Commensurability in Symmetric Nearest Neighbor Systems

J. J. P. Veerman* and J. Petrovic †

Abstract

These notes contain the calculations and proofs of some of the results of the accompanying paper [1].

1 Introduction

Given a differential equation on $\mathbb{R}^n$ of the form:

$$i\dot{x} = -Mx \tag{1.1}$$

where i the unit imaginary number and $\dot{x}$ denotes the time derivative of x. The variable x is understood to be in $\mathbb{R}^n$ and M is a real square symmetric matrix (in general Hermitian works). The symmetry derives from the physics problems that this applies to. The query now is: How can we be sure that the the solutions of this system are always periodic?

The answer of course is as follows. Since M is Hermitian, it is *normal*, ie has an orthogonal basis of eigenvectors, and its eigenvalues are real. Let us denote (for $j \in \{1, \cdots n\}$ the orthogonal eigenvectors by v_j and the associated eigenvalues by ω_j. Then solutions of Equation 1.1 can be written as:

$$x(t) = \sum_{j=n}^{n} c_j e^{i\omega_j t} v_j \tag{1.2}$$

Here the c_j are complex coefficients determined by the initial condition $x(0)$.

Lemma 1.1 *Let $T > 0$. Every solution of Equation 1.1 has a period T if and only T satisfies*

$$T(\omega_1, \cdots \omega_n) \in 2\pi\mathbb{Z}^n$$

Definition 1.2 *In this case we say that the eigenvalues $\{\omega_j\}_1^n$ are* commensurate *or, for short, that the matrix M is commensurate. The smallest positive T for which the above holds is called* the period.

*Fariborz Maseeh Dept. of Math. and Stat., Portland State Univ., Portland, OR, USA, and CCQCN, Dept of Physics, University of Crete, 71003 Heraklion, Greece; e-mail: veerman@pdx.edu

†Vinca Institute of Nuclear Sciences, University of Belgrade, 12-14 Mike Alasa, 11000 Belgrade, Serbia

Proof of Lemma: Every solution of Equation 1.1 has a period T if and only for all c_j

$$\sum_{j=n}^{n} c_j e^{i\omega_j(t+T)} v_j = \sum_{j=n}^{n} c_j e^{i\omega_j t} v_j$$

This equation holds if and only if it holds for each eigenvector:

$$\forall\ j\ :\ e^{i\omega_j(t+T)} v_j = e^{i\omega_j t} v_j$$

or

$$\forall\ j\ :\ e^{i\omega_j T} = 1$$

This is equivalent to saying that there are integers r_j such that

$$\forall\ j\ :\ \omega_j T = 2\pi n_j \quad \text{ or } \quad \frac{T}{2\pi}\omega_j = n_j$$

This proves the Lemma. ∎

Acknowledgments: J.J.P. Veerman's research was partially supported by the European Union's Seventh Framework Program (FP7-REGPOT-2012-2013-1) under grant agreement n316165.

2 Elementary Examples

We start with a real, symmetric, nearest neighbor interaction acting on $\mathbb{R}^3$:

$$A_3(a_1, a_2, a_3) \equiv \begin{bmatrix} 0 & a_1 & 0 \\ a_1 & 0 & a_2 \\ 0 & a_2 & 0 \end{bmatrix}$$

Its eigenvalues are easily calculated and are: $\{0, 0, \pm\sqrt{a_1^2 + a_2^2}\}$.

Lemma 2.1 *The matrix $A_3(a_1, a_2)$ is always commensurate. The period T is given by $\dfrac{2\pi}{\sqrt{a_1^2 + a_2^2}}$.*

Proof: According to Lemma 1.1, the period T must be the smallest positive number that satisfies:

$$T\sqrt{a_1^2 + a_2^2} = 2\pi n$$

Clearly $n = 1$ works and this immediately gives the period. ∎

Now we include a left-right symmetry (or invariance under $x_m \leftrightarrow x_{N-m}$). Here is the general matrix acting on $\mathbb{R}^5$:

$$M_5(a_1, a_2) \equiv \begin{bmatrix} 0 & a_1 & 0 & 0 & 0 \\ a_1 & 0 & a_2 & 0 & 0 \\ 0 & a_2 & 0 & a_2 & 0 \\ 0 & 0 & a_2 & 0 & a_1 \\ 0 & 0 & 0 & a_1 & 0 \end{bmatrix}$$

Its eigenvalues are easily calculated and are: $\{0, \pm a_1, \pm\sqrt{2a_2^2 + a_1^2}\}$.

Lemma 2.2 *The matrix $M_5(a_1, a_2)$ is commensurate if and only if:*
1. If $a_1 \neq 0$: there are integers n_1 and n_2 with GCD equal to 1 and such that

$$a_1^2 = q^2 n_1^2$$
$$a_2^2 = q^2(n_2^2 - n_1^2)$$

where q is an arbitrary (strictly) positive real. In this case the the eigenvalues relate to another as $(0, \pm n_1, \pm n_2)$ and the period T is given by $2\pi/q$.
2. If $a_1 = 0$ and $a_2 \neq 0$: In this case the system is periodic with period $T = \dfrac{2\pi}{\sqrt{2a_2^2}}$.

