\dot{z})] \quad .$$

Now if $\rho \leq \frac{1}{2}$ the LHS is the derivative of a non-negative quantity, while the RHS has the same sign as g if $r_- + r_+ < 1$.

Modeling of Material Aging

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Feb 5, 2pm

Material aging is a dissipative process where an increase in entropy leads to the degradation of the material. This can lead to dangerous situations unless a material is fixed or replaced. In one instance, a pilot was flying over Colorado when he noticed a fly in the cockpit. He smashed it against the plexiglass with a newspaper, and shattered the glass completely. After further review, it was found that the glass cleaner used was seeping down to where the glass and the rest of the plane connected. The cleaner caused a mechano-chemical reaction and caused a premature aging of the material.

One can think about material aging in terms of two different types of time: physical time, t, and the internal time of the material, τ. As an example of the effect that the use of material time can have, consider the standard equation of the harmonic oscillator in the material time τ

$$\partial_\tau^2 U + \omega^2 U = 0$$

Now rewrite this in terms of the physical time t where $\dfrac{d\tau}{dt} = S(t)$, assuming that $S(t) \neq 0$:

$$\partial_t^2 U - \partial_t \ln(S(t))\, \partial_t U + S(t)^2 \omega^2 U^2 = 0$$

An oscillator with constant frequency in material time becomes a damped (or forced depending on the sign of $S'(t)$) oscillator in physical time.

We introduce a new function, s, which represents entropy density, and σ is the entropy production. The entropy balance written in terms of the material time leads to the relation

$$S(t) = \exp\left(-\int_{t_0}^{t} \frac{\sigma}{s}\, dt'\right) \quad .$$

When the process is reversible, $\sigma = 0$ and $S(t) = 1$ and thus in this case physical and material time are the same. For irreversible processes, $S(t)$ decreases and so one hour in physical time actually goes by much faster in material time, hence the aging. When entropy flux is negligible, $s' = \sigma$. In an experiment with Groningen sandstone $S(t)$ has the form $S(t) = C\left(\sqrt{t+1}\right)^{-1}$ which can be derived from that hypothesis and the above equation. **Reference:** See also Preston's lecture in the 2006 NOSYGS seminars.

Trafficking Stationary Phase Integrals

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Feb 12, 2pm

We consider a finite number of vehicles on a one-lane highway. The position of the k-th vehicle is given by z_k, $k = 0, \ldots, N$ and all vehicles are assumed to be unit mass. Vehicle z_k is assumed to only receive information from vehicle x_{k-1}, $k = 1, \ldots, N$. The trajectory $z_(t)$ of the lead vehicle is a priori given. With some simplification the relevant system of equations is:

$$\ddot{z}_k = f\{z_k - z_{k-1}\} + g\{\dot{z}_k - \dot{z}_{k-1}\} \qquad \text{and} \qquad z_0(t) \text{ given} \quad .$$

where f and g are negative real constants.

Substituting $z_k(t) = e^{i\omega t} a_k(\omega)$ into the equation, gives:

$$a_k(\omega) = \left(\frac{f + i\omega g}{f + i\omega g + \omega^2} \right)^N \quad .$$

In this lecture we will simplify and set $f = 0$, thereby neglecting the contribution of the relative positions of the cars. It now turns that if the flock is in rest at $t = 0$ at which point the leader starts moving with unit velocity, then for for the N-th car:

$$\ddot{z}_N(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a_N(i\omega)\, e^{i\omega t}\, d\omega = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left(\frac{g}{g - \nu} \right)^N e^{\nu t}\, d\nu \equiv \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{N(a(\nu) + ib(\nu))}\, d\nu \quad .$$

where a and b are the real and imaginary parts of the exponent $\Phi(\nu)$.

Let $\hat{\mathbb{C}}$ the complex plane *minus* the singularities of the integrand. The stationary phase method consists of replacing the original path of integration $\gamma(t) = it$, for $t \in \mathbb{R}$ by another path $\hat{\gamma}$ along which the integral is easier to evaluate, and such that or $\gamma - \hat{\gamma}$ is contractible in $\hat{\mathbb{C}}$ and so:

$$\oint_{\gamma - \hat{\gamma}} e^{N\Phi(\nu)}\, d\nu = 0 \qquad \text{or} \qquad \oint_{\gamma} e^{N\Phi(\nu)}\, d\nu = \oint_{\hat{\gamma}} e^{N\Phi(\nu)}\, d\nu \quad .$$

This is achieved by choosing the path $\hat{\gamma}$ in such a way that in addition either

$$a(\hat{\gamma}(t)) < 0 \qquad \text{or} \qquad b(\hat{\gamma}(t)) = const \quad .$$

