Listing of the Summaries of the

Non-Linear Systems Group Seminar Winter and Spring 2006

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Parsimonious Cleavage and Homology I

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January 20

A *surface* is a compact 2-dimensional manifold without a boundary. There is a well known classification of surfaces based on genus (number of holes) and orientability (due to H. R. Brahana in 1921). Homology theory is an algebraic tool to rigorously study those concepts and stems from a later date than the classification itself.

There are several different versions of homology. The simplest and most intuitive version is known as simplicial homology $\mod 2$, and is defined for *simplicial complexes.* Informally, a simplicial complex is a collection of vertices in, say, $\mathbb{R}^n$, edges between the vertices, and faces bounded by edges, and so on for higher dimensions, subject to the condition that any two vertices determine at most one edge and three vertices determine at most one face (and so on). More formally, let C be a collection of finite subsets (of points) of some finite reference set R. Then C is a simplicial complex if for any set $A \in C$, every subset of A also belongs to C. An element of C containing exactly $p+1$ points from R is called a *p–simplex.* Each simplicial complex is also considered a topological space. (It is common to replace simplicial complexes by simpler CW complexes, essentially by replacing simplices by a smaller number of cells.)

Take one basis element for each p-simplex and let V_p the vector space generated by the linear combinations with coefficients in $\mathbb{Z}/2$ (no relations between the generators are allowed). The elements of V_p are essentially unions of p-simplexes and are called p-chains. V_0, for example, is the space generated by all the 1-element sets belonging to C; i.e., the vertices of C. The *boundary map* $\partial_p : V_p \to V_{p-1}$ sends a p-simplex to its boundary, that is: the sum of its $(p-1)$-dimensional faces.

For a given "nice" space S there may be many possible simplicial complexes realizing it, called *triangulations*, but the quantity known as the *Euler characteristic*, and defined as

$$\chi(S) = \sum (-1)^i \dim V_i \quad ,$$

depends only on the space S and not any particular triangulation. For a triangulation of a compact surface of genus g this of course corresponds to the well-known formulas

$$\chi(S) = F - E + V = 2(1-g) \quad \text{(orientable)}; \quad \chi(S) = F - E + V = 2 - g \quad (\text{non} - \text{orientable}),$$

where g is the numbers of handles (orientable) or the number of cross-caps (non-orientable).

One can show that homology is invariant under subdivision of the simplices. Thus Tietze formulated around 1905 his "Hauptvermutung" (or Main Conjecture) according to which any two triangulations of homeomorphic manifolds have a common refinement. While this is true in many cases, in 1961 J. Milnor showed this to be false in general. Nonetheless, in the late 1920's it was established without using the Hauptvermutung that simplicial homology $\mod 2$ and simplicial homology with integer coefficients are true topological invariants.

Parsimonious Cleavage and Homology II

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January 27

The construction of the homology groups depends on the coefficients used in the p-chains. The coefficients $\mathbb{Z}_2 = \mathbb{Z}/2$ are convenient, since they have the advantage of ignoring orientation and being easier to compute. The Universal Coefficient Theorem says that from a homology formulated with coefficients in $\mathbb{Z}$ (or integral homology) the homologies with other coefficients can be computed.

An example of this principle can be found when calculating the homology of the real projective plane represented by the unit disk with antipodal points on the boundary identified. As a CW-complex, this space can be considered as having a single 2-cell, the face f. Denote the "northern" half-circle bounding this disk by e (the edge). Then, by identification, the boundary of f equals *twice* the edge e. Thus $2e$ is homologous to zero, expressing that this surface is not orientable. It turns out that the integral homology for non-orientable surfaces has a finite subgroup (the torsion) of order 2. Using rational homology (with coefficients in $\mathbb{Q}$) one loses this information. Here is a table of homology groups of the projective plane, the Klein Bottle, and the torus:

Projective Plane:

Coeff.	H_0	H_1	H_2
$\mathbb{Z}$:	$\mathbb{Z}$	$\mathbb{Z}_2$	0
$\mathbb{Q}$:	$\mathbb{Q}$	0	0
$\mathbb{Z}_2$:	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Klein Bottle:

Coeff.	H_0	H_1	H_2
$\mathbb{Z}$:	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	0
$\mathbb{Q}$:	$\mathbb{Q}$	$\mathbb{Q}$	0
$\mathbb{Z}_2$:	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$

Torus:

Coeff.	H_0	H_1	H_2
$\mathbb{Z}$:	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$
$\mathbb{Q}$:	$\mathbb{Q}$	$\mathbb{Q} \oplus \mathbb{Q}$	$\mathbb{Q}$
$\mathbb{Z}_2$:	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$

Note that $\mathbb{Z}_2$ homology does not distinguish the Klein Bottle from the torus. Integral homology is a fine enough invariant to distinguish all the compact, connected 2-manifolds from one another.

Relative homology is a technique used to relate the homology of a simplicial complex X and the homology of a subcomplex Y of X. The relative homology groups are defined as the homology of the chain-complex $C_m(X,Y) = C_m(X)/C_m(Y)$. Since Y is a subcomplex of X, the inclusion map $i : C_m(Y) \to C_m(X)$ induces a map between homology groups $i_* : H_m(Y) \to H_m(X)$. Likewise, the natural projection $p : C_m(X) \to C_m(X)/C_m(Y)$ induces another map $p_* : H_m(X) \to H_m(X,Y)$. Finally, the boundary operator $\partial_m : C_m(X) \to C_{m-1}(X)$ also induces a map between homology groups of adjacent dimensions $\partial_{*m} : H_m(X,Y) \to H_{m-1}(Y)$. Putting these operators together yields the *long exact sequence*

$$... \xrightarrow{\partial_{*m+1}} H_m(Y) \xrightarrow{i_*} H_m(X) \xrightarrow{p_*} H_m(X,Y) \xrightarrow{\partial_{*m}} H_{m-1}(Y)...$$

Since the sequence is exact, by definition the kernel of each map is equal to the image of the preceding map. However, it is important to note that i_* is not necessarily an epimorphism and p_* is not necessarily a monomorphism. Besides being a useful tool, relative homology is used for characterizing homology by axioms (the Eilenberg - Steenrod axioms).

