Statistics of a Family of Piecewise Linear Maps

J. J. P. Veerman, P. J. Oberly, L. S. Fox

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Abstract

We study statistical properties of the truncated flat spot map $f_t(x)$ defined in Figure 1. In particular, we investigate whether for large $n$, the deviations $\sum_{i=0}^{n-1} \left( f_t(x_0) - \frac{1}{2} \right)$ upon rescaling satisfy a $Q$-Gaussian distribution if $x_0$ and $t$ are both independently and uniformly distributed on the unit circle. This was motivated by the fact that if $f_t$ is the rotation by $t$, then [2] found that in this case the rescaled deviations are distributed as a $Q$-Gaussian with $Q = 2$ (a Cauchy distribution). This is the only case where a non-trivial (i.e. $Q \neq 1$) $Q$-Gaussian has been analytically established in a conservative dynamical system.

In this note, we prove that for the family considered here, $\lim_n S_n/n$ converges to a random variable with a curious distribution which is clearly not a $Q$-Gaussian. However, the tail of the distribution is very reminiscent of a $Q$-Gaussian with $Q \approx 0.7$.

1 Introduction

In this note, we study statistical properties of the family of truncated flat spot maps $f_t(x)$ defined in Figure 1. One starts with $x \rightarrow 2x$ and truncates horizontally to obtain a “flat spot circle map” as illustrated in Figure 1 and given by

$$f_t(x) = \begin{cases} 2x \mod 1 & x \in \left[0, \frac{1}{2}\right], \\ t & x \in \left[\frac{1}{2}, \frac{1+t}{2}\right], \\ \frac{1+t}{2} & x \in \left[\frac{1+t}{2}, 1\right). \end{cases}$$

In particular, we wish to study the following quantity:

$$S_n := S(n, t, x_0) := \sum_{i=0}^{n-1} \left( f_t^i(x_0) - \frac{1}{2} \right).$$
By this we mean the histogram of $S(n,t,x_0)$ while keeping $n$ fixed and varying $t$ and $x_0$. We are interested in the limiting behavior of $S_n$ as $n$ tends to infinity. Thus, in computations using $S_n$, one requires that $n$ is “large”. It is easy to generalize all results of this paper to truncations of $x \to \tau x$ for any $\tau > 1$ (see [10, Section 5] for some of the details). For simplicity, however, we will stay with $\tau = 2$ in this note. We will refer to the distribution of $S$ as the distribution of the deviations of the family $f$. The question we want to investigate is whether the distribution of the values of $S_n$ upon rescaling satisfy a nontrivial $Q$-Gaussian distribution (defined below).

![Figure 1: The construction of $f_t(x)$, a) for $t = 1/3$ and b) for $t = 2/3$.](image)

In the 80’s, C. Tsallis and others suggested ([9] and references therein) that many physical systems, in particular those with long range interactions, the values of certain observables might have distributions that are not Gaussian, but a generalization thereof called $Q$-Gaussian. In recent years, the question whether the deviations of systems that are not completely chaotic satisfy a $Q$-Gaussian distribution has gained renewed attention [5]. In particular, this question has been studied for low-dimensional systems such as the standard map from the unit torus $S^1 \times S^1$ to itself

$$f(x,y) = (x + y + k \sin(2\pi x), y + k \sin(2\pi x)).$$

In that case, for large $k$, i.e. $k = 10$, the dynamics tends to have large Lyapunov exponents and correlations die out quickly. Thus, in this case, one expects the distribution of $S$ to be the standard Boltzmann-Gibbs (or Gaussian) one, which coincides with the $Q$-Gaussian distribution for $Q = 1$. Numerically this has been verified [7]. For low $k$, on the other hand, correlations die out more slowly, and one might expect a non-trivial $Q$-Gaussian distribution. Indeed for $k = 0.2$, numerics show that that is the case [7]. Somewhat surprisingly, in [8], it is argued that this curious behavior appears to persist for $k = 0$. Numerically, the authors measured $Q = 1.935$ in that case.