The first condition says that the asymptotic (for N large) contribution will exponentially small. If the first fails, the second says that b (the *phase*) is constant, so that cancelations in the integrand do not occur. In this case the asymptotic contribution is calculated by expanding $\Phi(\hat{\gamma}(t))$ around the(positive) local maxima of $a(\hat{\gamma}(t))$. We expand $\Phi(\hat{\gamma}(t))$ to first order if a positive local maximum is located at and endpoint of the path $\hat{\gamma}(t)$, to second order if the positive local maximum is located in the interior of the path. The only other thing one needs to know is that a local maximum of $a(\hat{\gamma}(t))$ along a stationary phase curve must be a saddle point ν_s of $a(\nu)$ and of $b(\nu)$. Therefore ν_s is a root of the second derivative of the exponent $\Phi(\nu) = \frac{\nu t}{N} - \ln\left(1 - \frac{\nu}{g}\right)$. The integrals resulting from such (lowest order) expansion are standard.

In the case at hand the integrand has only one pole (located at $\nu = g$) and one saddle located at $\nu_0 = g + \frac{N}{t}$. We choose the following path $\hat{\gamma}$ (see Figure 0.1). Branches I and III of $\hat{\gamma}$ are tangent to the imaginary at $\pm i\infty$ and connect the imaginary axis to a stationary phase curve given by II. We let $\hat{\gamma} \equiv III \circ II \circ I$. Now it is easily seen that the curves I and III give no contribution. Around the saddle ν_0 we expand:

$$\int_{III} \approx \int_{\nu_0 - i\infty}^{\nu_0 + i\infty} e^{N[\Phi(\nu_0) + \frac{1}{2}\Phi''(\nu_0)(\nu - \nu_0)^2]}\, d\nu \quad .$$

Finally use the fact that $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\, dx = \sqrt{2\pi}$ to establish that if $f = 0$ and $z_0(t) = \delta(t)$, then

$$\ddot{z}_n(t) = \frac{1}{t}\sqrt{\frac{N}{\pi}} \left(\frac{-gt}{N} \right)^N e^{gt + N}$$

is exponentially well approximated (in N) as N tends to ∞.

Reference: See also Veerman's lecture earlier in this episode and the one in the Rockefeller Math Seminars in 2008.

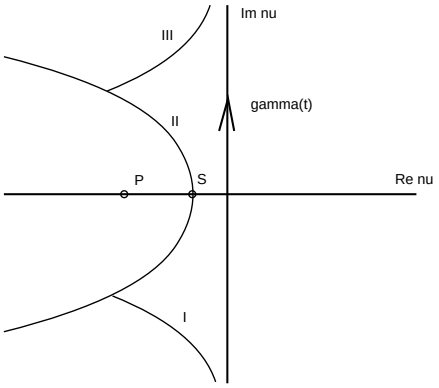

Figure 0.1: *The path γ along the imaginary axis is replaced by $\hat{\gamma}$ which follows first I (where a is negative), then II, a stationary phase curve, and finally III where a is negative again. P and S indicate the pole and the saddle respectively.*

Traffic Dynamics

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Feb 19, 2pm

If we have a set of sensor platforms scattered in an area, it is often useful to know their exact locations, but with large numbers of sensors, it may be problematic to accurately locate all of them manually. If the sensors are also equipped with the ability to measure the range and azimuth to neighboring sensors however, we could gather that information to then calculate their locations.

Along with any measurement is an error in that measurement, so let x_i be the i^{th} sensor and the measured distance between adjacent sensors u and v be given by: $\xi_{uv} = x_u - x_u + \varepsilon_{uv}$

If we are then measuring the distance between non-adjacent sensors, or verifying the distance already measured by comparing data from a path of measurements linking the two sensors, we might get a different value. In this case, we may want to test all possible paths in an attempt to get the best possible measurement (that is, to reduce the error as much as possible), so given a collection of such distances, we would want to calculate the average estimate as well as the covariance of the estimate.

In the one-dimensional problem, we would look at a group of sensors all at varying positions along a line. If every sensor has an error of s, then the errors simply add, so the calculated distance from x_1 to x_3 would have error $2s$. In two or three dimensions, the errors combine in more complex ways.

One way to calculate the positions (and errors thereof) would be to recreate the position graph as a weighted resistance network. (This is inspired by the work of Peter Doyle and Doylen Snell (P. G. Doyle and J. L. Snell, Random Walks and Electric Networks. New York: Math. Assoc. of Amer., 1984.) See also http://humdoi.mae.ufl.edu/ prabirbarooah/PBresearch.html for three papers related to this problem from Barooah et al..

