Asymmetric Decentralized Flocks

F. M. Tangerman, J. J. P. Veerman, B. D. Stošić,

***Abstract*—This paper analyzes the transient dynamics of one dimensional flocks, platoons, *i.e.* a finite collection of identical vehicles moving on the line, with a single leader with independent motion. We show for a class of platoon control laws that if the information flow is asymmetric then a motion change of the leader will cause system transients with amplitudes that grow at an exponential rate as the length of the platoon increases. With suitable choice of the control parameters the system is asymptotically stable and in steady state all vehicles move at the same velocity as the leader and at the required separation. When the leader changes velocity, over very long time scales the vehicles in the platoon tend to the steady state dictated by the leader's new velocity. The transient dynamics in the intermediary regime can however appear quite unstable, where the trailing vehicle can undergo oscillatory motion with amplitudes that grow exponentially large with the number of cars N in the platoon, or may be irresponsive over an exponentially long time to the change in the motion of the leader. In this paper we prove that if the control law is *asymmetric* then such transient errors, measured in terms of displacement between the leader and the trailing car, grow at an exponential rate in N, the length of the platoon. This contrasts sharply with the *symmetric* (bidirectional) case when such transient errors grow only linearly in the length of the platoon, the theoretical minimum for decentralized linear time-invariant platoon control systems with a constant vehicle spacing policy. These results suggest that symmetry of the information flow is an important design parameter for safe control laws for platoons.**

***Index Terms*—Stability of linear systems; Agents and autonomous systems; Traffic Control.**

I. INTRODUCTION

IN In the context of automated vehicular control (see [1], [2], [3], [4], [5]), biology (to understand the methods employed by flocks of birds or schools of fish to maintain formation), various motion control mechanisms and the resulting stability properties of the formation have been gaining interest. Many approaches assume an underlying information graph, which describes some of the agents as leaders that act on the basis of extraneous information and goals while the motion control of the remaining agents is determined from sensed motion (differences) from the agents linked by the information graph. Furthermore the system should admit a range of stable steady state solutions where every agent moves at the same velocity, at prescribed spacings. Finally, in order to have a scalable design, it is often assumed that the communication range is limited: the information graph is suitably local and the control law decentralized. In particular the leaders do not directly communicate their velocity or desired velocity to the formation, and thus motion changes of the leaders propagate indirectly through the formation producing transients. With increasing formation size such transients tend to increase in magnitude and to display a 'bullwhip' effect: to be largest furthest removed from the leaders ([6], [7], [8]). Large transients can mean that collisions can occur in the formation or that following distances become large, effectively disconnecting the formation ([9]), and are therefore undesirable.

In order to maintain formation cohesion it is therefore important to understand what additional design factors mitigate transient effects. The class of formations that has been studied most extensively (see for instance [6], [9], [10], [11], [12], [13], [14], [15]), and which represents the focus of this paper, is that of a platoon consisting of identical cars modeled as point masses moving on the line, with a linear time invariant and decentralized control law.

If one also assumes a constant spacing policy then (see [13]) in the transient regime between two steady state solutions the spacing errors will grow in amplitude at least polynomially in the number of cars N of the platoon, particularly when the motion of the lead agent is sinusoidal of low frequency. Reference [13] concludes that at least one car has to communicate with $\mathcal{O}(N)$ members of the platoon in order that spacing errors remain bounded.

The goal of this paper is to present a class of platoon models for which the transients can be explicitly analyzed in the time domain with increasing length of the platoon. This appears for general models a difficult task and the models studied here are therefore kept simple. We assume only nearest neighbor communication and a constant spacing policy. We find that in the transient regime the spacing errors are significantly larger in amplitude, namely exponentially in the number of cars N, with one exception, namely when the communication is symmetric. In the symmetric case (see [15]), spacing errors grow at most linearly in N, at the theoretical minimum.

Specifically, let $x_0(t)$ denote the position of the leader, and let $x_k(t)$ $k = 1, ...N$ be the positions of the cars following, with $x_N(t) < x_{N-1}(t) < ... < x_1(t) < x_0(t)$. Let Δ be the desired following distance between neigboring vehicles, and let f, g, and ρ be constants. Assume

$$
\begin{aligned}
for \quad & k = 1, ..., N-1 : \\
\ddot{x}_k \quad = \quad & f(\rho(x_k - x_{k+1} - \Delta)) & (1)\\
& + f(1-\rho)(x_k - x_{k-1} + \Delta)) & (2)\\
& + g(\rho(\dot{x}_k - \dot{x}_{k+1}) + (1-\rho)(\dot{x}_k - \dot{x}_{k-1})) & \\
\ddot{x}_N \quad = \quad & f(x_N - x_{N-1} + \Delta) + g(\dot{x}_N - \dot{x}_{N-1}) & (3)
\end{aligned}
$$

This control model determines for any vehicle, except the lead vehicle and the trailing vehicle, its acceleration by weighting of the relative positions and relative velocities of the

F. Tangerman consultant, 15 Stanton Street, Northport, NY, 11768, USA email: fmtangerman@gmail.com.

J.J.P. Veerman, Fariborz Maseeh Dept. of Math. and Stat.,Portland State Univ., Portland, OR 97201, USA. USA email: veerman@pdx.edu.

B. D. Stošić Dept. Estat. & Inf., Univ. Federal Rural de Pernambuco, 52171-900, Recife-PE, Brazil. e-mail: borkostosic@gmail.com.

Manuscript received February 11, 2011; revised March 10, 2012.

vehicles immediately behind by a factor of ρ and immediately in front by a factor of $1-\rho$. The trailing vehicle determines its acceleration from the relative position and relative velocity of the vehicle in front. This model is Galilean invariant, the parameter f corresponds to a spring constant, while g corresponding to a damping constant. When both f and g are negative this control system is asymptotically stable. Its steady states correspond the leader moving at constant velocity and with every vehicle following moving at the same velocity and at the prescribed following distance Δ.

Figure 1 illustrates this class of systems.

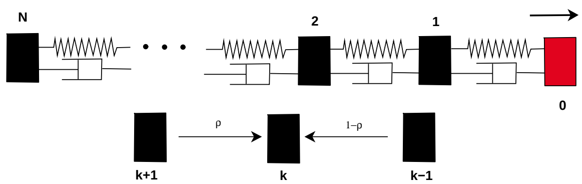

Fig. 1. *In the upper figure the 'symmetric' ($\rho = \frac{1}{2}$) platoon is illustrated. Each vehicle is linearly coupled to its nearest neighbor via springs and dampers. At $t = 0$ the leader, labeled 0, undergoes a forced motion: either an oscillation or a kick in the direction of the arrow above it. The asymmetric interaction is suggested in the lower figure, which shows the $k-th$-vehicle with its interactions and their weights. The arrows give the direction of the information flow.*

We say that the control law is *symmetric* if the weights ρ and $1-\rho$ are the same, equal to $\frac{1}{2}$, and *asymmetric* otherwise. The symmetric case is also known as the bidirectional case. The asymmetric case contains in particular the 'ahead only' case ($\rho = 0$), also known as the unidirectional case, when the data from the car in front are weighted fully while the data from the car behind is ignored. When $\rho = 1$, no weight is given to the vehicle in front, and the the leader is effectively isolated from the formation. This case is therefore not further considered here.

Assume that the platoon is initially in steady state. Since the control law is Galilean we may assume that all the velocities are equal to zero. Consider what we dub the 'canonical traffic problem': assume that at some time, say $t = 0$, the leader quickly accelerates, say impulsively $\ddot{x}_0(t) = \delta(t)$, to a new terminal velocity $v_0 = 1$. Since the control law is asymptotically stable the platoon will converge to the new steady state determined by the terminal velocity of the leader. Figure (2) shows the distinctive and different transient behaviors for three values of ρ: .45, .50 and .55, when $N = 100$.

In order to capture the exponential growth of transients, exponential in the length N of the platoon, we introduce two notions of (in)stability. The first is stated in the frequency domain: *harmonic instability* (see also [16]), where the growth is characterized by the *exponential* growth (or lack thereof) of the frequency response function of the trailing agent. Harmonic stability or instability can be relatively easily established. We show that asymmetric systems are harmonically unstable.

The second notion, *impulse instability* (see also [17]), is stated for the time domain, in terms of the exponential growth of the maximal displacement between the leader and the trailing vehicle as a function of N, the length of the platoon. We show that asymmetric systems are impulse unstable. By contrast, the symmetric case is both harmonically stable and impulse stable. We observe that these two notions of stability are different from, and in fact unrelated to, standard notions that express stability properties in terms of the location of eigenvalues of a linear operator at an equilibrium point.