Remarkably, even though the standard map with $k = 0$ would seem to be excruciatingly simple (namely a lamination of pure rotations), the analysis of its statistics is far from easy. In fact, its analysis relies on a difficult theorem by Kesten [3]. Indeed, in this case [2], the distribution of $S_n/\ln(n)$ tends to a Cauchy distribution, which again, as luck would have it, is a special case of the $Q$-Gaussian distributions, namely $Q = 2$ (not 1.935 as the measurement initially seemed to suggest). To date this seems to be the only
conservative dynamical system where analytic proof can be given that the deviations $S_n$ follow a non-trivial (i.e. $Q \neq 1$) $Q$-Gaussian distribution. (For overdamped many body systems, there are various results known, see [4] and references therein.)

This brings us back to the the family $f_t$. Just like the $k = 0$ standard map, it is a family of (weakly) monotone circle maps. So the question is, are its statistics similar to that of the $k = 0$ standard map?

There are various other reasons to study this family. The most important for our purposes is that the dynamics is analytically tractable. Furthermore, like the low $k$ standard map, its dynamics is somewhere between chaotic and completely simple. For any fixed $t$, the dynamics on any invariant set is semi-conjugate to a rotation, and thus definitely not chaotic. But on the other hand, the dynamics is not quite as tame the family of rotations that constitute the standard map with $k = 0$. For the image of the flat spot is a point, and so each map has a non-trivial attractor and repeller. Thirdly, the dynamics of this family is intimately related to that of the large $k$ standard map (see [14]) One can show that the geometry of the repelling invariant sets of irrational rotation number is the same as the asymptotic geometry of the Aubry Mather sets in the standard map [6]. This becomes clear if one locally ‘renormalizes’ the standard map around these orbits [13]. In view of the numerical results for the standard map just mentioned, a study of this family thus becomes doubly interesting.

A few remarks are in order before we set out to do so. Just like the family of rotations – the standard map with $k = 0$ – each $f_t$ is a map with a fixed rotation number $\rho(t)$. The big difference with the integrable standard map is this: It is well known that the set of parameters $t$ for which the map $f_t$ has irrational rotation number has Lebesgue measure zero and, in fact, Hausdorff dimension zero [12]. This implies that in terms of measure theory, the contribution of the irrational rotation numbers to the sum $S_n$ is negligible. This simplifies our analysis substantially, because it allows us to restrict to rational rotation numbers. It is also known that the function that gives the rotation number as a function of $t$ is precisely encoded by the Farey tree [10, 11]. Thus, using the tools provided by those papers, one can, in principle, compute the distribution of the partial sums $S_n$, assuming that $t$ and $x_0$ are uniformly distributed, by summing over the rationals.

In Sections 2 and 3, we prove that the distribution $S_n/n$ converges to a fixed distribution. We also obtain an exact expression for that distribution as an infinite sum. In Section 4, we compute the error caused by truncating the sum. Then, in the concluding section, we exhibit approximations of the distribution and discuss its characteristics. Our conclusion is twofold. The first one is that this distribution is much too “spiky” to be close to any $Q$-Gaussian. On closer investigation, however, it becomes clear that the behavior bears a striking resemblance to that of a $Q$-Gaussian with $q$ roughly equal to 0.70. It might be possible to argue that for smooth perturbations of our system the distribution of the $S$ will be smoothed and might perhaps be closer to a $Q$-Gaussian, but the mathematical tools to prove or disprove such an assertion seem out of reach today.

It is tempting to consider computing something like the above, but now restricting exclusively to the unstable orbits of this family. The thought here is that this is even
closer to the actual dynamics restricted to the minimal energy orbits (Aubry Mather sets) in the standard map for large \( k \). But now the problem is that the distribution if the corresponding \( S_n \) do not have a well-defined density.

For the record, we present some of the definitions relevant to our discussion here.

**Definition 1.1.** We define the function \( e_q \) for \( q > 0 \)

\[
e_Q : (−∞, 0] → (0, ∞) \quad \text{by} \quad e_Q(x) = (1 + (1 - Q)x)^{1/(1-Q)},
\]

and its inverse \( \ln_Q \) (for \( Q > 0 \)) as

\[
\ln_Q : (0, ∞) → \mathbb{R} \quad \text{by} \quad \ln_Q(y) = \frac{1 - y^{Q^{-1}}}{Q - 1}.
\]

It is straightforward to check that these functions are indeed inverses of one another and that setting \( Q = 1 \) (and taking a limit) gives the usual exponential and (natural) logarithm.