The basic method is to set up a weighted graph as an electrical network where the weights correspond to the resistance between any two nodes. Then, inject a unit current at one node and connect any other node to ground and use Kirchoff's and Ohm's Laws to create a system of linear equations that can then be solved for the voltage drop across each resistance. This corresponds to the desired distance [N.B. - I am not *sure* of that! ER] To estimate the covariance, we instead replace real-valued resistances with matrix-valued resistances and inject an identity-valued current at one node.

The analysis of the network can be simplified with the k-Fuzz operation on a graph. For every nonadjacent pair of vertices u and v, add an edge if there is a path of length k from u to v. This makes the network look more like a lattice, simplifying the analysis by patching 'holes' in the network. We then use that adding a resistor (edge) can only decrease the total resistance between any two nodes. (Similarly, removing a resistance, only increases the resistance

between any two points.)

References: P. G. Doyle and J. L. Snell, *Random Walks and Electric Networks*, New York: Math. Assoc. of Amer., 1984.

P. Barooah and J. P. Hespanha, *Estimation from Relative Measurements: Electrical Analogy and Large Graphs*, IEEE Transactions on Signal Processing, vol. 56(6), pp. 2181-2193, June 2008.

Separating Sufaces By Graphs

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February 26, 2pm

Let G be a graph which is a subcomplex of a surface M. The questions we wish to answer are: When will G separate M? When will G divide M into r regions where $r \geq 2$?

Let N be a regular neighborhood of G in M. We say G separates M if $M \setminus N$ is not connected. The connected components of $M \setminus N$ will be called regions and the number of connected components will be given by r. Using the *Exact Homology Sequence of a Pair* given by

$$\cdots \to H_2(G;2) \xrightarrow{i_*} H_2(M;2) \xrightarrow{j_*} H_2(M,G;2) \xrightarrow{\delta_*} H_1(G;2) \xrightarrow{i_*} H_1(M;2) \xrightarrow{j_*} H_1(M,G;2)$$
$$\to H_0(G;2) \to H_0(M;2) \to H_0(M,G;2) \to 0$$

where i_* is the map induced by the inclusion map $i : G \hookrightarrow M$. $H_p(M,G;2)$ is the relative homology group in dimension p, of M modulo G with coefficients in $\mathbb{Z}/2\mathbb{Z}$. (See Munkres for definition.) There is a "version" of the Lefschetz duality that states:

If M is a closed orientable surface, G is a subcomplex of M and V is the collection of regions into which G divides M, then

$$H_0(V;2) \cong H_2(M,G;2).$$

This means $\text{rank}[H_2(M,G;2)] = \text{rank}[H_0(V;2)] = r$ (the number of regions into which G divides M).

Let G be a simple closed loop and M be an orientable closed surface. This gives the exact sequence

$$\cdots \to 0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{j_*} H_2(M,G;2) \xrightarrow{\delta_*} \mathbb{Z}/2\mathbb{Z} \xrightarrow{i_*} H_1(M) \to \cdots$$

where the map i_* determines the homology group $H_2(M,G;2)$. For if $i_* = 0$ then $H_2(M,G;2) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. If $i_* \neq 0$ then $H_2(M,G;2) \cong \mathbb{Z}/2\mathbb{Z}$. That is, $i_* = 0$ gives $r = 2$ and $i_* \neq 0$ gives $r = 1$. This leads to a proposition by P.J. Giblin.

Proposition: A simple closed loop separates an orientable surface if and only if the 1-cycle (mod 2) given by the loop is homologous to zero on the closed surface.

Now suppose M is a genus 1 torus and we want to divide M into two regions with some graph G. Then $H_2(M,G;2) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $H_1(M;2) \cong \mathbb{Z}/2\mathbb{Z}$. This gives an exact sequence

$$\cdots \to 0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{j_*} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\delta_*} H_1(G;2) \xrightarrow{i_*} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \cdots$$

and from this sequence we can find an upper bound for how "big" G can be by limiting how many cycles are in G. In this case if G separates M into two regions then $\text{rank}[H_1(G;2)] \leq 3$.

References:
J. R. Munkres, *Elements of Algebraic Topology*, Perseus Publishing, 1984.
P. J. Giblin, *Graphs, Surfaces and Homology*, Chapman and Hall Ltd., 1977.

Stable Feedback Models for Electrosensory Filtering

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Mar 5, 2pm

The behavior of neurons is measured by their membrane voltage, and the biologically relevant behavior is when that voltage undergoes a 'spike' in the positive direction. These spikes are complex to model directly, so rather than attempt to do so, this model of neural adaptation views every spike as functionally identical with the only important aspect being what the time of occurrence is. By heavily abstracting the system and discretizing the time, we can find stable points of the system and show that the system is feedback controllable.