On Parsimonious Cleavages

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February 3

Suppose that M is a compact surface. A subset $L \subset M$ is called a *minimally separating set* (or a parsimonious cleavage) if $M - L$ has exactly two components, and if for any proper $\tilde{L} \subset L$, $M - \tilde{L}$ is connected. If M is the unit sphere, the Jordan Curve Theorem states that up to homeomorphism, there is exactly one minimally separating set: the circle. However, on compact connected surfaces of higher genus the classification of minimally separating sets (up to homemorphisms of the surface M) is an open problem. Assume that M is triangulable and that L is a subcomplex of that triangulation.

Now let M_- and M_+ denote the closure of the two components of $M - L$, and $N(L)$ a so-called regular neighborhood of L (a collar around L, cut out of the manifold M). Then

$$\chi(M) = \chi(M_+) + \chi(M_-) + \chi(N(L)).$$

Since the Euler characteristic is invariant under homotopy, $\chi(N(L)) = \chi(L) = V - E$, where E and V denote the number of edges and vertices of L. Suppose that g is the genus of M, and that $g_\pm$ are the genera of $M_\pm$. For the sake of simplicity, assume furthermore that M is orientable. After we cut $N(L)$ out of M, we obtain two surfaces $M_\pm$ with (respectively) $n_\pm$ holes. After we sew n_+ disks into M_+ and n_- disks into M_-, we obtain surfaces of Euler characteristic $2-2g_\pm$ (respectively). Thus we see that $\chi(M_\pm) = 2-2g-n_\pm$. Since $\chi(M) = 2-2g$, the main equation boils down to

$$(2+2g) + (V-E) = 2\sum g_\pm + \sum n_\pm \quad .$$

Now note that $n_\pm \geq 1$ and $g_\pm \geq 0$, and use this to classify the possibilities.

There may be multiple possibilities for the regular neighborhood of L, depending on how it is embedded in M. For example, if M is the torus, and L is the graph with two edges and one vertex, i.e. the figure "8", then $N(L)$ is a thickened figure "8". Now one can obtain a torus by sewing either 1 disk — if one considers the two cycles in L as homologically distinct in M — or a disk and twice punctured torus — if one considers the two cycles "parallel" in M. Thus, depending on the embedding of L in M, $N(L)$ may be homemorphic (in M) to either a "pair of pants" (three-punctured disk) or a once-punctured torus. A conjecture enumerates the possible regular neighborhoods:

Conjecture 0.1 *Suppose that L is a graph. Then for any surface F such that $\partial F \neq \emptyset$ and $\chi(F) = \chi(L)$, there exists a regular neighborhood $N(L)$ such that F is homeomorphic to $N(L)$.*

(NOTE: This was an ad-lib performance by the speaker based on suggestions from the audience, notably: the formulation of the problem (JJPV) and the conjecture (Steve Bleiler). The two main equations had been independently suggested by S. Anisov in a personal communication to JJPV. The conjecture turns out to be false as stated.)

Parsimonious Cleavage and Homology III

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February 10

Given a compact triangulable complex S, let C_p denote the p-th chain group of S with coefficients from a field F. The Euler characteristic $\chi(S) = \sum_{i=1}^{P}(-1)^i \dim C_i$ of S is an invariant and is independent of the triangulation of S. To see this, let $d_p : C_p \to C_{p-1}$ be the boundary operator, and define $Z_p = \ker d_p$ and $B_p = \operatorname{im} d_{p+1}$. These quantities are related by a short exact sequence:

$$0 \to Z_p \hookrightarrow C_p \to B_{p-1} \to 0.$$

Since the coefficients form a field (which allows division, as opposed to, say, $\mathbb{Z}$), this sequence splits, that is: $C_p \cong Z_p \oplus B_{p-1}$ and $Z_p \cong B_p \oplus H_p$, where H_p is the p-th homology group of S with coefficients in the field). Therefore,

$$\dim C_p = \dim Z_p + \dim B_{p-1} \qquad \text{and} \qquad \dim Z_p = \dim B_p + \dim H_p$$

Combining these two equations yields $\dim C_p - \dim H_p = \dim B_p + \dim B_{p-1}$. We now calculate:

$$\sum_{i=1}^{P}(-1)^i(\dim C_i - \dim H_i) = \sum_{i=1}^{P}(-1)^i(\dim B_i + \dim B_{i-1}) = 0.$$

Therefore,

$$\chi(S) = \sum_{i=1}^{P} (-1)^i H_i(S).$$

Since the homology groups $H_i(S)$ are invariant under different triangulations of S, this formula shows that $\chi(S)$ shares this invariance. Note that in the special case where $F = \mathbb{Z}_2$, the $c_k = \dim H_k(C)$ are referred to as the *connectivity numbers.*

One of the main tools used in homology is the *Mayer-Vietoris sequence*. Suppose that K is a simplicial complex, and that K_1 and K_2 are subcomplexes of K such that $K = K_1 \cup K_2$ and $K_0 = K_1 \cap K_2 \neq \emptyset$. The inclusion maps $j_n : K_0 \hookrightarrow K_n,\ i_n : K_n \hookrightarrow K,\ n \in \{1, 2\}$ induce the following relationship on the p-chain groups:

$$\begin{array}{ccc} C_p(K_1) & \xrightarrow{(i_1)_*} & C_p(K) \\ {\scriptstyle (j_1)_*}\uparrow & & \uparrow{\scriptstyle (i_2)_*} \\ C_p(K_0) & \xrightarrow{(j_2)_*} & C_p(K_2) \end{array}$$

Now, define the *chain complex* $\mathcal{C}(K)$ as the set of all p-chain groups $C(K)$ together with all boundary operators on those p-chain groups ($\mathcal{C}(K) = \{(C_p, d_p)\}$). A *chain map* between two chain complexes $C(K)$ and $C(S)$ is a collection of maps between the respective p-chain groups. Following these definitions, the diagram above induces a short exact sequence of chain complexes

$$0 \to \mathcal{C}(K_0) \xrightarrow{((J_1)_*, -(J_2)_*)} \mathcal{C}(K_1) \oplus \mathcal{C}(K_2) \xrightarrow{(I_1)_* + (I_2)_*} \mathcal{C}(K) \to 0$$

where $(J_n)_*$ is induced by $(j_n)_*$, and so on. This short exact sequence of chain complexes can be used to construct the following long exact sequence of homology groups, which is known as the Mayer-Vietoris sequence:

$$\begin{aligned} ... \to H_{p+1}(K) &\xrightarrow{\delta} H_p(K_0) \to H_p(K_1) \oplus H_p(K_2) \\ \to H_p(K) &\xrightarrow{\delta} H_{p-1}(K_0) \to ... \end{aligned}$$

The δ operators in this sequence are known as *connecting homomorphisms*; they are not the same as the boundary operators d_p of the p-chain groups, and are often quite complicated to compute.