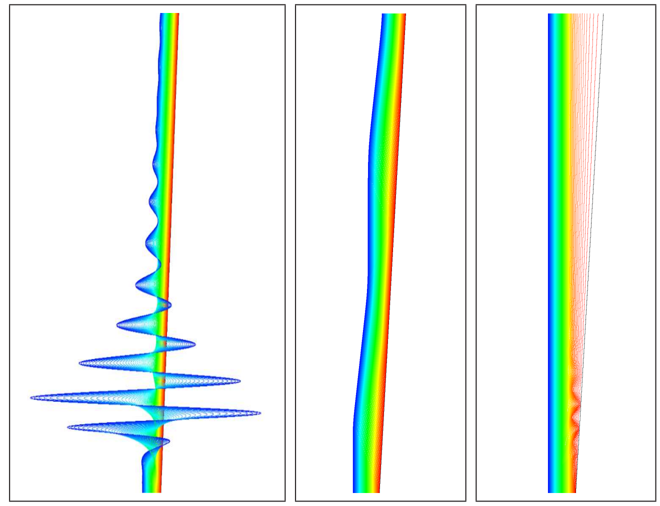

Fig. 2. *Images of time series for three values of ρ: $\rho = 0.45$ on the left, $\rho = 0.50$ in the middle and $\rho = 0.55$ on the right. In each of the figures the horizontal direction is the spatial direction, while the vertical direction is the time direction. In each is shown the trajectory of the leader, the right most - red - curve, and those of 100 cars following with the terminal car -blue - initially left most. In this simulation $f = -1$ and $g = -2$, the cars are initially at rest, each one unit apart, with the leader impulsively moving a speed of .1 m/s. The (vertical) time runs for 1000 seconds. The figures show dramatic differences. In the leftmost figure the trajectory of the terminal car is initially highly oscillatory and then converges to the desired velocity and separation. In the middle figure, the symmetric case, the trajectory of the terminal car is slowly oscillatory about the desired trajectory. In the rightmost figure the terminal car appears to be immobile on the time scale of the figure. Furthermore there are additional oscillations in the beginning of the platoon (that are not analyzed in this paper). While asymptotic stability guarantees that trajectories will converge to the steady state trajectories as time tends to infinity, in the rightmost figure one has to wait $\sim 10^5$ seconds before the terminal car moves at speed comparable to that of the leader.*

A. Related Work

A number of stability criteria were developed to develop controls for platoons (and other coupled systems), to ensure that perturbations (modeled as changes in initial conditions, or as stochastic fluctuations) diminish over time, or at least have a bounded effect. For systems of finite size with state vector $z = (z_1, ..., z_N)^T$ described by a linear homogeneous differential equation:

$$\dot{z} = Mz \quad , \tag{4}$$

such conditions amount to the requirement that the spectrum of M is in the left open or closed half plane.

For infinite linear systems Chu ([18]) defined, with further refinements in ([9], [19]), 'string stability' using the supremum norm: if for some B $\sup_k |z_k(0)| \leq B$, then the solutions $z_k(t)$ are uniformly bounded in t and tend to 0 as t tends to infinity. This form of stability requires that the closure of the spectrum of M be in the open left half plane. The notion *mesh stability* was introduced in ([19]) as a generalization of string stability for flocks in 2 or 3 dimensions.

Cook (see [20]) introduced a subtler variant, referred to as *practical string stability*, requiring an l_1 bound on the initial conditions: $\sum_k |z_k(0)| \leq B$ implies that the solutions $z_k(t)$ are uniformly bounded in t, and defined *practical asymptotic string stability* by the additional requirement that the solutions $z_k(t)$ tend to 0 as t tends to infinity. For bi-infinite, symmetric, systems of the form considered here [20] demonstrated practical asymptotic string stability.

For formations with independent leaders [21] introduced the concept of leader-to-formation stability, a weaker notion than string stability, characterizing the formation temporal response in terms of leader input and initial conditions in terms of certain inequalities and show how leader-to-formation stability can be maintained as platoons grow .

While this paper shows that front-back asymmetry causes exponential growth in system transients, it may be possible to improve some other platoon stability properties, for instance the stability of equilibrium states. Reference [22] discovered for a control model that is different from the one analyzed here (it has besides a desired spacing also a desired absolute vehicle velocity with damping based on the difference between vehicle velocity and desired velocity) that a slight asymmetry can influence the location of the least unstable eigenvalue and move it further in the left half plane. The impact on system transients was however not analyzed.

Specific attention to transient growth with increasing platoon size is provided in the following references. In [13] it is shown that spacing errors increase at least polynomially in the platoon size N under a constant spacing policy in the transient regime. Reference [6], using Bode's complementary sensitivity integral, and generalizing the results of [7], constructs many examples where in the transient regime spacing errors increase exponentially fast in the platoon size N and examines the effect of communication graph and constant spacing versus constant time headway policies in platoon dynamics. Reference [11] shows for the asymmetric example studied here, and assuming that the motion of both the leading car and the trailing car is given, that instabilities occur with amplitudes that become exponentially large as the length of the platoon increases. In the model studied in this paper the motion of the trailing car is determined by its coupling to the platoon and is far from independent. Most of the complexity of the analysis presented here is in fact the determination of the motion of the trailing car.

The generalization of these approaches to dimensions 2 and 3 poses numerous problems, not least of which are to guarantee asymptotic stability, system non-linearity and non-holonomy, see [23], [24].

B. Organization of the Paper

Section II defines the model, and we derive the requirement for the model to be asymptotically stable. Section III provides the definitions of harmonic stability and we show that the asymmetric systems are harmonically unstable. Section IV defines the 'canonical traffic problem' and defines impulse stability. We show that asymmetric systems are impulse unstable. Section V states the final conclusions of this paper. Some of the more calculational steps in the proofs are relegated to the Appendix.

Notational Conventions: For a complex number z its square root $\sqrt{z}$ is defined as the root with argument in the interval $[0, \pi)$ (branch cut along the positive real axis). The number of cars N is a parameter in this problem. When there is no risk of confusion we do not carry this parameter into the notation. Instead, the subscript N will always refer to the last car in the platoon. Similarly we do not carry the dependence on model parameters f, g, and asymmetry ρ, explicitly into the notation.

II. The Model

It is advantageous to write Equation (3) as a first order linear system, and to eliminate the following distance Δ. Let $z_0(t) = x_0(t)$ be the motion of the leader and consider the vector

$$\begin{aligned} z &\equiv (z_1, \dot{z}_1, z_2, \dot{z}_2, \cdots, z_N, \dot{z}_N) \\ &\equiv (x_1 + \Delta, \dot{x}_1, x_2 + 2\Delta, \dot{x}_2, \cdots, x_N + N\Delta, \dot{x}_N)^T \quad . \end{aligned}$$

Then Equation (3) can be written as a first order linear inhomogeneous differential equation (see also [14]):

$$\dot{z} = Mz + \Gamma_0(t) \quad . \tag{5}$$

The motion of the leader appears in the forcing term $\Gamma_0(t)$:

$$\Gamma_0(t) = \begin{pmatrix} 0 \\ (1-\rho)\left(f z_0(t) + g \dot{z}_0(t)\right) \\ 0 \\ \vdots \end{pmatrix} \quad . \tag{6}$$

The matrix M and its spectral properties can be explicitly analyzed using the following approach. Let Q_ρ denote the following $N \times N$ matrix:

$$Q_\rho = \begin{pmatrix} 0 & \rho & & & \\ 1-\rho & 0 & \rho & & \\ & \ddots & \ddots & \ddots & \\ & & 1-\rho & 0 & \rho \\ & & & 1 & 0 \end{pmatrix} , \tag{7}$$

Let I be the $N \times N$ identity matrix and $P \equiv I - Q_\rho$ be the reduced graph Laplacian. The entries of P describe the flow of information among the following cars in the platoon. Using the 2×2 matrices A and K given by:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & 0 \\ f & g \end{pmatrix} \quad . \tag{8}$$

and the Kronecker product ($\otimes$) M of Equation (5) can be written as:

$$M \equiv I \otimes A + P \otimes K \quad . \tag{9}$$

The eigenvalues of Q_ρ and thus of P are known in many cases and therefore easily related to the spectrum of M.

We note that Equation (3) remains invariant under Galilean transformations, and that steady state solutions in the reference frame of the leader take the form: $z = 0$. We can also fix a reference frame in which all the agents are initially at rest ($z =$

0), consider a nonzero motion of the leader in this reference frame, and determine the motion of the entire flock relative this reference frame.

A. Asymptotic Stability

Definition 2.1: The system in Equation (5) is called 'asymptotically stable' if all eigenvalues of M have negative real part.

Asymptotic stability implies for a given motion $z_0(t)$ of the leader that the difference between two solutions, determined by different initial conditions of the following cars in the platoon, tends to zero as time tends to infinity.

In what follows we show that 'asymptotic stability' is equivalent to the requirement that the coefficients f and g are negative.