**Definition 1.2.** We say that \( y \) satisfies nontrivial \( q \)-Gaussian statistics if the density of \( y \) is given by \( C e_Q(−β(y − y_0)^2) \) for some \( Q \neq 1 \), where \( C \) is a normalization constant.

There are a few special cases of note. We say that it satisfies Boltzmann-Gibbs statistics if this holds for \( Q = 1 \). This is the case for very chaotic systems. When \( Q = 2 \), it satisfies Cauchy statistics. The only dynamical system for which non-trivial (i.e. \( Q \neq 1 \)) \( Q \)-Gaussian statistics has been established is the integrable standard map [2].

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## 2 Distribution of \( S_n/n \)

We start with a result that summarizes some of the considerations in [10, 11]. In what follows, \( \{x\} \) denotes the fractional part of \( x \).

**Proposition 2.1.** For each \( \rho ∈ (0, 1) \) in lowest terms, there is an interval \( I_\rho ⊂ [0, 1] \) (which is a point iff \( \rho \) is irrational) such that for \( t ∈ I_\rho \):

1. For each \( t \), the flat spot map \( f_t \) has a unique rotation number \( \rho(t) \).
2. \( \rho(t) \) is continuous and (weakly) monotone increasing.
3. If \( t ∈ \text{int}I_{p/q} \) (with \( \gcd(p, q) = 1 \)), then there is a unique unstable orbit \( O^u_{p/q} \) which is \( q \)-periodic.
4. If \( t ∈ \text{int}I_{p/q} \) (with \( \gcd(p, q) = 1 \)), then there is a unique stable orbit \( O_{p/q} = \{t, \{2t\}, \ldots \{2^{q-1}t\}, \{t\}, \ldots \} \) which is \( q \)-periodic.
5. All \( x \in [0, 1] \setminus O^u_{p/q} \) eventually reach the stable periodic orbit.

6. The set \( [0, 1] \setminus \bigcup_{\substack{p/q \in (0,1) \\gcd(p,q)=1}} I_{p/q} \) has Hausdorff dimension 0.

We remark that if \( t \) is in the boundary of a rational rotation interval, then the stable and unstable orbits coincide (becoming stable in one direction and unstable in the other). If \( \rho \) is irrational, something similar happens, and there is a unique invariant set, which is stable in one direction, and unstable in the other. The details of the correct description are quite cumbersome, and we will not need all the details, so we leave them out.

For the remainder of this section, denote the flat spot \( [0, t^2] \cup [1+t^2, 1) \) by \( F_1 \). Restricted to the closure of its complement \( F_C^1 \) (see Figure 1), \( f_t : F_C^1 \to [0,1] \) is a bijection. Thus for every positive \( i \), there is a unique inverse image \( f_t^{-i}(F_1) \) of \( F_1 \). Denote this inverse image by \( F_{i+1} \).

**Lemma 2.2.** The intervals \( F_i \) intersect at most in a measure zero set.

**Proof.** The length of \( F_i \) equals \( 2^{-i} \). Thus if they intersect in sets of at most zero measure, then the union of the \( F_i \) must have full measure. Proposition 2.1 items 4. and 5. imply that this must be the case. \( \square \)

**Theorem 2.3.** Fix \( \ell \in \mathbb{N} \). If \( t \) is in the resonance interval \( I_{p/q} \), then for all \( x_0 \in [0,1] \) in a set of measure \( 1 - 2^{-\ell} \)

\[
\frac{S(n,t,x_0)}{n} = \frac{1}{q} \sum_{i=0}^{q-1} \{2^i t\} - \frac{1}{2} + \frac{\Delta_{p/q}}{n},
\]

where \( |\Delta_{p/q}| \) is no greater than \( \frac{q+\ell}{2} \).