Induction on Parsimonious Cleavages

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February 17

Suppose that M is a compact surface. A minimally separating set L of M was defined in the lecture on February 3, and the problem of classifying all minimally separating sets of M was raised. Ontaneda proposed an interesting possibility for approaching this problem: induction on the number of edges of a non-separating graph L' embedded in M. The inductive algorithm is as follows: suppose that L' is a non-separating graph embedded in M. Define $L' \cup e$ to be L' with two (not necessarily distinct) vertices of L' with a new edge e, such that e is not homotopic to a point. If $L' \cup e$ is separating, it must be minimally separating, since neither L' nor e is separating. If $L' \cup e$ is not separating,

simply repeat the edge addition on $L' \cup e$. This inductive algorithm will eventually terminate, since there is only a finite number of ways to add edges that are not homotopic to a point.
(NOTE: This was an ad-lib performance by the speaker based on suggestions from the audience.)

Mandelgraphs I

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February 24

An undirected graph G is a finite collection $V(G)$ of vertices together with a symmetric subset of $V \times V$ coding the edges of the graph. A graph isomorphism between two graphs G_1 and G_2 is a bijection $f : V(G_1) \to V(G_2)$ that preserves the edge relation (and its inverse does too). The minimal number of colors needed to color the vertices such that no two adjacent vertices have the same color (the vertex chromatic number) is an isomorphism invariant as is the edge chromatic number. In what follows 'loops' are disallowed (edges whose endpoint equals the initial point. Furthermore, adjacency of two vertices v_1 and v_2 is expressed by $v_1 \sim v_2$.

An *independent set* in a graph G is a subset of the vertices such that no two of them are adjacent. The cardinality of the largest independent set is called its *independence number* $\alpha(G)$. The *independence polynomial*

$$i_G(x) = \sum_{k=0}^{\alpha(G)} i_k x^k$$

encodes the number i_k of independent sets of cardinality k. Contrary to convention the empty set is not counted as an independent set, so $i_0 = 0$ for any graph. Additionally, because of the no-loop condition one-element vertex sets are always independent. So i_1 equals the cardinality of the vertex set. The *lexicographic product* $G[H]$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$, where the vertices $\{g, h\}$ and $\{g', h'\}$ are adjacent if $g \sim g'$ or if $g = g'$ and $h \sim h'$.

The following result connects these concepts. If G and H are graphs, then the independence polynomial of $G[H]$ is $i_{G[H]}(x) = i_G(i_H(x))$. This relationship between lexicographic products and independence polynomials means that taking a lexicographic power G^k corresponds to iterating the independence polynomial $f_G^k(x) = f_G \circ f_G \circ ... \circ f_G(x)$ (k times).

In turn this can be related to dynamics in the complex variable. Think of $z \to f(z)$ as a (conformal) dynamical system in the complex plane $\mathbb{C}$. Denote by $K(f)$ the set of points z in $\mathbb{C}$ such that $f^k(z)$ does *not* tend to infinity. For polynomials of degree greater than 1, this is clearly a bounded set. Its boundary is called the Julia Set $J(f)$.

Mandelgraphs II

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March 3

The Julia Set $J(f)$ of $z \to f(z)$ either is connected (and bounds a set $K(f)$ of positive measure in the complex plane), or else it is totally disconnected (and so has empty interior). In the first case $K(f)$ contains an attractor for the dynamics and the complement of $K(f)$ is attracted to ∞. In the second case $J(f)$ and all points in its complement iterate to ∞. Either way it is a repelling set (nearby orbits iterate away from the set) with periodic orbits forming a dense subset. One can distinguish between these cases by iterating the critical points (z_0 such $f'(z_0) = 0$). $J(f)$ is connected if and only if all orbits $\{f^n(z_0)\}_n$ are bounded.

This gives us two methods to calculate the Julia Set of an independence polynomial $f(z) = i_G(z)$. The naive one is to calculate high-periodic orbits — the roots of $f^n(z) - z = 0$ — and eliminate orbits that are attracting. The second one is to iterate backwards *any* point z_0 *not* in the attractor, so that it will be attracted to the Julia Set. Thus the set $R_n(z_0) \equiv i_G^{-n}(z_0)$ of points that map to z_0 under $i_G^n(z)$ will approximate $J(f)$.

In our case we can even do slightly better: since $i_0 = 0$, $i_G(z)$ has a factor z. Thus $i_G(0) = 0$. However, 0 is *not* an attracting fixed point since $i'_G(0) = i_1$ which equals the number of vertices in the graph (and which is greater than 1). Thus $0 \in J(i_G)$. Because the Julia Set is backward invariant, this implies that the set $R_n(0)$ not only approximates $J(i_G)$ but that in addition $R_n(0) \subseteq J(i_G)$.

The central question in all of this is whether there is any non-trivial natural relation between the Julia Set of a polynomial and the graph it was derived from. The answer appears to be negative. In particular one can show that there are connected graphs whose Julia Set is connected, and others whose Julia Set is disconnected. The same holds for graphs that are not connected. If G' is obtained from G by subdividing a single edge, there also appears to be no relation between $J(i_G)$ and $J(i_{G'})$. However there are some positive results, of which we will mention a few in what follows.

Recall that if G^k denotes the k-fold lexicographical product of the graph G, we have: $i_{G^k}(z) = i_G^k(z)$, so that $J(G)$ (the Julia Set associated with G) is the same as $J(G^k)$. Slightly more exciting is the fact that one can show that

$$J(K_p[G]) = p \cdot J(G[K_p]) \quad ,$$

where $K_p[G]$ denotes the lexicographical product of the complete graph of p vertices (K_p) with a graph G and $p\cdot$ denotes multiplication (rescaling) in the complex plane. Together withe observation that a Julia Set of a polynomial is bounded, this implies (just set $G = K_p$, $p > 1$) that the Julia Set of K_p must be point. This in turn says that $i_{K_p}(z) = nz$ for some $n > 1$. It is easily checked from the definition of the independence polynomial (February 24) that, in fact, $i_{K_p}(z) = pz$.

