Start Thinking about Numerics

J. J. P. Veerman*

November 12, 2008

There are essentially two ways to prove existence and uniqueness of solutions of ODE. The usual one is the Picard method based on functional analysis and the contracting mapping theorem. See, V.I. Arnold, ODE, green edition pg 215, 3rd edition pg 270. The other method is related to the Euler Step numerical method. A description of this proof can be found in W. Hurewicz, Lectures of ODE, MIT Press (2nd paperback edn), 1970. The following questions are related to this second method.

1) Write a "computer code" to numerically solve the equation $\dot{x} = kx$ with initial condition $x(t_0) = x_0$, by means of Euler's Method. By "computer code" I mean a clear series of mathematical statements (formulated in standard mathematical language) that can be translated unequivocally into any computer language by someone trained in both mathematics and a computer language. Do not, I repeat, *not*, give me a print out of some incomprehensible assembly or other language.

2) Pretend you are the computer: carry out, explicitly and by hand, the program you wrote in 1) for the equation $\dot{x} = 2x$ with initial condition $x(0) = 3$, for t in $[0, 1]$. (Use 2 or, at most, 3 intervals!!)

3) Now, take a step back and use n intervals for problem 2. What is the expression for $x(1)$ you get? Make sure that you check this carefully. What is the exact solution of the differential equation of problem 2? So what is the exact value of $x(1)$?

4) (THE COOL QUESTION:) Historical records (written in an ancient language) show that in the distant past you showed great interest and talent in your calculus courses. In those glorious days it was revealed to you that

$$\lim_{n\to\infty} x_0 \left(1 + \frac{a}{n}\right)^n = x_0 e^a$$

and you immediately understood its proof. Recover this proof (from your calculus book for example), and explain what this has to do with problem 3.

5) (THE REALLY, REALLY COOL QUESTION:) In the far future a secretive organization not to be named writes an Euler Method program to numerically integrate $\dot{x} = x$ with initial condition $x(0) = 1$ for $t \in [0, 1]$. In defiance of numerous congressional hearings and widespread rioting, they refuse to publicize the number of time steps their algorithm uses to numerically integrate the equation. You are by now a gray eminence in the field of ODE, and are called in by the White House for advice.
You know only a few things:
a) The exact solution to the problem.
b) Billy Eaverbeager, a thoroughly reliable mathematical aide to the Senate majority leader has run the infamous program and found that its outcome differs by Δ, a number very close to 0 indeed, from what you know is the exact solution.
c) The president at that point is an Independent.
d) On National Public Radio, a socialist syndicate called "Car Guys' Grandchildren" based out of Yellow Knife, Canada, runs a show a show dedicated to spousal troubles and ordinary differential equations.
e) The number of steps used by the secretive organization is always the same.
After the problem is explained to you, just before they whisk you off to the Embassy Suite in the Four Seasons Hotel,

*e-mail: veerman@pdx.edu

you mutter "AHA, the number of time steps is approximately $\frac{e}{2|\Delta|}$, perhaps a little more but not by much". Early next morning the following mathematical explanation hits the press.....
(The hints are that there are some red herrings in this story, and that problem 1 is really useful.)

6) (THE SO COOL YOU CAN'T BEAR IT ANYMORE QUESTION:) I start this question with a comment (which you don't have to prove, though given its importance to national politics, you might want to). According to MAPLE (a software program that manipulates expressions with unknowns in it), the following holds:

$$\lim_{n\to\infty} \frac{e}{2\left(e - \left(1+\frac{1}{n}\right)^n\right)} - n = \frac{11}{12} \quad .$$

Just for the heck of it, assume that statement is true. What does it imply for the difference between your estimator $\frac{e}{2|\Delta|}$ of n — in question 5 — and the actual number n of steps itself? (Hint: Write an asymptotic ($n \to \infty$) expression for their difference.)

7) (THE BEWARE THE BANKERS ISSUE:) Exponential growth is the basis of mathematical interest theory, which is part of actuarial science. Some of the same issues come up. We introduce just a few concepts here (taken from Lecture 10 of *P.Tannenbaum*, **Excursions in Modern Mathematics**, Prentice Hall, 2010.)
The *principal* P is the amount of money at time zero. The (nominal) *Annual Percentage Rate* (or APR) is the basis for calculating the interest per year on the principal. So the future value F of the principal P compounded monthly for n years at a (nominal) APR of r is:

$$F = P\left(1+\frac{r}{12}\right)^{12n}$$

If it was compounded weekly, it would be:

$$F = P\left(1+\frac{r}{52}\right)^{52n}$$

If it was compounded *continuously*, it would be:

$$F = \lim_{m\to\infty} P\left(1+\frac{r}{m}\right)^{mn} = Pe^{rn}$$

Note that interest is not always compounded. *Simple interest* means that the earned money does not revert to the principal, ie: does not itself draw interest. The *Annual Percentage Yield* (or APY) is the actual interest accrued per year (as a fraction of the principal. From this you can check that the APY α for an investment that receives interest at APR r, compounded m times a year, satisfies:

$$\alpha = \left(1+\frac{r}{m}\right)^{m} - 1$$

Show that $\alpha \geq r$ and that equality occurs iff $m = 1$, that is: iff interest is compounded once a year.
(*Note that frequently different names or conventions for the quantities discussed are used.*)