**Proof.** First, take \( x_0 \) in \( F_1 \). The orbit of \( x_0 \) is

\[
\{x_0, t, \{2t\}, \{2^2 t\}, \ldots \{2^{q-1} t\}, t, \ldots \}.
\]

Periodicity ensues since by hypothesis \( \{2^{q-1} t\} \in F_1 \). So in this case we have

\[
S(1+kq,t,x_0) - \left( x_0 - \frac{1}{2}\right) = k \sum_{i=0}^{q-1} \left( \{2^i t\} - \frac{1}{2}\right).
\]

Now \( n \) may not be of the form \( 1+kq \). So more generally, we obtain for some \( j \in \{1, \ldots q-1\} \)

\[
S(n,t,x_0) = k \sum_{i=0}^{q-1} \left( \{2^i t\} - \frac{1}{2}\right) + \sum_{i=0}^{j} \left( \{2^i t\} - \frac{1}{2}\right). \tag{2.1}
\]

This leads to an estimate of the error term

\[
\Delta_{p/q} \leq \frac{q+1}{2}. \tag{2.2}
\]
Now, take \( x_0 \) in \( F_i \) with \( r \leq \ell \). By Lemma 2.2, this set has measure \( 1 - 2^{-\ell} \). It takes \( \ell - 1 \) iterates to get in \( F_1 \), after which we apply (2.1). Thus by the same reasoning as before, we get

\[
S(n, t, x_0) - \sum_{i=0}^{r-1} \left( f^i(x_0) - \frac{1}{2} \right) = k \sum_{i=0}^{q-1} \left( \{2^i t\} - \frac{1}{2} \right) + \sum_{i=0}^{j} \left( \{2^i t\} - \frac{1}{2} \right).
\]

Using (2.2), this gives \( \Delta_{p/q} \leq \frac{q+r}{2} \). By hypothesis, we may take \( r \leq \ell \).

Notice that the initial starting point \( x_0 \) has little influence on the distribution of \( S_n/n \) for large \( n \). Thus to study the distribution \( S(n, t, x_0)/n \), it suffices to consider the contribution by \( t \) alone. By Proposition (2.1) item 6., this can be achieved by considering \( t \in I_\rho \) only for rational \( \rho \), and then summing over all rationals to obtain the full distribution. We fix some notation. Let \( \nu \) be the density of the distribution of \( S(n, t, x)/n \) as \( n \) tends to infinity, with \( t \) and \( x \) uniformly distributed in \([0,1]\), and let \( \nu_{p/q} \) be the density of the distribution of \( S(n, t, x)/n \) as \( n \) tends to infinity when \( t \in I_{p/q} \).

Then

\[
\nu(z) = \sum_{q=2}^{\infty} \sum_{1 \leq p < q \atop \gcd(p,q)=1} \nu_{p/q}.
\]

We therefore turn our attention to the computation of \( \nu_{p/q} \).

### 3 Computing \( \nu_{p/q} \)

The first step is to compute the resonance intervals \( I_{p/q} \). Set \( \rho := p/q \). From now on, we will always assume that a rational \( p/q \) is given in lowest terms, i.e. \( \gcd(p, q) = 1 \). Let \( s^{-}(\rho) = s^{-} \) be the binary string defined [10, 11] by

\[
s^{-}_i = \lfloor i\rho \rfloor - \lfloor (i - 1)\rho \rfloor.
\]

Notice that \( s^{-}(\rho) \) is periodic with period \( q \), and that \( s^{-}_1 = 0 \). Furthermore, the first \( q \) entries of \( s^{-} \) contain exactly \( p \) “1”’s. Let \( s^{+}(\rho) \) denote the image of \( s^{-}(\rho) \) under the shift map, so that \((s^{+}_1, s^{+}_2, ...) = (s^{-}_2, s^{-}_3, ...) \). Equivalently,

\[
s^{+}_i = \lfloor (i + 1)\rho \rfloor - \lfloor i\rho \rfloor.
\]

Finally, we need to define

\[
t_0(\rho) = \sum_{i=1}^{q} s^{+}_i 2^{-i}.
\]