Graphs with no independent sets of order 3 or higher give rise to quadratic polynomials $i_G(z) = mz^2 + nz$, where m is the number of edges needed to make the graph complete ("non-edges"), and n is the number of vertices. These polynomials can be conjugated by a Möbius transform to $z^2 - n(2-n)/4$. Thus their Julia Sets are also conjugates.

Note: These two talks comprise the speaker's Master's Thesis, based on *The Independence Fractal of a Graph*, by J. I. Brown, C. A. Hickman, and R. J. Nowakowski, J. of Comb. Th. Series B 87 (2003) 209-230.

Parsimonious Cleavages in Orientable Surfaces

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March 10

Given a closed orientable surface M we want to classify all *minimally separating* subsets $L \subset M$ up to some sort of equivalence. Three possible equivalences are homotopy in M, homeomorphism of M, and graph isomorphism of L. As before we assume that M is triangulable and that L is a 1-dimensional subcomplex of that triangulation. Thus it seems easiest to classify L up to graph homeomorphism.

Let us examine the local characteristics of L. If e is an edge of L, then e must bound both components of $M - L$ (otherwise L is not minimally separating). It follows that if v is a vertex of L, it must have an even number of incident edges. One can show that L must contain at least one cycle. Either C is a connected component of L or there exists $1 \le i \le n$ such that $\deg(v_i) > 2$. Suppose that C has $v_1, ..v_n$ as vertices of even degree greater than 2. One is allowed to perform certain 'moves' that change L into a different set L' which however is still minimally separating. It turns out that these 'moves' allow us to reduce the number of incident edges in the vertices of C to 2. This implies that a new minimally separating set L' in which C is topologically a circle and is an isolated component of L'. Assume without loss of generality that $L' \neq C$.

Now cut M along C. We obtain a surface with two holes. Fill these holes with disks to obtain another closed surface M'. Furthermore the set $L' - C$ is minimally separating in M'. It is easy to that $\chi(M') = \chi(M) + 2$, where χ is the Euler characteristic. This implies that the genus $g(M')$ of M' equals $g(M) - 1$.

This argument shows that any (simplicial) minimally separating set in a surface of genus g can be obtained from a minimally separating set in a surface of genus $g - 1$. The procedure is as follows. Start with a minimally separating set L' in M' of genus $g - 1$. Glue a handlebar into M' that connects the two components of M' and call the new surface M. Draw a cycle C on the handlebar so that $L' \cup C$ separates M. Perform the reverse of the 'moves' alluded to earlier.

Here is an example. Since the surface of genus 0 the sphere has only one minimally separating set, it is fairly easy to generate the minimally separating sets of the surface of genus 1, the torus. There are five, and they are the circle, two disjoint circles, the figure-eight (a circle glued to another circle in one place), the bouquet of three circles (a circle glued to another circle twice in the same place), and the graph with two vertices and four edges between them (a circle glued to another circle in two different places).

Stochastic Loewner Evolution

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April 7

Motivated by the possible solution of the Bierberbach Conjecture, Karel Löwner (a Czech who later adopted the amercanized name Charles Loewner) in 1923 was led to study the singularities of conformal maps by consideration of the differential equation

$$\frac{d}{dt} f_t(z) = \frac{2}{f_t(z) - \xi(t)} \qquad f_0(z) = z$$

where $z \in \mathbb{C}, t \in \mathbb{R}$ and $\xi(t)$ is a real function called the *forcing*. The evolution of $f_t(z)$ in time t is called *Loewner Evolution*. With certain basic assumptions about the asymptotic behavior of f, this equation defines a family of conformal mappings of domains $\mathcal{D}_t \subset \mathcal{H}$ (the upper half plane) to $\mathcal{H}$ itself that map the boundary of $\mathcal{D}_t$ onto the real line. The domains $\mathcal{D}_t$ have the additional property that $\mathcal{D}_t \subset \mathcal{D}_s$ for $s < t$ so that as time evolves the domains $\mathcal{D}_t$ shrink.

For example if $\xi(t) \equiv 0$, we find an explicit solution $f_t(z) = \sqrt{z^2 + 4t}$. There is a branch point for this solution at $z = 2i\sqrt{t}$, so $\mathcal{D}_t$ is $\mathcal{H}$ with the segment from $z = 0$ to $z = 2i\sqrt{t}$ removed (the branch cut). It is evident in this example that as t evolves, the region of singularity traces out the curve $\gamma(t) = 2i\sqrt{t}$ in $\mathcal{H}$, fittingly called the *trace*. The situation may be shown to be similar for general forcing $\xi(t)$, and thus we may view Loewner evolution as taking a forcing function $\xi(t)$ to a trace $\gamma(t)$ in $\mathcal{H}$.

The function $\xi(t)$ is called β-Hölder continuous (with constant A) if $|\xi(t) - \xi(s)| \le A|t - s|^\beta$. We may then state some amazing results relating the forcing ξ to its trace γ:

- if ξ is smooth ($\beta = 1$), γ is an arc and intersects the real axis only in $\gamma(0)$.
- if in addition $\xi(t)$ is periodic, $\gamma(t)$ is self-similar

• if ξ is 1/2-Hölder with constant at least 4 (or if $\beta < 1/2$), γ is a self-intersecting curve.

In the late 90's, Oded Schramm considered a particular forcing function: $\xi(t) = \sqrt{\kappa}B(t)$ where $B(t)$ is normalized 1-dimensional Brownian motion. It is a classical result that $B(t)$ is 1/2-Hölder in the following sense: $<(B(t) - B(s))^2 >= |t-s|$. He and others discovered a remarkable and useful relationship between κ and the trace:

• if $0 \leq \kappa \leq 4$, $\gamma(t)$ is an arc.

• if $4 < \kappa < 8$, $\gamma(t)$ intersects itself (with probability 1).

• if $\kappa \geq 8$, $\gamma(t)$ fills a region of $\mathcal{H}$.

This particularly type of Loewner Evolution is called *Schramm-Loewner Evolution* or SLE (sometimes also called Stochastic Loewner Equation). SLE has proved a useful geometrical tool for physicists. For example, when $\kappa = 2$, SLE will transform a random walk in one dimension into a non-self intersecting random walk in two dimensions. When $\kappa = 6$, SLE is related to critical cluster boundaries for percolation in statistical mechanics. Finally, when $\kappa = 8$, SLE is related to space-filling curves winding along uniform spanning trees.