First, the eigenvalues of P, as shown in [25], are (for the most part) implicitly given in terms of an angle ϕ that is a root of the following equation involving $\rho \in (0,1)$ and N:

$$(2\rho-1)\cot(\phi) = \cot(N\phi) \ . \tag{10}$$

Proposition 2.2: ([25]) For any $\rho \in (0,1)$, the $N \times N$ matrix P has N distinct eigenvalues $\{\lambda_\ell\}_{\ell=0}^{N-1}$ each contained in the interval $(0,2)$, and of the following form:

i) If $\rho \in (0, \frac{N+1}{2N}]$**:** for $\ell \in \{0,\dots,N-1\}$, $\lambda_\ell = 1 - 2\sqrt{\rho(1-\rho)}\cos\phi_\ell$, where $\phi_\ell \in \left(\frac{\ell\pi}{N}, \frac{(\ell+1)\pi}{N}\right)$ is a root of Equation (10).

ii) If $\rho \in (\frac{N+1}{2N}, 1)$**:** for $\ell \in \{1,\dots,N-2\}$, $\lambda_\ell = 1 - 2\sqrt{\rho(1-\rho)}\cos\phi_\ell$, where $\phi_\ell \in \left(\frac{\ell\pi}{N}, \frac{(\ell+1)\pi}{N}\right)$ is a root of Equation (10); $\lambda_0 = \frac{(2\rho-1)^2}{2\rho^2}\left(\frac{1-\rho}{\rho}\right)^{N-1} + \mathcal{O}\left(\left(\frac{1-\rho}{\rho}\right)^{2N-2}\right)$ and $\lambda_{N-1} = 2 - \lambda_0$.

In particular, if $\rho < \frac{1}{2}$ the eigenvalues of P are uniformly bounded away from zero, while if $\rho > \frac{1}{2}$ one eigenvalue, λ_0, is close to zero while the remaining ones are uniformly bounded away from zero. To every eigenvalue λ_ℓ of P (see also [14], [15], [26]) corresponds a pair of eigenvalues $\nu_{\ell\pm}$ of M that are the solutions of the equation

$$\nu^2 - \lambda_\ell g\nu - \lambda_\ell f = 0 \quad , \tag{11}$$

Specifically:

Theorem 2.3: Asymptotic stability of the system given in Equation (5) is equivalent to the condition that both parameters f and g in Equation (3) are negative.

Proof: The eigenvalues of M are

$$\nu_{\ell\pm} = \frac{1}{2}\left(\lambda_\ell g \pm \sqrt{(\lambda_\ell g)^2 + 4\lambda_\ell f}\right) = \frac{\lambda_\ell g}{2}\left(1 \pm \sqrt{1 + \frac{4f}{\lambda_\ell g^2}}\right) \quad ,$$

where λ_ℓ runs through the spectrum of P. Recall that each λ_ℓ is contained in the interval $[0,2]$ (see Proposition 2.2). If $f > 0$ then both $\nu_{\ell\pm}$ are real and have opposite signs. If $f = 0$ then $\nu_{\ell-} = 0$. In either of these cases the system is thus not asymptotically stable. If $f < 0$ and $g > 0$ the real parts of $\nu_{\ell\pm}$ are positive, while if $f < 0$ and $g < 0$ the real parts of $\nu_{\ell\pm}$ are negative. Therefore the system is asymptotically stable if and only if both f and g are strictly smaller than zero. ■

The following corollary provides a geometric picture of the location of the eigenvalues, when the model parameters f and g are negative:

Corollary 2.4: The eigenvalues $\nu_{\pm\ell}$ of M in the complex ν plane either lie on the circle $|\nu + \frac{f}{g}|^2 = \frac{f^2}{g^2}$, namely whenever $\frac{4|f|}{\lambda_\ell g^2} > 1$, or else are real numbers less than or equal to $-\frac{|f|}{|g|}$.

i. When $\rho < \frac{1}{2}$ all eigenvalues are uniformly (in the parameter N) bounded away from zero.
ii. When $\rho > \frac{1}{2}$, and N is sufficiently large, one pair of eigenvalues (corresponding to the index 0) is close to zero, while the remaining are uniformly bounded away from zero.

Proof: If $\frac{4|f|}{\lambda_\ell g^2} > 1$ the eigenvalues $\nu_{\ell\pm}$ are complex conjugates. If ν and $\overline{\nu}$ are the roots of Equation (11) then $\nu\overline{\nu} = -\lambda_\ell f$ and $\nu + \overline{\nu} = \lambda_\ell g$. Now $|\nu + \frac{f}{g}|^2 = \nu\overline{\nu} + (\nu+\overline{\nu})\frac{f}{g} + \frac{f^2}{g^2}$. Therefore the right hand side reduces to $\frac{f^2}{g^2}$.

If the eigenvalues are real, then they are the intersections of the graph of $y = \nu^2$ and the graph of $y = \lambda_\ell g\nu + \lambda_\ell f$ which has negative slope and is non-positive over the interval $[-\frac{|f|}{|g|}, 0]$. Therefore the intersection points must be to the left of $\nu = -\frac{|f|}{|g|}$. The statements (i) and (ii) follow immediately from the corresponding statements regarding the eigenvalues λ_ℓ of P. ■

III. Harmonic Instability of Asymmetric Flocks

Assume that Equation (5) is asymptotically stable. Also assume that the leader executes an oscillation of the form $z_0(t) = e^{i\omega t}$. Then the motion of the k^{th} car $z_k(t)$ tends to $a_k(i\omega)e^{i\omega t}$ as t tends to infinity. The functions $a_k(i\omega)$ are the frequency response functions, and have these properties:

Proposition 3.1: The frequency response functions $a_k(i\omega)$ satisfy:

1) $a_k(0) = 1$ and,
2) $a_k(-i\omega)$ is the complex conjugate of $a_k(i\omega)$

Proof: The first property expresses translational invariance of the Equation (5): when $z_0 = 1$ then the limiting steady state solution is $z_k = 1$. The second property follows since the system has real coefficients. ■

Instead of analyzing all cars in the platoon, we concentrate on the dynamics of the trailing car and its frequency response function: $a_N(i\omega)$. We are interested in its behavior as $N \to \infty$ in the supremum norm over ω: does this remain bounded, increase linearly or polynomially with N or does it increase at an exponential rate? We will see that at large frequencies $a_N(i\omega)$ tends to rapidly decay as a function of N, while at low frequencies it tends to rapidly increase.

Definition 3.2: Let $A_N \equiv \sup_{\omega\in\mathbb{R}} |a_N(i\omega)|$. The system is called 'harmonically stable' if it is asymptotically stable and if $\limsup_{N\to\infty} |A_N|^{1/N} \le 1$. Otherwise the system is called 'harmonically unstable'.

Harmonic instability implies that certain oscillatory motions of the leader will have their amplitude magnified through the platoon with amplitude that is exponentially large in N at the trailing agent. This definition of harmonic instability allows for the possibility that the set of frequencies ω where $a_N(i\omega)$ is on the order of A_N may change with N. Harmonic stability was established for the symmetric case ($\rho = \frac{1}{2}$) in [26]. In what follows we establish that Equation (5) is *harmonically unstable* in the asymmetric case.

The following constant will frequently simplify formulae:

$$\kappa \equiv \frac{1-\rho}{\rho} \qquad \text{or} \qquad \rho = \frac{1}{1+\kappa} \quad .$$

Transforming Equation (5) to the frequency domain via the Laplace Transform leads to recursion relations between the various frequency response functions that are easily solved, using the method that will be outlined next (see also Lemma 3.2 of [15]). Specifically, one obtains from the Laplace Transform of Equation (3) that:

$$\begin{aligned} \nu^2 a_k(\nu) &= f(a_k(\nu) - \rho a_{k+1}(\nu) - (1-\rho)a_{k-1}(\nu)) + \\ &\quad g\nu(a_k(\nu) - \rho a_{k+1}(\nu) - (1-\rho)a_{k-1}(\nu)), \\ &\quad k = 1, ..., N-1 \end{aligned}$$

and the recursion relation is obtained by expressing a_{k+1} in terms of a_k and a_{k-1}. As this relation is linear the solution is of the form $a_k = \alpha_+\mu_+^k + \alpha_-\mu_-^k$, where $\mu_\pm$ are the roots of a quadratic equation and the coefficients $\alpha_\pm$ are independent of k. The boundary conditions for the leader and for the trailing car allow one to determine the coefficients $\alpha_\pm$ and to obtain a_k in closed form. We note that essentially the same recursion relations occur in the determination of the eigenvalues and eigenvectors of the matrix Q_ρ. To state the result in the form needed involves a number of intermediary functions. Let

$$\gamma \equiv \gamma(\nu) = \frac{f + g\nu - \nu^2}{f + g\nu} \tag{12}$$

and define $\mu_\pm = \mu_\pm(\nu)$ as the roots of the following quadratic equation:

$$\rho\mu^2 - \gamma\mu \quad + \quad (1-\rho) = 0, \ i.e \tag{13}$$

$$\mu_\pm \equiv \mu_\pm(\nu) \quad = \quad \frac{1}{2\rho}\left(\gamma \pm \sqrt{\gamma^2 - 4\rho(1-\rho)}\right) \tag{14}$$

One obtains as specific expression:

Proposition 3.3: For $\rho \in (0,1)\backslash\{\frac{1}{2}\}$ the frequency response function of the last agent is given by

$$a_N(\nu) = \frac{1+\kappa}{\kappa}\, \kappa^N \, \frac{\mu_+ - \mu_-}{\left(\mu_+ - \mu_+^{-1}\right)\mu_+^N - \left(\mu_- - \mu_-^{-1}\right)\mu_-^N} \quad ,$$

with γ as in Equation (12) and $\mu_\pm$ as in Equation (14).