**Remark:** Since \( s^{-}_1 = 0 \), we have that \( s^{+}_q = 0 \).
Lemma 3.1 ([10, 11]). Let $s^+$ be the binary string associated to the rotation number $p/q \in (0, 1)$ as above. The $p/q$ resonance interval is given by

$$I_{p/q} = \left[ t^-_{p/q} - \frac{1}{2q-1}, t^+_{p/q} \right],$$

where

$$t^+_{p/q} = \sum_{i=1}^{\infty} 2^{-i} s^+_i = \frac{1}{1-2^{-q}} \sum_{i=1}^{q} 2^{-i} s^+_i$$

Figure 2: The second iterates of the maps in Figure 1a), $t = 1/3$, and 1b), $t = 2/3$. These values are the endpoints of the resonance interval $I_{1/2} = [t^-_{1/2}, t^+_{1/2}]$. Note that for $f_{t-}$, the left endpoint of the flat spot is 2-periodic, while for $f_{t+}$, the right endpoint of the flat spot is 2-periodic.

Consider for a moment the endpoints of the rotation interval and simplify the notation a bit: $I_{p/q} = [t^-, t^+]$. Figures 1 and 2 illustrate the fact that both $f_{t-}(t^-)$ and $f_{t+}(t^+)$ follow the $q$-periodic unstable orbit $O^u_{p/q}$. This immediately gives the following lemma.

Lemma 3.2. $\sum_{i=0}^{q-1} \{2^i t^-\} = \sum_{i=0}^{q-1} \{2^i t^+\}$.

Figure 3: Schematic picture of $\lim_n S(n,t,x) / n$ for $t \in I_{p/q} = [t^-, t^+]$ as $n$ tends to infinity.
**Theorem 3.3.** The density $\nu_{p/q}$ is given by

$$\nu_{p/q} = \frac{q}{2q - 1} 1_{J_{p/q}},$$

where $1_X$ is the characteristic function of $X$,

$$J_{p/q} = \left[ \frac{p - t_0}{q} - \frac{1}{2}, \frac{p - t_0 + 1}{q} - \frac{1}{2} \right],$$

and $t_0$ is as defined in (3.3).

**Proof.** For this proof, we will restrict $t \in I_{p/q}$. Thus $f_t$ must have a stable periodic orbit with a point in the flat spot. That implies that $f_t^q(t) = \{2^q t\} = t$.

By Theorem 2.3, we have

$$\lim_{n \to \infty} \frac{S_n}{n} = \frac{1}{q} \sum_{i=0}^{q-1} \{2^i t\} - \frac{1}{2}. \tag{3.4}$$

Thus, denoting this limit by $\langle s \rangle$

$$\partial_t \langle s \rangle = \frac{2^q - 1}{q}, \tag{3.5}$$

except at the discontinuities of $\langle s \rangle$.

We know, however, exactly where those discontinuities arise. For $t$ in the proscribed range, the collection $\{\{2^i t\}\}_{i=0}^{q-1}$ has precisely one discontinuity, namely for $i = q - 1$. Thus we have the situation where $\{2^{q-1} t^-\}$ equals the left endpoint of the flat spot (as in Figure 2a) while $\{2^{q-1} t^+\}$ equals the right endpoint of the flat spot (Figure 2b). In between, at $t = t_0$ defined in equation (3.3), we have

$$\lim_{t \searrow t_0} \{2^{q-1} t\} = 1 \quad \text{and} \quad \lim_{t \nearrow t_0} \{2^{q-1} t\} = 0. \tag{3.6}$$

Therefore, using (3.4), we get (see Figure 3)

$$\lim_{t \searrow t_0} \langle s \rangle = \lim_{t \nearrow t_0} \langle s \rangle - \frac{1}{q}. \tag{3.5}$$

Furthermore, this implies that the $q$th image of $t_0/2$ under $f_{t_0}$ equals 0. Thus $t_0$ as defined in equation (3.3) is the unique point of jump discontinuity.