Note: The Riemann Mapping Theorem says that for any simply connected open domain there is a injective conformal map onto the open unit ball. The Bieberbach Conjecture: The n-th coefficient in the power series of an injective analytic function from the unit disk to any domain should be no greater than n. (The latter was finally proved by Louis de Branges in 1985.)

Non-Orientable Surfaces and Double Covers

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April 14

Let M_1, M_2 be (closed, compact, connected) surfaces. Define an operation on these surfaces, called the connected sum, as follows: let $D_i \subset M_i$ be embedded disks; remove these disks to form two surfaces M_1', M_2' with "holes". Form the new surface $M_1 \# M_2$ by gluing (identifying via homeomorphism) the boundaries ∂D_1 and ∂D_2. It may be verified that every surface is homeomorphic to exactly one of the following $g \geq 0$ or $k \geq 1$ and all possibilities occur:

• if M is orientable, there is a g such that M is homeomorphic to $T^2 \# \ldots \# T^2$ (g times).

• if M is non-orientable, there is a k such that M is homeomorphic to $P^2 \# \ldots \# P^2$ (k times).

The numbers g and k are the orientable genus and the non-orientable genus, respectively. The case $g = 0$ is taken by convention to be S^2.

On the other hand it is not true that connected sums of distinct manifolds are distinct. For example: $P^2 \# K$ (where K is the Klein bottle) is homeomorphic to $P^2 \# T^2$. But $K \approx P^2 \# P^2$ is not homeomorphic to the torus. (The connected sum operation does not have an inverse.)

For every surface we have that $H_0(M; \mathbb{Z}_2) \approx H_2(M; \mathbb{Z}_2) \approx \mathbb{Z}_2$. Each torus in the connected sum results in two generators in H_1 and each projective plane results in one. Using that $\chi(M) = \sum (-1)^i \dim H_i$, we get that

$$H_1(M, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2^{2g} \text{ for orientable} \\ \mathbb{Z}_2^{k} \text{ for non-orientable} \end{cases} \quad \text{so that} \quad \chi(M) = \begin{cases} 2-2g \text{ for orientable} \\ 2-k \text{ for non-orientable} \end{cases}$$

Every non-orientable surface M has associated to it an orientable surface $\widetilde{M}$ called the *orientable double cover*. This surface is unique up to homeomorphism, and may be constructed via its fundamental group as follows: every "loop" in $\pi_1(M)$ is either orientable or non-orientable, a property unchanged under homotopy. Replace each non-orientable loop downstairs with half of a loop upstairs in such a way that going around the non-orientable loop twice downstairs amounts to traversing both segments of the loop upstairs. After this fundamental group is constructed, build the surface corresponding to it. It is also possible to recognize the orientable double cover as the covering space corresponding to the subgroup of π_1 of index two that is the kernel of the homomorphism $\pi_1(M) \to \{\pm 1\}$ that sends non-orientable (orientable) to -1 (1), respectively.

Consider a triangulation K of M. This triangulation can be lifted to a triangulation $\widetilde{K}$ of the double cover $\widetilde{M}$. Every simplex "downstairs" (in M) lifts to exactly two simplices "upstairs" (in $\widetilde{M}$). So (with the definition of the characteristic from January 20) $\chi(\widetilde{M}) = 2\chi(M)$.

An Odd Dynamical System

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April 21

The upper half plane $\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 | y > 0\}$ together with the Riemannian metric $g = \frac{dx^2+dy^2}{y^2}$ is a model for hyperbolic space known as the *Poincare upper half plane.* $SL(2,\mathbb{Z})$ is the set of 2×2 integer matrices with determinant 1. It acts on $\mathbb{H}^2$ by associating a fractional linear transformation f_A to a matrix A:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , \quad f_A : z \mapsto \frac{az+b}{cz+d} \quad .$$

The resulting group of transformations (identifying the actions of any matrix A and its negative $-A$) is called the modular group. Their composition satisfies $f_A \circ f_B = f_{AB}$. It is then easy to see that a transformation f_A maps $\mathbb{H}^2$ to itself, has inverse $f_{A^{-1}}$, and preserves the metric (is an isometry) of $\mathbb{H}^2$. The quotient manifold $M \equiv \mathbb{H}^2/SL(2,\mathbb{Z})$ consists of a fundamental domain for the groups with an identification of its sides. This fundamental region is usually taken to be the region $\{z \in \mathbb{H}^2 \,|\, |z| < 1/2, |z| > 1\}$. The projection $\pi : \mathbb{H}^2 \to \mathbb{H}^2/SL(2,\mathbb{Z})$ is a local isometry and maps geodesics to geodesics. The dynamical system is the geodesic flow in the quotient manifold M.

It is easily checked that $\gamma_0(t) = ie^t$ is a geodesic. All other geodesics in $\mathbb{H}^2$ can be obtained by applying the collection of all isometries of $\mathbb{H}^2$ to $\gamma(t)$. Thus the set of geodesics in $\mathbb{H}^2$ is given by

$$\{\, \gamma(t) = f_M(\{\gamma_0(t)\}) \mid M \in SL(2,\mathbb{R}) \,\} \quad .$$

These geodesics are either vertical lines or semi-circles intersecting the boundary $\Im z = 0$ orthogonally. Their projections, the geodesics on M, are closed, or they fill M, or one end fills M while the other end spirals into infinity. Consider the family of (unit speed) geodesics emanating from the point $z_0 \equiv (x_0, 0)$. The distance $d(\gamma_1(t), \gamma_2(t))$ between two geodesics with slightly different tangent vector will increase exponentially (a hallmark of chaotic behavior). Nonetheless, the collection of curves (called *horocyles*) orthogonal to this family of geodesics to a family of geodesics maintain a constant distance from each other. (It can be shown that the horocycles themselves are circles whose "southpole" is located in z_0 and that there is an isometry, namely circle-inversion, which maps them to parallel horizontal lines.)