Remark: Even though $\mu_+(\nu)$ and $\mu_-(\nu)$ are not rational functions of ν, $a_N(\nu)$ is a rational function of ν.

Remark: The eigenvalues $\nu_{\ell\pm}$ are poles of the function $a_N(\nu)$. The denominator of $a_N(\nu)$ is however expressed in terms of the μ variable and we point out the following properties. Let $\mu_{+\ell} = \mu_+(\nu_{\ell+})$ and $\mu_{-\ell} = \mu_-(\nu_{\ell+})$. Then $\mu_{-\ell} = \dfrac{\kappa}{\mu_{+\ell}}$. Moreover since $\gamma(\nu_{\ell+}) = \gamma(\nu_{\ell-})$ also $\mu_{+\ell} = \mu_+(\nu_{\ell-})$. Furthermore $\mu_{+\ell}$ and $\mu_{-\ell}$ are the roots of the quadratic equation $\rho\mu^2 - (1-\lambda_\ell)\mu + (1-\rho) = 0$. In particular, if $\lambda_\ell = 1 - 2\sqrt{\rho(1-\rho)}\cos(\phi_\ell)$ then $\mu_{+\ell} = \sqrt{\kappa}e^{i\phi_\ell}$ and $\mu_{-\ell}$ is the complex conjugate of $\mu_{+\ell}$. Such values of μ lie therefore on the circle centered at 0 of radius $\sqrt{\kappa}$.

Since $a_N(-i\omega)$ equals the complex conjugate of $a_N(i\omega)$) it is sufficient to study the magnitude of $a_N(i\omega)$ only for $\omega > 0$. In the next proposition we extract the leading growth terms in the frequency response.

Proposition 3.4: Suppose f, g, $\omega > 0$ and $\rho \in (0, 1/2) \cup (1/2, 1)$ are all fixed. Then there exists $r \in (0,1)$, such that as N tends to infinity:

$$a_N(i\omega) = \frac{1+\kappa}{\kappa}\, \mu_-^N \, \frac{\mu_+ - \mu_-}{\mu_+ - \mu_+^{-1}} \left(1 + \mathcal{O}(r^N)\right) \quad .$$

Proof: Use Proposition 3.3 and the fact that $\mu_-\mu_+ = \kappa$ to rewrite $a_N(i\omega)$ as:

$$\frac{1+\kappa}{\kappa}\, \mu_-^N \, \frac{\mu_+ - \mu_-}{\mu_+ - \mu_+^{-1}} \left(1 - \frac{\mu_- - \mu_-^{-1}}{\mu_+ - \mu_+^{-1}}\left(\frac{\mu_-}{\mu_+}\right)^N\right)^{-1}$$

The magnitude of the ratio $\frac{\mu_-(i\omega)}{\mu_+(i\omega)}$ is bounded by $r < 1$, as in Lemma A.3 in the Appendix. It therefore suffices to show that for fixed ω the factor $\frac{\mu_- - \mu_-^{-1}}{\mu_+ - \mu_+^{-1}}$ is finite.

Since

$$\frac{\mu_- - \mu_-^{-1}}{\mu_+ - \mu_+^{-1}} = \frac{-1}{\kappa}\, \frac{\mu_+^2 - \kappa^2}{\mu_+^2 - 1} \quad ,$$

it suffices to prove that if $\omega > 0$, then $\mu_+(i\omega)^2 \neq 1$. Now suppose that $\mu_+(i\omega)^2 = 1$, then since μ and γ are related by a quadratic equation with real coefficients:

$$\rho\mu^2 - \gamma\mu + (1-\rho) = 0 \quad . \tag{15}$$

$\gamma(i\omega) = \pm 1$ and so Lemma A.1 implies that $\omega = 0$. ■

Remark: It is important to emphasize that the bound $\mathcal{O}(r^N)$ is not uniform in ω. The factor analyzed in the proof of the preceding proposition becomes large if μ_+ is close to 1. The results in the appendix show that this can occur if and only if $\rho < \frac{1}{2}$ and ω is close to zero. We make use of this fact in the proof (for $\rho < \frac{1}{2}$) of the next and main result of this section, which shows that asymmetric systems are harmonically unstable.

Theorem 3.5: (Harmonic Instability) For all $\rho \in [0,1)\backslash\{\frac{1}{2}\}$, A_N grows exponentially in N.

Proof: The behavior exhibited in the two cases $\rho < \frac{1}{2}$ and $\rho > \frac{1}{2}$ is different and will be treated separately.

Fix $\rho < \frac{1}{2}$ and let $\omega_+ = \omega_+(f, g, \rho)$ be as in Equation 18 in the Appendix. Lemma A.4 and the remark thereafter now imply that $|\mu_-(i\omega)| > 1$ if $\omega \in (0, \omega_+)$. The claim then follows directly from Proposition 3.4.

Finally fix $\rho > \frac{1}{2}$, or $\kappa \in (0,1)$. First use Lemma A.2 to see that if $\omega^2 = \frac{1}{2}|f|(1-\kappa)^2\kappa^{N-1}$, then $\mu_+ = 1 - \frac{(1-\kappa^2)}{2}\kappa^{N-1} + \mathcal{O}(\kappa^{3N/2})$ and $\mu_- = \kappa(1 + \frac{(1-\kappa^2)}{2}\kappa^{N-1}) + \mathcal{O}(\kappa^{3N/2})$.

Substitute this into the denominator of a_N in Proposition 3.3. The leading order cancels. The next term is of order at least $\kappa^{3N/2}$. Thus A_N is of order at least $\kappa^{-N/2}$. ■

Remarks: The proof of Theorem 3.5 also shows a dichotomy: When $\rho < \frac{1}{2}$, then there is an interval of fixed size over which $a_N(i\omega)$ grows at an exponential rate. When $\rho > \frac{1}{2}$ there is for fixed N a small region close to zero where $|a_N(i\omega)|$ is of order at least $\kappa^{-N/2}$.

IV. Impulse Instability of Asymmetric Flocks

Definition 4.1: The 'canonical traffic problem' corresponds to setting the acceleration of the leader equal to the Dirac delta function, $\ddot{z}_0(t) = \delta(t)$, and $z_0(0) = \dot{z}_0(0) = 0$.

Definition 4.2: Consider Equation (5) for the canonical traffic problem and subject to the initial conditions $z_k(0) = \dot{z}_k(0) = 0$, $k = 1...N$. Let $Z_N^{(i)} \equiv \sup_{t>0} |\frac{d^i}{dt^i}(z_N(t) - z_0(t))|$. The system is called 'impulse stable' if it is asymptotically stable and if for $i = 0,\ 1,\ 2$, $\limsup_{N\to\infty} \left|Z_N^{(i)}\right|^{1/N} \leq 1$. Otherwise the system is called 'impulse unstable'.
Impulse instability means therefore that if the leader receives a 'unit-kick', then that perturbation travels through the flock to produce at some time $t = t_N$ relative displacements errors ($|z_0(t) - z_N(t)|$), relative velocities errors ($|\dot{z}_0(t) - \dot{z}_N(t)|$), or relative accelerations errors $|\ddot{z}_N(t)|$ that are exponentially large in N. As time further increases asymptotic stability will force the trailing agent, as well as the entire flock, to travel ultimately 'in formation'.

Remark: The notions 'harmonic stability' and 'impulse stability' are not unrelated, since the time domain and frequency domain are linked via the Laplace Transform. However these notions appear not to be equivalent.