Lemma 3.2 also establishes that that at the ends of the interval $I_{p/q}$

$$\lim_{n \to \infty} \frac{S(n, t^-, x)}{n} = \lim_{n \to \infty} \frac{S(n, t^+, x)}{n}. \tag{3.7}$$

Statements (3.5), (3.6), and (3.7) establish the description of $\langle s \rangle$ as given in Figure 3.
Thus for $t \in I_{p/q}$, the support $J_{p/q}$ of $\langle s \rangle$ is given by

$$J_{p/q} = \left[ \frac{1}{q} \sum_{i=0}^{q-1} \left( \{2^i t_0 \} - \frac{1}{2} \right), \frac{1}{q} \sum_{i=0}^{q-1} \left( \{2^i t_0 \} - \frac{1}{2} \right) + \frac{1}{q} \right].$$

By equation (3.3) and Proposition A.2, this is equal to

$$J_{p/q} = \left[ \frac{p - t_0}{q} - \frac{1}{2}, \frac{p - t_0 + 1}{q} - \frac{1}{2} \right].$$

This interval is the support of $\nu_{p/q}$. For fixed $p/q$ and $t \in I_{p/q}$, $S_n(t)/n$ is linear a.e. in $t$ (with one jump discontinuity). Therefore $\nu_{p/q}$ is a.e. equal to a constant on its support. As $\int \nu_{p/q}(z) \, dz = |I_{p/q}| = (2^q - 1)^{-1}$, and $|J_{p/q}| = 1/q$, this gives the constant $\frac{q}{2^q - 1}$. 

**Remark:** Theorem 3.3 implies that the support of $\nu_{p/q}$ is in $[-\frac{1}{2}, \frac{1}{2}]$. In fact, it immediately follows that the support of the distribution of $\langle s \rangle$ equals that interval.

### 4 Error estimate for $\nu$. 

The geometric decrease in the magnitude of $\nu_{p/q}$ suggests that the computations of the preceding section ought to be quite accurate. In this section, we prove that is the case for the absolute error.

Let

$$\nu_N(x) = \sum_{q=2}^{N} \sum_{1 \leq p < q \gcd(p,q)=1} \nu_{p/q}(x)$$

be the $N^{th}$ partial sum of $\nu$, and let $||f|| = \sup_{x \in [-1/2, 1/2]} |f(x)|$. The primary result for the absolute error is Theorem 4.2.

We first need the following simple lemma.

**Lemma 4.1.** Fix an integer $q \geq 2$. If $p_1$ and $p_2 = p_1 + i$, $i \in \mathbb{N}$, are positive integers relatively prime to $q$, then $J_{p_1/q} \cap J_{p_2/q} \neq \emptyset$ if and only if $i = 1$.

**Proof.** Suppose $\tau \in J_{p_1/q} \cap J_{p_2/q}$ (note that then $q \geq 3$). Let $t_i = t_0(p_i/q)$, where $t_0$ is defined in Equation (3.3). Then

$$q\tau + q/2 \in [p_i - t_i, p_i - t_i + 1]$$

Now, clearly, $p_1 - t_1 < p_2 - t_2$. So the two intervals overlap if and only if $p_2 - t_2 < p_1 - t_1 + 1$, or iff

$$p_1 + i - t_2 < p_1 - t_1 + 1 \iff i - 1 < t_2 - t_1.$$

Since $t_i \in (0,1)$ and $t_2 > t_1$, this is equivalent to $i = 1$. 


Theorem 4.2. Fix an integer \( N \geq 2 \). Then
\[
||\nu - \nu_N|| < \frac{4(N+2)}{2^{N+1} - 1}.
\] (4.1)

Proof. Let \( x \in [-1/2, 1/2] \). For any \( q \geq 2 \) then
\[
\sum_{1 \leq p < q \atop \text{gcd}(p,q)=1} \nu_{p/q}(x) \leq \frac{2q}{2^q - 1}
\]
by Lemma (4.1) and definition of \( \nu_{p/q} \). Thus
\[
0 \leq \nu(x) - \nu_N(x) = \sum_{N+1}^{\infty} \sum_{1 \leq p < q \atop \text{gcd}(p,q)=1} \nu_{p/q}(x) \leq \sum_{q=N+1}^{\infty} \frac{2q}{2^q - 1} \leq 2 \sum_{q=N+1}^{\infty} \frac{2^{N+1}}{2^{N+1} - 1} \sum_{q=N+1}^{\infty} \frac{q}{2^q}.
\]
The last sum can be explicitly evaluated via
\[
\sum_{q=N+1}^{\infty} \frac{q}{2^q} = -\partial_\alpha \left[ \sum_{q=N+1}^{\infty} e^{-q\alpha} \right]_{\alpha=\ln 2}.
\]
We leave the details to the reader, but the upshot is
\[
0 \leq \nu(x) - \nu_N(x) \leq 2 \frac{2^{N+1}}{2^{N+1} - 1} \frac{N+2}{2^N} = \frac{4(N+2)}{2^{N+1} - 1}.
\]