If V is a vector field on a manifold M, then for a given $f : M \to \mathbb{R}$, then (V, M, f) is *observable* if for any distinct integral curves γ_1, γ_2 of V, $f|_{\gamma_1} \neq f|_{\gamma_2}$. Informally, f distinguishes every two distinct integral curves of V. The pair (V, M) is said to be *universally observable* (another hallmark of chaotic behavior) if for every non-constant $f : M \to \mathbb{R}$, (V, M, f) is observable. It was proved by Doug McMahon that the family of geodesics of $\mathbb{H}^2/SL(2,\mathbb{Z})$ is universally observable. The behavior of this system is in general not very well understood. For example, the Laplacian of the quotient $\mathbb{H}^2/SL(2,\mathbb{Z})$ has two types of eigenvalues: discrete and continuous. The discrete ones are still quite mysterious, since unlike in other similar situations the lengths of closed geodesics have not yet been found to be related directly to the discrete spectrum.

Unconstrained and Constrained Aging in Rods

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April 28

Consider a homogeneous rod of a certain material, say plastic or metal. We wish to study the changes in the material when the rod is subjected to forces of the following types: *unconstrained aging*, in which the rod is simply allowed to grow old; *stress relaxation*, in which the rod is deformed quickly to a certain length, held there for a given amount of time, and released; and *creep*, in which the rod is slowly pulled with a constant force. In practice empirical relations between the strain and the stress (force per unit cross-sectional area) are often used for modeling. We present here an outline of a dynamical system for modeling these forces.

The changing metric properties of the rod over time are described by specifying a metric on the rod and its evolution in time. Much as one might make even marks on the rod and then measure these marks after the experiment to detect changes. This metric is given in the ordered basis induced by cylindrical coordinates (t, R, z, θ) via

$$\begin{pmatrix} s^2 & 0 & 0 & 0 \\ 0 & \lambda_v^2\lambda_d^{-1} & 0 & 0 \\ 0 & 0 & \lambda_v^2\lambda_d^2 & \\ 0 & 0 & 0 & R^2\lambda_v^2\lambda_d^{-1} \end{pmatrix}$$

where λ_v is related to uniform stretching??? and λ_d is related to constant volume expansion??? and s is rescaled time. Let $x = \log \lambda_v$ and $y = \log \lambda_d$.

Given certain physical considerations, one can define an action

$$A(x, y, s) = \int_0^T [F(x, y, s) + s\chi(s^{-1}x_t, s^{-1}y_t)]dt \qquad x(t), y(t), s(t) \in \mathbb{R}$$

where F is a function related to the structural strength (cohesiveness) of the material, the work done by the load and the strain energy and χ is related to the dissipative potential. Interpreted as a classical mechanical system, F is like a potential, and χ plays the role of kinetic energy. (With a little work this system can also be written as a two-degree-of-freedom Hamiltonian system.) The problem now is to solve the Euler-Lagrange equations of the action A.

With some simplifying assumptions one can do a qualitative analysis of this problem. In the case of unconstrained aging, this reveals that trajectories of solutions in the s, x plane begin at $s(0) > 0$ on the s-axis and progress in the negative x direction and positive s direction, tending at infinity toward the curve $F_s = 0$, not exceeding the maximum s value s^∞. In the case of stress relaxation, we have a qualitatively similar dynamical system involving now only the s and y variables: solutions begin at $s(0)$ on the s-axis and progress in the positive y direction and positive s direction, tending toward the curve $F_s = 0$ at infinity. For creep, there is a dramatic change; y values for solution curves now become unbounded in finite time, corresponding to the fact that the material under consideration will break after finite time.

Minimal Separating Sets in Non-Orientable Surfaces

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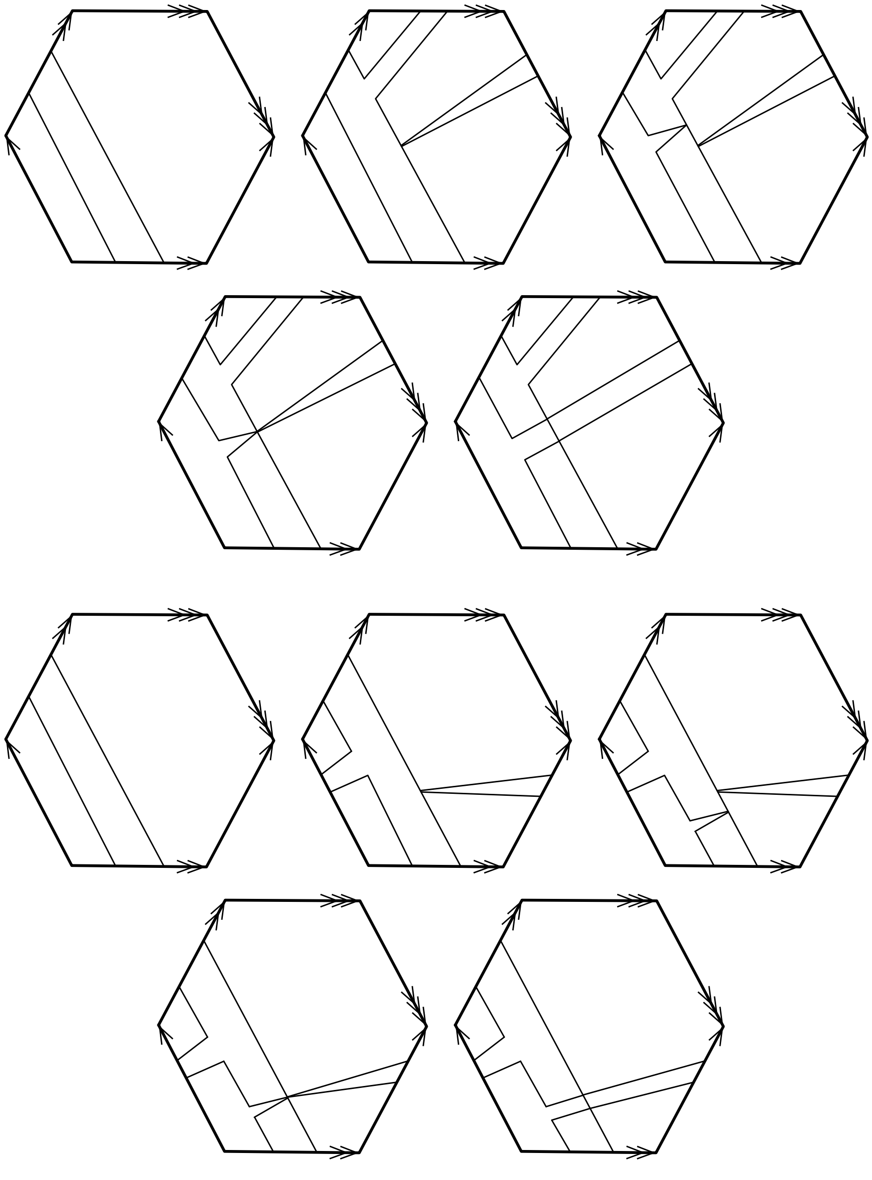
April 28

Some minimally separating sets in $P\#P\#P$, the non-orientable surface of (non-orientable) genus 3. The moves are also illustrated. Some of the graphs may be graph isomorphic, but there is no homeomorphism that also preserves the surface.