A. Residue Expansions

We will solve the canonical traffic problem, via the Inverse Laplace Transform. Precisely, with $\ddot{z}_0(t) = \delta(t)$ in Equation (5) and initial conditions: for all $k \geq 1$: $\dot{z}_k(0) = z_k(0) = 0$, the acceleration of the trailing agent is then given as the Inverse Laplace Transform of the frequency response funtion (see [15]):

$$\ddot{z}_N(t) \equiv \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} a_N(\nu) e^{\nu t}\, d\nu \quad . \tag{16}$$

In order to evaluate this inverse the strategy is to perform a residue expansion (or partial fraction expansion) of $a_N(\nu)$. Using Proposition 3.3 we write $a_N(\nu) = \frac{p(\nu)}{q(\nu)}$ as a quotient of polynomials with degree(q) at least that of degree(p). The zeros of q are the eigenvalues of M. Thus according to Theorem 2.3 the denominator in $a_N(\nu)$ has only simple roots located at $\{\nu_{\ell\pm}\}$, except when for some ℓ: $-4f = \lambda_\ell g^2$. Avoiding that case for simplicity, we have with residue $\text{Res}(a_N(\nu), \nu') = \frac{p(\nu')}{\frac{dq}{d\nu}(\nu')}$

$$a_N(\nu) = \sum_{\ell=0}^{N-1} \left(\frac{\text{Res}(a_N(\nu), \nu_{\ell-})}{\nu - \nu_{\ell-}} + \frac{\text{Res}(a_N(\nu), \nu_{\ell+})}{\nu - \nu_{\ell+}} \right) \tag{17}$$

The motion of the trailing car is then:

$$z_N(t) = \sum_{\ell=0}^{N-1} \left(\frac{\text{Res}(a_N, \nu_{\ell-})}{\nu_{\ell-}^2}\, e^{\nu_{\ell-} t} + \frac{\text{Res}(a_N, \nu_{\ell+})}{\nu_{\ell+}^2}\, e^{\nu_{\ell+} t} \right) + C_N + D_N t \quad .$$

The constants of integration C_N and D_N have to guarantee that $z_N(0) = 0$, $\dot{z}_N(0) = 0$.

Recall that the indexing has been chosen so that the pair $\nu_{\pm\ell}$ corresponds to $\mu_{+\ell}$.

Theorem 4.3: If the poles are simple then $\text{Res}(a_N(\nu), \nu_{\ell\pm})$ is given by

$$-\frac{(f + g\nu_{\ell\pm})^2}{\nu_{\ell\pm}\,(2f + g\nu_{\ell\pm})} \cdot \frac{\kappa^{N-1}\mu_{+\ell}^{N-3}(\mu_{+\ell}^2 - \kappa)^2}{2N\mu_{+\ell}^{2N-2}(\mu_{+\ell}^2 - 1) + 2\mu_{+\ell}^{2N} + 2\kappa^{N-1}}$$

Proof: The theorem is an application of the chain rule. Recall the following implicit relationships. γ is a function of ν:

$$\gamma = 1 - \frac{\nu^2}{f + g\nu} \quad .$$

Recall that μ_+ is a function of γ by (choose the '+'root):

$$\rho\mu^2 - \gamma\mu + (1 - \rho) = 0 \quad ,$$

This defines μ_+ and μ_- implicitly as a function of ν. Since μ_- equals κ/μ_+, the expression for a_N in Proposition 3.3 is a rational function of μ_+ alone:

$$\begin{aligned} a_N &= \frac{1+\kappa}{\kappa} \frac{\kappa^N \mu_+^N (\mu_+^2 - \kappa)}{(\mu_+^2 - 1)\mu_+^{2N} + (\mu_+^2 - \kappa^2)\kappa^{N-1}} \\ &\equiv \frac{1+\kappa}{\kappa} \frac{p_N(\mu_+)}{q_N(\mu_+)} \quad . \end{aligned}$$

(The polynomials p_N and q_N still have a factor $(\mu_+^2 - \kappa)$ in common, which is kept to simplify the calculation.) As a result $a_N(\nu) = a_N(\mu_+(\nu))$. Furthermore $\gamma(\nu_{\ell+}) = \gamma(\nu_{\ell-})$ and therefore $\mu_+(\nu_{\ell\pm}) = \mu_{+\ell}$. Now apply the chain rule to obtain the residues when the poles of a_N are simple:

$$\text{Res}(a_N(\nu), \nu_{\ell\pm}) = \frac{1}{\mu_+'(\nu_{\ell\pm})} \text{Res}(a_N(\mu), \mu_{+\ell})$$

with μ_+' the derivative of μ_+. Using the above relations, one obtains:

$$\mu_+'(\nu_{\ell\pm}) = -(1+\kappa)\, \frac{\mu_+(\nu_{\ell\pm})^2}{\mu_+(\nu_{\ell\pm})^2 - \kappa}\, \frac{\nu_{\ell\pm}(2f + g\nu_{\ell\pm})}{(f + g\nu_{\ell\pm})^2} \quad .$$

Using this and replacing the residue of $a_N(\mu_+)$ by $\frac{1+\kappa}{\kappa}\, \frac{p_N(\mu_{+\ell})}{q_N'(\mu_{+\ell})}$, we obtain that $\text{Res}(a_N(\nu), \nu_{\ell\pm})$ equals

$$-(1+\kappa)^{-1} \left(\frac{\mu_{+\ell}^2 - \kappa}{\mu_{+\ell}^2} \right) \frac{(f + g\nu_{\ell\pm})^2}{\nu_{\ell\pm}\,(2f + g\nu_{\ell\pm})} \cdot \frac{(1+\kappa)\kappa^{N-1}\mu_{+\ell}^N(\mu_{+\ell}^2 - \kappa)}{2N\mu_{+\ell}^{2N-1}(\mu_{+\ell}^2 - 1) + 2\mu_{+\ell}(\mu_{+\ell}^{2N} + \kappa^{N-1})}$$

which after some simplification gives the desired result. ■

We note that the residue in the previous theorem is a product of two factors. The first, the ν-factor, is a function of ν and equal to $\dfrac{1}{\gamma'(\nu_{\ell\pm})}$. The other, the μ-factor is expressed in the $\mu_{+\ell}$ variable is the same for $\nu_{\ell+}$ and $\nu_{\ell-}$ and can be bounded separately using the fact that in most cases $\mu_{+\ell}$ has norm equal to $\sqrt{\kappa}$. While in the previous we assumed that the poles are simple, it is clear that the residues can be arbitrarily large if $\nu_{\ell\pm}$ is very close to zero or if $2f + g\nu_{\ell\pm}$ is very close to zero. The first case occurs only, see Corollary 2.4, if $\rho > \frac{1}{2}$ and when $\ell = 0$, and N is large. In that case one finds the pair of eigenvalues $\nu_{0\pm}$ close to zero for which we have a bound. The contribution to the Inverse Laplace Transform is relatively easy to analyze, and is then dominant, see the results in subsequent sections. The second case occurs precisely when the pair $\nu_{\ell\pm}$ is close to the point $\nu^* \equiv -2\frac{f}{g}$. Furthermore it is quite possible that for large N there are many, $\mathcal{O}(N)$, pairs $\nu_{\ell\pm}$ close to ν^*. ν^* a the bifurcation point, the intersection of the circle of radius $\frac{f}{g}$, centered at the point $-\frac{f}{g}$, with the negative real axis, and is a critical point of γ: $\gamma'(\nu^*) = 0$. We now show that for each pair there is sufficient cancelation so that one can bound the large time behavior of its contribution to the Inverse Laplace Transform.

Proposition 4.4: Let $0 < \epsilon < 2\dfrac{f}{g}$. There exists a constant C so that if the pair $\nu_{\ell\pm}$ satisfies $Re(\nu_{\ell\pm}) < -\epsilon$ then the Inverse Laplace Transform of $\sum_\pm \dfrac{1}{\gamma'(\nu_{\ell\pm})}\dfrac{1}{(\nu - \nu_{\ell\pm})}$ is bounded by $Ce^{-\epsilon t}$

Proof: It suffices to consider the case that the pair $\nu_{\ell\pm}$ is close to ν^*. First observe that both $\nu_{\ell+}$ and $\nu_{\ell-}$ have the same value under γ: $\gamma(\nu_{\ell\pm}) = 1 - \lambda_\ell$. We claim that the pair $\nu_{\ell\pm}$ is essentially symmetric with respect to ν^*. Namely consider the Taylor expansion of γ near ν^*:

$$\gamma(\nu) = \gamma(\nu^*) + \frac{1}{2}\gamma''(\nu^*)(\nu - \nu^*)^2 + \mathcal{O}((\nu - \nu^*)^3)$$

Since $\gamma(\nu_{\ell-}) = \gamma(\nu_{\ell+})$ and since $\gamma''(\nu^*) \neq 0$ therefore: $\nu_{\ell-} - \nu^* = -(\nu_{\ell+} - \nu^*) + \mathcal{O}((\nu_{\ell+} - \nu^*)^2)$. Therefore also the pair $\gamma'(\nu_{\ell\pm})$ is essentially symmetric with respect to zero: $\gamma'(\nu_{\ell\pm}) = \gamma''(\nu^*)(\nu_{\ell\pm} - \nu^*) + \mathcal{O}((\nu_{\ell\pm} - \nu^*)^2)$ and $\gamma'(\nu_{\ell-}) = -\gamma'(\nu_{\ell+}) + \mathcal{O}((\nu_{\ell+} - \nu^*)^2)$