The primary idea is to look at the changes of the weights of connections between cells in the system over time as governed by a function of how soon after an input spike, the output spike occurs. By treating the time of the output spike as a random variable depending on a threshold θ and the voltage change caused by the input spike, we can condense the entire complicated description to the vector equation:

$$\mathbf{V}(t+1) = \mathbf{V}_n(t) + \alpha\mathbf{1} - M\mathbf{f}(\mathbf{V}(\mathbf{t}).$$

This shows the membrane voltage V for every cell at time $t+1$ as a vector function of the previous cycle's state. This reduces the problem to one of looking at the matrix M. This in turn allows us to show the existence of an equilibrium (depending on parameters α and β connected to the learning law that changes the synaptic weights), as well as show the conditions for local stability of the equilibria. A large part of this analysis follows from using the properties of the circulant matrix M.

By shifting the equilibrium to the origin, and scaling the nonlinearity of the system, we can show the conditions under which the equilibrium is globally asymptotically stable.

By writing f as a function of θ and V, we can look at θ as a variable whose value is adjusted via feedback from the other cells. Following similar techniques to the above, we find that only one eigenvalue can be changed with this type of feedback, but that the feedback law does allow us to change (improve) the rate of convergence.

If we allow θ to change within a cycle, then it too becomes a vector Θ whose components are θ_i being the value at x_i. This allows us to write the system as:

$$z(t+1) = (I - M)z(t) + Mu(t),$$

where $z(t) = V - \hat{V}$ and $u(t) = \Theta - \hat{\Theta}$ where $\hat{V}$ and $\hat{\Theta}$ are the equilibrium points for the system. This system is asymptotically controllable with controls in $\mathcal{U}$ (a bounded subset of $\mathbb{R}^N$ containing zero in its interior) if and only if the eigenvalues of $I - M$ have magnitude less than or equal to 1.

Finally, if we can close the feedback loop with a second sigmoid function (the first being arising from the probability distribution of the spike time), we can give conditions on the existence of either one globally asymptotically stable, or three locally stable equilibria.

Reference: G. Lafferriere, P.D. Roberts, *Stable Feedback Models for Electrosensory Filtering in Mormyrid Fish*, Proc. of the 46th Conf. on Decision and Control (2007) pp. 609-614.

Quotients of Surfaces by Cyclic Groups

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Let X be an orientable surface of any genus, and consider a finite group G of orientation preserving symmetries of X. We denote the set of the orbits under G of the elements of X by X/G, and we call this the quotient space of X by G. Suppose $m = |G|$. We are interested in the case where X/G satisfies two conditions:

- X/G has genus 0, and
- X/G contains exactly three orbits of size less than m.

When this happens, we say that G is a quasiplatonic group, X is a quasiplatonic surface, and that G is acting on X quasiplatonically.

A tool used to classify such group actions is called the signature. Suppose $x_1, x_2, x_3 \in X$ are points lying in distinct orbits of size less than m. The stabilizer of $x \in X$ is the subgroup of G whose elements fix x. Let $n_1 = |\operatorname{Stab}(x_1)|, n_2 = |\operatorname{Stab}(x_2)|$, and $n_3 = |\operatorname{Stab}(x_3)|$, and without loss of generality assume $n_1 \leqslant n_2 \leqslant n_3$. Then, the signature of G acting on X is the triple (n_1, n_2, n_3). We call the n_i the periods of the signature.

Quasiplatonic groups acting on surfaces of genus 0 or 1 are well known. A complete classification of Abelian quasiplatonic groups has been previously discovered. In this talk, we are only concerned with the classification of cyclic quasiplatonic groups for surfaces of arbitrary genus, which follows below:

Theorem: *Fix a signature (n_1, n_2, n_3) and let $M = \operatorname{lcm}(n_1, n_2, n_3)$. There is a quasiplatonic surface X with quasiplatonic cyclic group G and signature (n_1, n_2, n_3) if and only if the following conditions are met:*
- $|G| = M = \operatorname{lcm}(n_1, n_2) = \operatorname{lcm}(n_1, n_3) = \operatorname{lcm}(n_2, n_3)$ — where lcm *means lowest common multiple;*
- if M is even, then exactly 2 of the periods n_i must be divisible by the maximum power of 2 that divides $|G|$.

To summarize this theorem, for a cyclic group of order m there are only three forms that a signature may take, namely (m, m, m), (n, m, m) and (n_1, n_2, n_3) where n, n_1, n_2, and n_3 all divide m (and satisfy all the other conditions listed above).

This takes us to the problem addressed in this talk. It is not currently known how many topologically inequivalent ways that an Abelian quasiplatonic group can act upon a surface. There is a counting tool previously developed that we can use to answer this question for cyclic groups acting on quasiplatonic surfaces of genera greater than or equal to 2. We used this to develop formulae where given a group G and a signature, we can count the number of topologically inequivalent ways in which G can act with that signature.