The Octonion Projective 2-Space.

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April 28

It is commonly known that one may construct topological real projective space $\mathbb{R}P^n$ by identifying antipodal points in the sphere S^n. Because antipodal identification is geometric, it is easy to miss the underlying algebraic process, which becomes more evident in the construction of complex projective space. $\mathbb{C}P^n$ is the set of all equivalence classes of unit length $(n+1)$-tuples of complex numbers where two vectors are equivalent if they are unit length scalar multiples of one another:

$$[z_1, \ldots, z_{n+1}] \sim [w_1, \ldots, w_{n+1}] \text{ if } (z_1, \ldots, z_{n+1}) = \lambda(w_1, \ldots, w_{n+1}) \text{ where } \lambda \in \mathbb{C} \text{ and } |\lambda| = 1.$$

In the real case, the only length 1 scalars are 1 and -1, and thus the equivalence relation boils down to antipodal identification. Similarly, we may construct $\mathbb{H}P^n$ through equivalence classes of quaternion vectors under multiplication by norm 1 quaternions.

These constructions yield fiber bundles *fiber* $\rightarrow$ *total space* $\rightarrow$ *base space*, all locally trivializable (that is: the preimage (in the total space) of a small enough neighborhood of any point in the base space is a direct product of the base-space and the fiber):

$$\begin{aligned}
&S^0 \rightarrow S^n \rightarrow \mathbb{R}P^n \text{ in the real case with } S^n \cong \mathbb{R}P^n \times S^0\\
&S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{C}P^n \text{ in the complex case with } S^{2n+1} \cong \mathbb{C}P^n \times S^1 \text{ and}\\
&S^3 \rightarrow S^{4n+3} \rightarrow \mathbb{H}P^n \text{ in the quaternion case with } S^{4n+3} \cong \mathbb{H}P^n \times S^3.
\end{aligned}$$

This notation means that, for example, $S^n \rightarrow \mathbb{R}P^n$ is a covering map where each inverse image (fiber) looks like S^0. The "local trivialization" of S^n through this map looks like two copies of $\mathbb{R}^n$, hence the pairing with $S^0 = \{-1, 1\}$.

It is natural to attempt the same construction for the octonions $\mathbb{O}$. This will fail however as the octonions are not associative and multiplication by unit length octonions will not be a group action on octonion $(n+1)$-tuples and therefore will not define an equivalence relation: If α and β are unit octonions, and $x \in S^{8\cdot 2+7}$, then it is not always true that $\alpha(\beta x) = (\alpha\beta)x$. Thus if $y = \beta x$ and $z = \alpha y$, it is not clear that z is equivalent to x. Thus it is impossible to define in this way an identification map $S^{16+7} \rightarrow \mathbb{O}P^2$.

However $\mathbb{O}P^2$ can be constructed by other means. For example, via Lie group theory, $\mathbb{O}P^2 = F_2/\mathrm{Spin}(9)$, but this characterization is completely algebraic and obscures many geometrical properties. Happily, a simpler process is given by the following

Theorem: Let $V = \{(o_1, o_2, o_3) : o_i \in \mathbb{O}, (o_1 o_2)o_3 = o_1(o_2 o_3), [o_1, o_2, o_3] \neq 0\}$. Then $\mathbb{O}P^2$ is homeomorphic to $V/\sim$ where $(o_1, o_2, o_3) \sim (p_1, p_2, p_3)$ if there is a k in the subalgebra generated by o_1, o_2, o_3 such that $(p_1, p_2, p_3) = k(o_1, o_2, o_3)$.

Since $\mathbb{O}P^2$ is a Riemannian manifold, the above theorem will perhaps afford a way to calculate the metric and curvature tensor.

Combinatorial Quandaries

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April 28

A fundamental result in combinatorics states that for any set S with n (distinct) elements there are $\binom{n}{k}$ ways of choosing a k-element subset from S, where $\binom{\cdot}{\cdot}$ is the binomial coefficient, and that, on the other hand, the total number of distinct subsets is 2^n (the cardinality of the power-set). This then gives the following relation:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

There is another situation in which a judicious choice of notation produces a similar relation. Suppose that $L(n_1, n_2)$ is an n_1 by n_2 2-dimensional lattice of integer-points (from $(1,1)$ to (n_1, n_2)), and define a partial ordering on the elements $a = (a_1, a_2)$ and $b = (b_1, b_2)$ of L by

$$a \leq b \quad \Leftrightarrow \quad a_1 \leq b_1 \text{ and } a_2 \leq b_2 \quad ,$$

and a strict inequality is said to hold when at least one of these last two relations is also strict. A *chain* in L is a subset $C \subset L$ such that for any two elements $x, y \in C$ either $x < y$ or $y < x$. In other words, a chain is a (strictly) ordered subset of L. In direct analogy with the above, define $\binom{n_1, n_2}{k}$ to be the number of k-element chains in L, and $2^{(n_1, n_2)}$ to be the total number of chains in $L(n_1, n_2)$. This time these definitions lead to relation characterizing ordered subsets of the lattice, namely:

$$\sum_{k=0}^{n_1+n_2} \binom{n_1, n_2}{k} = 2^{(n_1, n_2)} \text{ where } 2^{(n,1)} = 2^{(1,n)} = 2^n \quad .$$

At this point several interesting things happen. First of all, *computationally explicit* expressions for both the LHS and the RHS of the principal equation above are available and therefore have to be equal (see the paper in the note below). Second, previous proofs of this equality have been extremely long-winded and indirect. Again, the referred paper shows a direct *combinatorial* proof, which, though somewhat tricky, is short, intuitive, and elementary. Third, there are a few unexpected applications made easy by this notation. Fourth, there seem to be no real world, readily available application for some sort of continuum limit of this, the statistics of which could easily be developed; it would be akin to deriving the Poisson distribution from the binomial distribution and the calculational effort would be similar. Finally, there is a most curious thing which ended up calling our attention to this train of thought and which will be described next.