The Inverse Laplace Transform is equal to $\sum_\pm \dfrac{e^{\nu_{\ell\pm} t}}{\gamma'(\nu_{\ell\pm})} = e^{\nu^* t}\sum_\pm \dfrac{e^{(\nu_{\ell\pm} - \nu^*)t}}{\gamma'(\nu_{\ell\pm})}$. Let $f(t) = \sum_\pm \dfrac{e^{(\nu_{\ell\pm} - \nu^*)t}}{\gamma'(\nu_{\ell\pm})}$. First, consider $f(0) = \dfrac{1}{\gamma'(\nu_{\ell+})} + \dfrac{1}{\gamma'(\nu_{\ell-})}$. This quantity is uniformly bounded: $|f(0)| \leq K_1$ when the pair $\nu_{\ell\pm}$ is close to ν^*. Second, consider its derivative

$$f'(t) = \sum_\pm \frac{(\nu_{\ell\pm} - \nu^*)e^{(\nu_{\ell\pm} - \nu^*)t}}{\gamma'(\nu_{\ell\pm})}$$

The coefficients $\dfrac{(\nu_{\ell\pm} - \nu^*)}{\gamma'(\nu_{\ell\pm})}$ are also bounded. Fix $\beta > 0$, small, and assume that the real parts of the complex numbers $\nu_{\ell\pm}$ are within β of the real part of ν^*, then we obtain a bound for $f'(t)$ of the form:

$$|f'(t)| \leq K_2 e^{\beta t}$$

By integration we obtain therefore $|f(t)| \leq K_1 + \dfrac{K_2}{\beta}e^{\beta t} \leq K_3 e^{\beta t}$. Multiplication by $e^{\nu^* t}$ proves the proposition. ■

We next consider the μ-factor in the residue formula. Recall that when $\lambda_\ell = 1 - 2\sqrt{\rho(1-\rho)}\cos(\phi_\ell)$, then $\mu_{+\ell} = \sqrt{\kappa}e^{i\phi_\ell}$ and has magnitude $\sqrt{\kappa}$. In the next proposition we consider therefore the μ-factor on the circle of radius $\sqrt{\kappa}$ in the complex plane.

Proposition 4.5: The maximum of the absolute value of the function $\frac{\kappa^{N-1}\mu^{N-3}(\mu^2-\kappa)^2}{2N\mu^{2N-2}(\mu^2-1)+2\mu^{2N}+2\kappa^{N-1}}$ on the circle of radius $\sqrt{\kappa} \neq 1$ is bounded from above by $\mathcal{O}(\dfrac{\kappa^{N/2}}{N})$.

Proof: The numerator is bounded by $\mathcal{O}(\kappa^{3N/2})$. Since $\kappa \neq 1$ then for N sufficiently large the absolute value of the denominator is greater than $N\kappa^N$. The result follows. ■

Combining these two results we obtain the following corollary bounding the contribution of a pair $\nu_{\ell\pm}$ to $z_N(t)$.

Corollary 4.6: Let $0 < \epsilon < 2\dfrac{f}{g}$. If the pair $\nu_{\ell\pm}$ satisfies $Re(\nu_{\ell\pm}) < -\epsilon$ then its contribution to $z_N(t)$ for $t > 0$ is bounded by $\mathcal{O}(\dfrac{\kappa^{N/2}}{N})e^{-\epsilon t}$.

1) $\rho > 1/2$ (or $\kappa < 1$): When $\rho > 1/2$ (or $\kappa < 1$), the weighting is more on the agent following. We will show that in this case two poles dominate the frequency response $a_N(\nu)$ and we can estimate the impulse response.

Remark: In the symmetric case (see [15]) on the order $\sqrt{N}$ poles are close to zero.

Fix the parameters f, g and ρ and let $N \to \infty$. The first goal is to derive an asymptotic expansion for the trajectory of the trailing car.

Proposition 4.7: The impulse response for the trailing car is given by:

$$z_N(t) \;=\; t - \frac{1}{\sqrt{|f|}\,\sqrt{\lambda_0}}\, e^{\lambda_0 g t/2}\, \sin(\sqrt{\lambda_0 |f|}\, t) + \mathcal{O}(\kappa^{N/2})$$
$$where \quad \lambda_0 = \frac{1}{2}(1-\kappa^2)\kappa^{N-1}.$$

Proof: The first term is a linear term obtained by integrating the acceleration twice and using the boundary conditions $z_N(0) = \dot{z}_N(0) = 0$. The sinusoidal term is the contribution of a pair of poles close to zero, and the error term is the contribution from the remaining poles.

Proposition 2.2 implies that when $\kappa < 1$ the eigenvalue $\lambda_0 = \frac{1}{2}(1-\kappa^2)\kappa^{N-1}$ is exponentially small. Theorem 2.3 implies that the sign of $1 + \frac{4f}{\lambda_0 g^2}$ determines whether the corresponding pair of eigenvalues $\nu_{0\pm}$ are real or complex. Therefore, we may assume that $\nu_{0\pm}$ are complex with small negative real part, It is straightforward to obtain to leading order: $\nu_{0\pm} = \frac{1}{2}\lambda_0 g \pm i\sqrt{\lambda_0|f|}$. As a result the ν-factor in the residue is equal to $-\pm\dfrac{i\sqrt{|f|}}{2\sqrt{\lambda_0}}$. Furthermore, to leading order $\mu_{+0} = 1 - \frac{1}{2}(1-\kappa^2)\kappa^{N-1}$, and the corresponding μ-factor equals λ_0. As a result the residue $Res(a(\nu), \nu_{0\pm}) = - \pm \frac{i}{2}\sqrt{\lambda_0 |f|}$. Taking the Inverse Laplace Transform, and integrating twice, produces the second term in the asymptotic expansion. For small time its contribution is equal to $-t$ and in order to maintain the boundary condition, the first term is needed.

The pairs $\nu_{\ell\pm}$, $\ell = 1, ..., N-2$ are contained in the left half plane $Re(\nu) < -\epsilon$ with ϵ independent of N. We can apply Corollary 4.6 to bound the contribution of these pairs.

Finally we have a pair corresponding to $\ell = N-1$. Since $\lambda_{N-1} = 2 - \lambda_0$, this produces also a pair in the left half plane, far from zero. It is straightforward to see that the corresponding μ-factor is $\mathcal{O}(\kappa^N)$. As a result, this pair contributes no more than $\mathcal{O}(\kappa^N)e^{-\epsilon t}$ to $z_N(t)$. ∎

Note that λ_0 is exponentially small, yet positive, in N. The motion $z_N(t)$ for a substantial time interval, as long as $\sqrt{\lambda_0|f|}\ t$ is sufficiently small, remains small,*i.e.* the last car appears to be immobile, while the leading car has traveled a significant amount.

Theorem 4.8: If $\rho > 1/2$ (or $\kappa < 1$) then Equation (5) is impulse unstable.

Proof: It suffices to show that one can find times so that the distance between the leader and the trailing car is exponentially large. The term $'t'$ in the asymptotic expansion for $z_N(t)$ in Proposition 4.7 provides the location of the leader. The magnitude of the second sinusoidal term describes the distance between the leader and the trailing car. If one chooses $t = \frac{\pi}{2\sqrt{\lambda_0|f|}} = \mathcal{O}(\kappa^{-N/2})$ then this distance is equal to $\frac{e^{\lambda_0 g t/2}}{\sqrt{\lambda_0|f|}} = \mathcal{O}(\kappa^{-N/2})$. Since $\kappa < 1$ the distance is then exponentially large in N. The system is therefore impulse unstable. ∎

2) $\rho < 1/2$ (or $\kappa > 1$): When $\rho < 1/2$ (or $\kappa > 1$), the weighting favors the agent in front. In this case no poles appear to be negligible. However, there exists $\epsilon > 0$ so that all eigenvalues $\nu_{\ell\pm}$ $\ell = 0, ..., N-1$ are in the left half plane $Re(\nu) < -\epsilon$. Their residues are however not small and tend to increase with N. As a result all eigenvalues contribute and a useful asymptotic expansion appears intractable. We can apply the logic behind Corollary 4.6 to deduce an upper bound on the acceleration: $|\ddot{z}_N(t)| \leq \mathcal{O}(\kappa^{N/2})e^{-\epsilon t}$ which is enough to control the large time behavior.

Theorem 4.9: If $\rho < 1/2$ (or $\kappa > 1$) then Equation (5) is impulse unstable.