5 Conclusion

Using Theorem 3.3, we can approximate \( \nu \) by the sum \( \sum_{q=2}^{N} \sum_{1 \leq p < q \atop \text{gcd}(p,q)=1} \nu_{p/q} \) for some \( N \).

We end this note with some numerical experiments to gain some insight in the structure of \( \nu(z) \).

In Figure 4, we illustrate the way the density is generated from the densities \( \nu_{p/q} \) for \( q \) is 2, 3, 4, and so forth. It is interesting to note that, indeed, whenever both \( p \) and \( p+1 \) are relative prime to \( q \), then the support \( \nu_{p/q} \) and that of \( \nu_{(p+1)/q} \) overlap (see Lemma 4.1). This phenomenon seems to be responsible a lot of the “spiky-ness” of the final distribution.

The first thing we note, of course, is (as predicted by Theorem 3.3) that the contributions \( \nu_{p/q} \) of the individual rotation numbers is not merely a scaled version of \( \nu \). This, at least, is similar to the situation for the \( k = 0 \) standard map [2], where the distribution of the deviations for the individual rotation numbers may be very different from the average over all rotation numbers.
Figure 4: First row from left to right: $\nu_{1/2}$, $\nu_{1/3} + \nu_{2/3}$, and the sum of all $\nu_{p/q}$ for $q \leq 3$. Second row: the sum of $\nu_{p/4}$ and the sum of all $\nu_{p/q}$ for $q \leq 4$. Third row: the sum of $\nu_{p/5}$ and the sum of all $\nu_{p/q}$ for $q \leq 5$. Fourth row: the sum of $\nu_{p/6}$ and the sum of all $\nu_{p/q}$ for $q \leq 6$. (Note the difference of scales in the graphs.)

The lower left plot of Figure 5 was generated by summing the $\nu_{p/q}$ for all $q$ in $\{2, \ldots, 50\}$. Theorem 4.2 tells us that this figure approximates the true density $\nu(z)$ with an (absolute) error of at most $10^{-13}$. Clearly, the graph of this density resembles the profile of the building in the “Ghostbusters” movie much more than it resembles a $Q$-Gaussian density (see Definition 1.2) for any $Q$.

To illustrate the “spiky-ness” of $\nu$, we draw the same function, but with $\nu_{p/q}$ of Theorem 3.3 replaced by $\frac{q}{16^{q-1}} J_{p/q}$, so that the scaling goes slower. Since now more details are visible, in Figure 5, we summed $nu_{p/q}$ for all $p, q$ relatively prime with $q \leq 150$. In the interest of comparison, we note that if left and right endpoints of $\nu_{p/q}$ do not
coincide, it is easy to see that the total length of the “up” or “down” travel in the left plot of Figure 5 equals

$$\ell = \sum_{q \geq 2} \frac{q \phi(q)}{2q - 1} \approx 3.59,$$

where $\phi$ is Euler’s phi function. The valuation was done numerically. The total length of the “up” travel in the right plot of Figure 5 equals

$$\ell = \sum_{q \geq 2} \frac{q \phi(q)}{1.4q - 1} \approx 36.34.$$

![Figure 5: Left: the sum of all $\nu_{p/q}$ for $q \leq 50$. Right: the sum of all $\nu_{p/q}$ for $q \leq 150$ but where the $\nu_{p/q}$ are rescaled so that they decay less quickly. This way the variability of the function is emphasized. (We did not normalize $\nu$ to integrate to 1.)](image)

