Invert by Contraction and Live to Tell the Story

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based on V. I. Arnold’s ODE (3rd edn) Section 31.9, problem 1. References are to that book.

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In this problem, Vladimir Arnold means to teach you a useful strategy in mathematics. We prove the (local) inverse function theorem. At first glance there is absolutely no connection with contractions in metric spaces. The art here consists in fiddling with this problem, and recognizing, out of the blue as it were, that Equation (0.3) given below does give you a contraction. The technique then consists of massaging the problem, reformulating it, until you can finally apply the contraction principle. This is the most demanding part and makes up most of the questions below.

**Theorem 0.1** Given a continuously differentiable $f : \mathbb{R}^n \to \mathbb{R}^n$ and a point $r_0 \in \mathbb{R}^n$ so that $Df_{r_0}$ is invertible, there is an open neighborhood $V$ of $r_0$ in $\mathbb{R}^n$ on which $f$ has an inverse $f^{-1}$.

Thus we have:

$$f : V \to W \quad \text{and} \quad z = f(r) \Rightarrow r = f^{-1}(z) .$$

We will prove this using the strategy outlined in the referred problem.

1) Use Taylor’s Theorem to write

$$z = f(r_0) + Df_{r_0} \cdot (r - r_0) + \xi(r) ,$$

where $\xi$ is continuous and $\xi(r_0) = 0$ and $D\xi_{r_0} = 0$.

2) Define

$$y \equiv (Df_{r_0})^{-1} \cdot (z - f(r_0)) , \quad x \equiv r - r_0 , \quad \phi(x) \equiv (Df_{r_0})^{-1} \cdot \xi(r) .$$

Show that now

$$y = x + \phi(x) .$$

3) Suppose a function $\psi$ exists so that (locally)

$$x = y + \psi(y) .$$

Derive:

$$\psi(y) = -\phi(y + \psi(y)) .$$

4) Still assuming Equation (0.2) and using the notation in Equation (0.1), express the (local) inverse $r = f^{-1}(z)$ in terms of $\psi$.

This gives us the inverse. Now we prove Equation (0.2). In the following $B_d$ means the closed ball with radius $d$ in $\mathbb{R}^n$.

5) For any two given positive constants $a$ and $b$, define $M$ to be the space of continuously differentiable functions $h : \{y \in \mathbb{R}^n \quad \text{and} \quad \|y\| \leq a\} \to \mathbb{R}^n$ such that:

$$h(0) = 0 \quad \text{and} \quad Dh_0 = 0 \quad \text{and} \quad \max_{y \in B_a} \|Dh_y\| \leq b .$$

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Give $M$ the sup-norm: $\|h\| \equiv \max_{y \in B_a} |h(y)|$. Prove $M$ is a complete metric space (Arnold, page 275).

6) Use that $h(y) = \int_0^1 Dh_{ty} \cdot y \, dt$ to show that

$$\forall h \in M \quad \|h\| \leq ab.$$ 

Also show that

$$\|D\phi\|_{y \in B_{a+b}} \leq K \quad \text{where} \quad K \to 0 \quad \text{if} \quad a \to 0.$$ 

Now define the functional $\mathcal{F} : M \to M$ as

$$\mathcal{F}h(y) \equiv -\phi(y+h(y)).$$ 

7) Check that if $a$ is small enough, $\mathcal{F}$ maps $M$ into itself. (You could use that $D(\mathcal{F}h)_y = \int_0^1 D\phi_{t(y+h(y))} \cdot (y+f(y)) \, dt$.)

8) Check that if $a$ is small enough, $\mathcal{F}$ is a contraction on $M$. (You could use the following. Take $h_1$ and $h_2$ in $M$. Then $\mathcal{F}h_2 - \mathcal{F}h_1 = \phi(y+h_1(y)) - \phi(y+h_2(y)) = \int_0^1 D\phi_{y+th_1(y)+(1-t)h_2(y)} \cdot (h_1(t) - h_2(y)) \, dt.$)