Proof: We show that as $N \to \infty$ the sup norm of the acceleration $\sup_t |\ddot{z}_N(t)|$ grows at an exponential rate. The proof of Theorem 3.5 shows that when $\rho \in (0, 1/2)$, then there is an interval of fixed size in the frequency domain, contained in the interval $[0, \omega+]$ over which $a_N(i\omega)$ grows at an exponential rate: there is $\beta > 1$, independent of N, so that $\sup_\omega |a_N(i\omega)| \geq \beta^N$. From the Fourier Transform:

$$a_N(i\omega) = \frac{1}{2\pi}\int_0^\infty \ddot{z}_N(t)e^{i\omega t}dt$$

and as a result, using the general L^1 bound:

$$|a_N(i\omega)| \leq \frac{1}{2\pi}\int_0^\infty |\ddot{z}_N(t)|dt$$

one obtains that: $\beta^N \leq \frac{1}{2\pi}\int_0^\infty |\ddot{z}_N(t)|dt$. Since $|\ddot{z}_N(t)| \leq \mathcal{O}(\kappa^{N/2})e^{-\epsilon t}$ we can find a constant C so that $\int_{CN}^\infty \kappa^{N/2}e^{-\epsilon t}dt \to 0\ as\ N \to \infty$. Thus one can find σ so that for N large $\int_{CN}^\infty |\ddot{z}_N(t)|dt \leq \sigma$. With $T_N = CN$

$$\frac{1}{T_N}\int_0^{T_N} |\ddot{z}_N(t)|dt \geq 2\pi\frac{\beta^N - \sigma}{T_N}$$

Since T_N is linear in N the right hand side of the inequality grows at an exponential rate. Since the average:

$$\frac{1}{T_N}\int_0^{T_N} |\ddot{z}_N(t)|dt$$

grows at least at an exponential rate, $\sup_t |\ddot{z}_N(t)|$ grows also at least at an exponential rate in N and proves impulse instability. ∎

V. Conclusion

In this paper we have analyzed a simple decentralized linear platoon motion model on the line in which the influence of neighboring agents is weighted asymmetrically. We have shown that although these systems are asymptotically stable, transients due to the response to changes in the motion of the leader, tend to grow at a rate that is exponential in the length N of the platoon (impulse instability).

These results contrast to the symmetric case, *i.e.* equally weighted, where such transients grow at a rate that is roughly linear in the length of the platoon. It may thus be argued that symmetry of information flow is an important consideration in the design of a stable decentralized control law for platoons.

These results have been proven rigorously for the canonical traffic problem where the leader accelerates impulsively. In this case the acceleration of the leader produces a uniform distribution in the frequency domain. Similar results should hold for a much larger class of motions of the leader. What should be important is that the support of the leader acceleration in the frequency domain contains frequencies ω where the frequency response function $a_N(i\omega)$ grows at an exponential rate.

A critical component of the approach in this paper and also [15] has been to understand the location of the eigenvalues of the system with increasing length of the platoon, the residues of the frequency response function at these eigenvalues, and a method to control the influence of nearly double poles (pairs of eigenvalues that are close). The authors expect that this approach generalizes to much larger classes of decentralized, linear and time invariant, Galilean invariant motion models. This suggests a program to characterize those control laws that are impulse stable.

A particular example is the following. The weighting parameters in the control law considered in this paper are the same for relative position and velocity. One can ask if there are benefits in choosing them differently, say of the form:

$$\begin{aligned}\dot{v}_i &= f\{\delta x_i - (1-\rho)\delta x_i - \rho\delta x_{i+1}\}\\ &\quad + g\{v_i - (1-r)v_{i-1} - r v_{i+1}\}\\ &\quad \delta x_i \equiv x_i - h_i\end{aligned}$$

with ρ and r distinct, while maintaining Galilean invariance. While it is certainly possible (P. Barooah and H. Hao, private communication) to influence the spectrum of M and in particular the location of the eigenvalue in the left half plane closest to zero, we conjecture that such systems will exhibit both harmonic and impulse instability, unless the system is symmetric: $r = \rho = \frac{1}{2}$.

We believe that is the first paper to rigorously demonstrate impulse instability in platoon models. It is of great interest to analyze a wider class of platoon models, for instance heterogeneous platoons with relaxed headway policies (see for instance [6]) in the time domain and to verify impulse instability of such systems.

It is in general of interest to define a notion of 'flock stability' that properly quantifies geometric or dynamic characteristics of 'flock transients' as the size of the flock grows, in one or more dimensions. Such a notion should incorporate what effect a motion change that occurs on the boundary of the flock, a response to an external factor, has on the flock, as a function of time and size of the flock. Our proposal for the notion of 'flock stability' is simple: we say a system is *flock stable* if it is harmonically stable and impulse stable. Flock stability thus requires sub-exponential growth rates of transients, and in particular dynamic stability. When these conditions are not satisfied we call the system *flock unstable*.

Appendix A
Technical Results

For completeness we collect a number of straightforward results that are necessary for development of the theory, but would clutter the exposition in the main text. Various relevant quantities are evaluated for $\nu = i\omega$ where ω is real and non-negative. As observed in the main text, by symmetry of the impulse response function, we may assume $\omega \geq 0$ without loss of generality.

Lemma A.1: $\gamma(i\omega) = 1 - \dfrac{\omega^2|f|}{f^2+\omega^2 g^2} + i\,\dfrac{\omega^3|g|}{f^2+\omega^2 g^2}$.

Proof: This follows immediately from the definition of γ in Equation (12). ∎

Lemma A.2: i): For $\rho \in (0, \frac{1}{2})$ and $\omega \geq 0$ small:

$$\begin{aligned}\mu_+ &= \frac{1-\rho}{\rho}\left(1 + \frac{\omega^2}{(2\rho-1)|f|} - i\,\frac{|g|\ \omega^3}{(2\rho-1)f^2}\right) + \mathcal{O}(\omega^4)\\ \mu_- &= 1 - \frac{\omega^2}{(2\rho-1)|f|} + i\,\frac{|g|\ \omega^3}{(2\rho-1)f^2} + \mathcal{O}(\omega^4)\end{aligned}$$

ii): For $\rho \in (\frac{1}{2}, 1)$ and $\omega \geq 0$ small:

$$\begin{aligned}\mu_+ &= 1 - \frac{\omega^2}{(2\rho-1)|f|} + i\,\frac{|g|\ \omega^3}{(2\rho-1)f^2} + \mathcal{O}(\omega^4)\\ \mu_- &= \frac{1-\rho}{\rho}\left(1 + \frac{\omega^2}{(2\rho-1)|f|} - i\,\frac{|g|\ \omega^3}{(2\rho-1)f^2}\right) + \mathcal{O}(\omega^4)\end{aligned}$$

Proof: A calculation, using the definition of $\mu_\pm$, see Equation (14). ∎

Remark: This expansion diverges for $\rho = 1/2$.

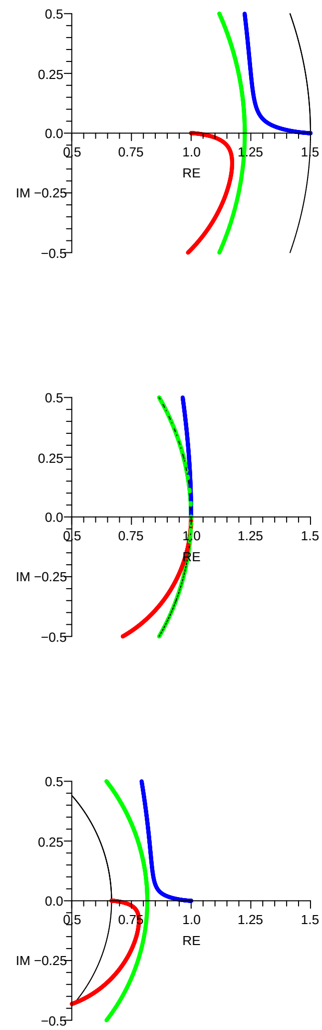

Fig. 3. *The eigenvalues $\mu_+(i\omega)$ (blue) and $\mu_-(i\omega)$ (red) of C when $f = g = -1$ for ω positive. From left to right: $\rho = 0.4$, 0.5, and 0.6. In addition the circles with radii $\sqrt{\kappa}$ and κ are drawn in green and black, resp., where $\kappa \equiv \frac{1-\rho}{\rho}$.*

Lemma A.3: For each $\rho \in (0,1)\backslash\{\frac{1}{2}\}$, the number $r \equiv \sup_\omega \frac{|\mu_-(i\omega)|}{|\mu_+(i\omega)|}$ is smaller than one. (See Figure 3.)

Proof: From Lemma A.1, $\gamma(i\omega) \approx -\frac{i\omega}{g}$ when ω is large. Substitute this into the expression for $\mu_\pm$ in Equation 14 to see that for large ω, in fact $\frac{|\mu_-(i\omega)|}{|\mu_+(i\omega)|}$ becomes very small. When $\omega = 0$, Lemma A.2 implies that $\frac{|\mu_-(i\omega)|}{|\mu_+(i\omega)|} = \min\{\kappa, \kappa^{-1}\}$.