Even though the ghostbuster density is not a $Q$-Gaussian, we nonetheless tried to find a “best” fitting $Q$-Gaussian to it. To this end, we plotted $\ln Q(\nu)$ against $z^2$ for many values of $Q$. We then took that value of $Q$ for which the graph is most linear and $-\beta$ is the slope of the line of best fit. The process is illustrated in the left of Figure 6, where we plotted We found that $Q = 0.65$ and $Q = 0.75$ give considerably worse results (left of Figure 6). The way the $Q = 0.7$-Gaussian traces the average behavior of the data is, in our view, remarkable, especially in view of the fact that in earlier numerical simulations $Q < 1$ had already been observed in (critical) circle maps [1]. Because our relative errors grow quickly as $|z|$ gets past 0.45, we aim to approximate $\ln Q(\nu)$ with a linear function of $z^2$ as well as possible in the interval $[0, 0.2025]$ (0.2025 being the square of 0.45). Note that $\ln Q(0) = -(1 - Q)^{-1}$. This explains the leveling out of $\ln Q(\nu)$ as $z^2$ tends to $1/4$. The right side of Figure 6 shows the density along with the best $Q$-Gaussian-fit. In spite of the in some sense remarkable fit, the support of the best $Q$-Gaussian is $[-(\beta(1 - Q))^{-1/2}, (\beta(1 - Q))^{-1/2}] \approx [-0.455, 0.455]$ underlining again that the fit is not perfect.
Figure 6: Left: The graph of $\ln_q(\nu)$ with (from top to bottom) $Q = 0.65, 0.70$ and $0.75$ against $z^2$ and a linear approximations with $\beta = 16.1$. Right: The distribution $\nu$ with the $q$-exponential fit ($Q = 0.7, \beta = 16.1$) discussed in the text.

References


A Summing fractional parts of a geometric series

Assuming we avoid infinite repetitions of 1, a number $t \in [0,1)$ has a unique binary expansion (with $d_j \in \{0,1\}$):

$$t = \sum_{j=1}^{\infty} d_j 2^{-j}.$$ 

The expression $d_j 2^{-j}$ will be denoted by $t_{-j}$, so that we have $t = \sum_{j=1}^{\infty} t_{-j}$.

**Lemma A.1.** For $1 \leq j \leq n + 1$:

$$\sum_{i=0}^{n} \{2^i t_{-j}\} = d_j - t_{-j}.$$ 

**Proof.** For $i \in \{0, \cdots, j-1\}$, we have $\{2^i t_{-j}\} = 2^i t_{-j}$. After that, $\{2^i t_{-j}\}$ equals zero. Thus the sum in the lemma equals:

$$\sum_{i=0}^{j-1} 2^i t_{-j} = \frac{2^j - 1}{2 - 1} d_j 2^{-j} = d_j - d_j 2^{-j},$$

which equals the expression in the RHS of the lemma. \qed

**Proposition A.2.** Now let $t := \sum_{j=1}^{q} d_j 2^{-j} = \sum_{j=1}^{q} t_{-j}$. Then

$$\sum_{i=0}^{q-1} \{2^i t\} = \sum_{j=1}^{q} d_j - t \quad \text{and} \quad \sum_{i=0}^{q-1} [2^i t] = 2^q t - \sum_{j=1}^{q} d_j.$$
Proof. That the formulae in the proposition are equivalent can be seen by summing them, which yields the usual geometric sum. So it suffices to prove the first formula.

We have

\[
\sum_{i=0}^{q-1} \{2^it\} = \sum_{i=0}^{q-1} \left\{ 2^i \sum_{j=1}^{q} t_{-j} \right\}
\]

Furthermore,

\[
\left\{ 2^i \sum_{j=1}^{q} t_{-j} \right\} = \left\{ 2^i \sum_{j=1}^{i} t_{-j} + 2^i \sum_{j=i+1}^{q} t_{-j} \right\}
\]

and \(2^i \sum_{j=1}^{i} t_{-j}\) is an integer while \(2^i \sum_{j=i+1}^{q} t_{-j}\) is in \([0, 1)\). Thus Lemma A.1 then gives

\[
\sum_{i=0}^{q-1} \sum_{j=1}^{q} \{2^it_{-j}\} = \sum_{j=1}^{q} d_j - t_{-j},
\]

which in turns simplifies to the required expression. \(\square\)