A path in which only the moves right $(p, q) \to (p+1, q)$, up $(p, q) \to (p, q+1)$, and diagonal $(p, q) \to (p+1, q+1)$ are present is called a *king's path.* The number of paths that contain exactly d diagonal moves is given by the multinomial (again, see reference below) $\binom{n_1+n_2-2-d}{d, n_1-d-1, n_2-d-1}$. Summing this number over d yields the total number of possible king's paths in the lattice $L(n_1, n_2)$ (from $(1,1)$ to (n_1, n_2)). These numbers,

$$D(n_1, n_2) = \sum_{d=0}^{\min\{n_1, n_2\}-1} \binom{n_1 + n_2 - 2 - d}{d, n_1 - d - 1, n_2 - d - 1}.$$

are referred to as Delannoy numbers.

Clearly, this relation can be stated in any d-dimensional lattice L. So we have the following conjecture:

$$\text{For any dimension } d, \text{ we have:} \qquad 2^n D(n, \cdots n) = 2^{(n, \cdots n)} \quad .$$

In dimension 1, this is trivial as there is only one king's path. In dimension 2 there are complicated algebraic proofs in the literature, however a simple conceptual proof appeared in the reference below. In dimension 3 and higher numerical results indicate that this relation holds. However, so far no proofs have been found.

Note: The above appeared in the preprint: J. S. Caughman, C. R. Haithcock, J. J. P. Veerman, *Lattice Chains and Delannoy Numbers.* The notation there is a little different from the one here; in particular here we denote the lower left point of the lattice by $(1,1)$ (not $(0,0)$) to emphasize the naturality of the conjecture.

Fractional Graph Colorings.

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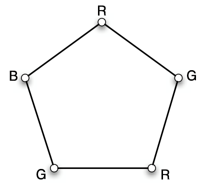
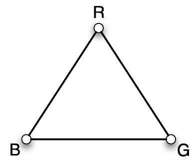
June 2

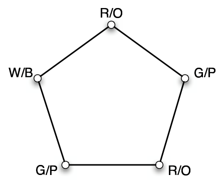
A (proper) *vertex coloring* of a graph X is an assignment of one color to each of the vertices of X such that no two adjacent vertices are assigned the same color.

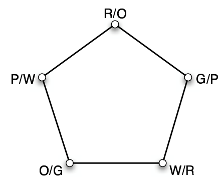




The *chromatic number* $\chi(X)$ of a graph X is the least number of colors required to color X. It is clear that 3 is the chromatic number of the above graphs (Red, Blue, and Green). One of the reasons graph colorings are important is they specify certain graph homomorphisms in the following sense: a graph X is *n-colorable* if and only if there is a graph homomorphism from X to K_n (the complete graph on n vertices).

But a three coloring of the pentagon still seems a little "loose"; this can be tightened with the concept of *fractional colorings.* An $\frac{n}{k}$-coloring is a vertex coloring using n colors with k colors assigned to each vertex such that no two adjacent vertices share any colors. The 3-coloring of the pentagon above trivially produces a $\frac{6}{2}$-coloring, but a little thought can actually reduce this to a $\frac{5}{2}$-coloring as shown below. A $\frac{6}{2}$-coloring will not necessarily produce a 3-coloring: take the graph with vertices given by all 2-element subsets of 6 elements. Join two vertices with an edge if their corresponding subsets are disjoint. This graph is $\frac{6}{2}$-colorable but not 3-colorable.





The *fractional chromatic number* $\chi_F(X)$ of a graph X is the infimum of $\frac{n}{k}$ over all $\frac{n}{k}$-colorings of X. We describe an algorithm to compute this number. Notice first that, given a coloring, the subset of vertices corresponding to a single color C is an independent set, called the *color class* of C. Now, given a graph X, construct a matrix A_X with a column for every independent set of X and a row for every vertex. Enter a 1 in the matrix if the vertex of the row is in the independence set of the column and a zero otherwise. We then apply this matrix to a vector $v = (v_1, \ldots, v_l)^T$ whose rows correspond again to the independence sets. This vector is given as follows: for an $\frac{n}{k}$-coloring, we put an entry of $\frac{1}{k}$ in the row if it corresponds to an independence set that is a color class, and a 0 if not. One can verify that $\sum v_i = \frac{n}{k}$, and that $A_X v$ equals the vector of all 1's. Finding $\chi_F(G)$ then becomes a linear programming problem: minimize $\sum v_i$ subject to the constraints $(A_X v)_i \geq 1$. It is a theorem in linear programming that solutions exist and are rational.

The final consideration is the location of $\chi_F(X)$ in the inequality $\omega(X) \leq \chi_F(X) \leq \chi(X)$, where $\omega(X)$ is the clique number of X (the size of the maximal clique). For example, can difference between two terms in this inequality be arbitrarily large? The answer is yes, as is demonstrated by the following sequence of graphs, due to Mycielski: given a graph X with $V(X) = \{v_1, \ldots, v_l\}$ form a new graph $\mu(X)$ By taking a new vertex set $V(\mu(X)) = \{x_1, \ldots, x_l, y_1, \ldots, y_l, z\}$ and connecting vertices by the rules

$$x_i \sim x_j \iff v_i \sim v_j, \quad x_i \sim y_j \iff v_i \sim v_j, \quad y_i \not\sim y_j \text{ and } y_i \sim z.$$

Let $G_1 = K_2$, and let $G_{n+1} = \mu(G_n)$. One can then prove that $\omega(G_n) = 2$, $\chi(G_n) = \chi(G_{n-1}) + 1$ and $\chi_F(G_n) = \chi_F(G_{n-1}) + 1/\chi_F(G_{n-1})$. Thus ω is constant, χ is linear, and (as can be verified) χ_F grows like $\sqrt{2n}$. Thus the differences $\chi_F(G_n) - \omega(G_n)$ and $\chi(G_n) - \chi_F(G_n)$ grow arbitrarily large.

Note: The information in this summary is based on the article *The fractional chromatic number of a graph and a construction of Mycielski*, M. Larsen, J. Propp, D. Ullman, J. Graph Th. 19, 411-416, 1995.