It is now sufficient to prove that for $\omega \in \mathbb{R}^+$ the absolute values $|\mu_\pm|$ are never equal. So suppose there are ω_0 and $\theta \in \mathbb{R}$ so that $\mu_+(i\omega_0) - \mu_-(i\omega_0)e^{i\theta} = 0$. Using Equation 14 then gives:

$$\gamma(1 - e^{i\theta}) = -\sqrt{\gamma^2 - 4\rho(1-\rho)}\,(1 + e^{i\theta}) \qquad .$$

Dividing this by $1+e^{i\theta}$, squaring the equation, and noting that $\frac{(1-e^{i\theta})^2}{(1+e^{i\theta})^2} = -(\tan(\frac{\theta}{2}))^2$, we see that

$$\gamma^2\left(1 + \left(\tan\frac{\theta}{2}\right)^2\right) = 4\rho(1-\rho) \quad .$$

This implies that γ^2 is a positive real and therefore γ is real for some $\omega \neq 0$, which is impossible by Lemma A.1. ■

Lemma A.4: For each $\rho \in (0, 1/2)$, there is a unique $\omega_+ > 0$ such that

$$\begin{aligned} \omega \in (0, \omega_+) \qquad &\Longrightarrow \qquad |\mu_-(i\omega)| > 1 \\ \omega > \omega_+ \qquad &\Longrightarrow \qquad |\mu_-(i\omega)| < 1 \end{aligned}$$

Proof: We know that $\mu_-(0) = 1$ and (from the proof of the previous Lemma) for large ω: $|\mu_-(\omega)|$ is small. It is sufficient to prove that ω_+ is the unique solution in $(0, \infty)$ of $|\mu_-(i\omega)| = 1$ and that it is simple.

Consider the characteristic equation (Equation 14) $\rho\mu^2 - \gamma\mu + (1-\rho) = 0$ and suppose that there is a root $\mu = e^{i\theta}$. Then $\gamma = \rho e^{i\theta} + (1-\rho)e^{-i\theta} = \cos(\theta) + i(2\rho - 1)\sin(\theta)$. Equate this to the expression given in Lemma A.1 and use the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ to obtain:

$$\left(1 - \frac{\omega^2|f|}{f^2 + \omega^2 g^2}\right)^2 + \frac{1}{(2\rho-1)^2}\left(\frac{\omega^3|g|}{f^2 + \omega^2 g^2}\right)^2 = 1$$

This equation factors as follows:

$$\omega^2\left(\frac{g^2}{(2\rho-1)^2}\,\omega^4 + (f^2 - 2|f|g^2)\omega^2 - 2|f|^3\right) = 0$$

The second factor gives exactly one simple positive root for ω^2, yielding a unique simple positive root $\omega = \omega_+$. ■

Remark:

$$\frac{\omega_+^2}{(1-2\rho)|f|} \;=\; \left(1 - \frac{|f|}{2g^2}\right)(1-2\rho) + \sqrt{\left(1 - \frac{|f|}{2g^2}\right)^2 (1-2\rho)^2 + \frac{2|f|}{g^2}}$$

ACKNOWLEDGMENTS

The authors are grateful for the constructive comments of the reviewers.

REFERENCES

[1] W.S. Levine, M. Athans, *On the Optimal Error Regulation of a String of Moving Vehicles*, IEEE Trans. Automatic Control, Vol. AC-11, No. 3, pp 355-361, July 1966.

[2] S.M. Melzer, B.C. Kuo, *Optimal regulation of systems described by a countably infinite number of objects*, Automatica, Vol 7, Issue 3, pp 359-366, May 1971.

[3] S.E. Shladover, *Review of the State of Development of Advanced Vehicle Control Systems (AVHS)*, Vehicle System Dynamics, 23(8), pp 597-625, November 1995.

[4] R. E. Chandler, R. Herman, E. W. Montroll, *Traffic Dynamics: Studies in Car Following*, Op. Res 6, pp 165-184, 1958.

[5] R. Herman, E. W. Montroll, R. B. Potts, R. W. Rothery, *Traffic Dynamics: Analysis and Stability in Car Following*, Op. Res 7, pp 86-106, 1959.

[6] R. H. Middleton, J. H. Braslavsky, *String Instability in Classes of Linear Time Invariant Formation Control with Limited Communication Range*, to appear IEEE Trans. Aut. Contr, Vol 55, Issue 7, pp 1519-1530, 2010.

[7] P. Seiler, A. Pant, K. Hedrick, *Disturbance propagation in vehicle strings*, IEEE Trans. Automatic Control Vol 49, Issue 10, pp 1835 - 1842, 2004.

[8] P. Barooah, J. P. Hespanha, *Error Amplification and Disturbance Propagation in Vehicle Strings with Decentralized Linear Control*, IEEE Conference on Decision and Control 44, vol 5, pp 4964-4969, 2005.

[9] D. Swaroop, J. K. Hedrick, *String Stability of Interconnected Systems*, IEEE Trans. Aut. Contr. 41, No 3, pp 349-357, 1996.

[10] D. Swaroop, J. K. Hedrick, *Constant Spacing Strategies for Platooning in Highway Systems*, J. Dyn. Sys., Measurement, and Contr. 41, No 3, pp 462-470, 1999.

[11] P. A. Cook, S. Sudin, *Convoy Dynamics with Bidirectional Flow of Control Information*, 10th IFAC Symposium on Control in Transportation Systems, Tokyo, pp 433-438, 2003.

[12] D. Yanakiev, I. Kanellakopoulos, *A simplified framework for string stability analysis in AHS*, Proc. 13th IFAC World Congress, San Francisco, Vol Q, pp 177-182, 1996.

[13] S. K. Yadlapalli, S. Darbha, and K. R. Rajagopal, *Information Flow and Its Relation to Stability of the Motion of Vehicles in a Rigid Formation*, IEEE Trans. on Automatic Control, vol. 51, no. 8, pp 1315-1319, August 2006.

[14] J. J. P. Veerman, John S. Caughman, G. Lafferriere, A. Williams, *Flocks and Formations*, J. Stat.Phys. 121, Vol 5-6, pp 901-936, 2005.

[15] J. J. P. Veerman, B. D. Stosic, F. M. Tangerman, *Automated Traffic and The Finite Size Resonance*, J Stat Phys 137, Issue 1, pp 189-203, 2009.

[16] J. J. P. Veerman, *Stability of Large Flocks: an Example*, arXiv:1002.0768.

[17] J. J. P. Veerman, F. M. Tangerman, *Impulse Stability of Large Flocks: an Example*, arXiv:1002.0782.

[18] K-C. Chu, *Decentralized Control of High-Speed Vehicle Strings*, Transportation Science 8, No 4, pp 361-384, 1974.

[19] A. Pant, P. Seiler, K. Hedrick, *Mesh Stability of Look-Ahead Systems*, IEEE Trans. Aut. Contr. 47, No2, pp 403-408, 2002.

[20] P. A. Cook, *Conditions for String Stability*, Syst. Contr. Letters 54, pp 991-998, 2005.

[21] H. G. Tanner, G. J. Pappas and V. Kumar, *Leader-to-Formation Stability*, IEEE TRA, 20(3), pp 443-455, 2004.

[22] P. Barooah, P. G. Mehta and J. P. Hespanha, *Mistuning-based Decentralized Control of Vehicular Platoons for Improved Closed Loop Stability*, IEEE Transactions on Automatic Control, vol. 54, no.9, pp 2100-2113, Sept. 2009.

[23] J.Baillieul and A. Suri, *Information patterns and hedging Brockett's theorem in controlling vehicle formations*, 42nd IEEE Conf. Decision and Control, Vol 1, pp 556–563, 2004.

[24] J. J. P. Veerman, *Symmetry and Stability of Homogeneous Flocks (a Position Paper)*, Proc 1st Int'l Conf on Pervasive and Embedded Computing and Communication Systems, Algarve, 2010.

[25] C. M. da Fonseca, J. J. P. Veerman, *On the Spectra of Certain Directed Paths*, Applied Mathematics Letters, 22, pp 1351-1355, 2009.

[26] J. J. P. Veerman, B. D. Stosic, A. Olvera, *Spatial Instabilities and Size Limitations of Flocks*, Networks and Heterogeneous Media, Volume 2, Number 4, pp 647-660, December 2007.

Folkert Tangerman Finished his thesis at Boston University in 1986. He has been professor at the Math and Applied Math departments at Stony Brook, NY. He worked for Raytheon-Photon Research Associates, and is now a consultant.

J.J.P. Veerman Finished his thesis at Cornell University in 1986. Since then he has lived and worked in the US, Europe, and South America. He is currently Full Professor of Mathematics at Portland State University in Portland, Oregon.

Borko Stosic Obtained a doctoral degree in theoretical physics from Belgrade University in 1991. Since then, he has worked in Europe, USA, and Brazil. He is currently an associate professor at the Department of Statistics and Informatics of the Federal Rural University of Pernambuco, Brazil.