Proof: According to Lemma 1.1, the period T must be the smallest positive number that satisfies:

$$Ta_1 = 2\pi n_1 \qquad \text{and} \qquad T\sqrt{2a_2^2 + a_1^2} = 2\pi n_2$$

In both cases of the lemma T cannot be zero. In case 1, n_1 is also not equal to zero. In this case, divide the second expression by the first. Clearly if $a_1 a_2 \neq 0$ then neither n_1 nor n_2 is zero, and T is minimal iff $\text{GCD}(n_1, n_2) = 1$. ■

3 Commensurability of A_4 and A_5

We look at the real, symmetric, nearest neighbor interaction with left-right symmetry acting on $\mathbb{R}^4$ and $\mathbb{R}^5$. Here are their general forms:

$$A_4(a_1, a_2, a_3) \equiv \begin{bmatrix} 0 & a_1 & 0 & 0 \\ a_1 & 0 & a_2 & 0 \\ 0 & a_2 & 0 & a_3 \\ 0 & 0 & a_3 & 0 \end{bmatrix} \qquad \text{and} \qquad A_5(a_1, a_2, a_3, a_4) \equiv \begin{bmatrix} 0 & a_1 & 0 & 0 & 0 \\ a_1 & 0 & a_2 & 0 & 0 \\ 0 & a_2 & 0 & a_3 & 0 \\ 0 & 0 & a_3 & 0 & a_4 \\ 0 & 0 & 0 & a_4 & 0 \end{bmatrix}$$

We start with the commensurability of $A_4(a_1, a_2, a_3)$. It is easy to verify that its eigenvalues are given by:

$$\begin{bmatrix} \frac{1}{2}\sqrt{2{a_3}^2 + 2{a_2}^2 + 2{a_1}^2 + 2\sqrt{{a_3}^4 + 2{a_3}^2{a_2}^2 - 2{a_3}^2{a_1}^2 + {a_2}^4 + 2{a_2}^2{a_1}^2 + {a_1}^4}} \\ -\frac{1}{2}\sqrt{2{a_3}^2 + 2{a_2}^2 + 2{a_1}^2 + 2\sqrt{{a_3}^4 + 2{a_3}^2{a_2}^2 - 2{a_3}^2{a_1}^2 + {a_2}^4 + 2{a_2}^2{a_1}^2 + {a_1}^4}} \\ \frac{1}{2}\sqrt{2{a_3}^2 + 2{a_2}^2 + 2{a_1}^2 - 2\sqrt{{a_3}^4 + 2{a_3}^2{a_2}^2 - 2{a_3}^2{a_1}^2 + {a_2}^4 + 2{a_2}^2{a_1}^2 + {a_1}^4}} \\ -\frac{1}{2}\sqrt{2{a_3}^2 + 2{a_2}^2 + 2{a_1}^2 - 2\sqrt{{a_3}^4 + 2{a_3}^2{a_2}^2 - 2{a_3}^2{a_1}^2 + {a_2}^4 + 2{a_2}^2{a_1}^2 + {a_1}^4}} \end{bmatrix}$$

We can simplify this by writing

$$\begin{aligned} x^2 &\equiv a_1^2 + a_2^2 + a_3^2 \\ u^2 &\equiv a_1 a_3 \end{aligned} \tag{3.1}$$

And this gives for the eigenvalues:

$$\left[\ \pm\tfrac{1}{2}\sqrt{2x^2 \pm 2\sqrt{x^4 - 4u^4}}\ \right] \tag{3.2}$$

Lemma 3.1 *The a_1 and a_3 can be calculated from a_2, x and u as follows:*

$$\begin{aligned} a_1 &= \frac{1}{2}\left(\epsilon_1\sqrt{x^2 + 2u^2 - a_2^2} + \epsilon_2\sqrt{x^2 - 2u^2 - a_2^2}\right) \\ a_3 &= \frac{1}{2}\left(\epsilon_1\sqrt{x^2 - 2u^2 - a_2^2} - \epsilon_2\sqrt{x^2 - 2u^2 - a_2^2}\right) \end{aligned}$$

where ϵ_1 and ϵ_2 in $\{-1, +1\}$

Proof: Equation 3.1 is equivalent with:

$$(a_1 + a_3)^2 = x^2 + 2u^2 - a_2^2 \qquad \text{and} \qquad (a_1 - a_3)^2 = x^2 - 2u^2 - a_2^2$$

Upon taking the roots this is equivalent to (ϵ_1 and ϵ_2 in $\{-1, +1\}$):

$$a_1 + a_3 = \epsilon_1\sqrt{x^2 + 2u^2 - a_2^2} \qquad \text{and} \qquad a_1 - a_3 = \epsilon_2\sqrt{x^2 - 2u^2 - a_2^2}$$

Now taking the sum and difference of the equations gives the result. ■

Theorem 3.2 *The system given by $A_4(a_1, a_2, a_3)$ (a_i real) is commensurate if and only if: there are integers n_1, n_2 with GCD equal to 1 such that the a_i satisfy:*

$$\begin{aligned} a_1 &= \tfrac{q}{2}\left(\epsilon_1\sqrt{(n_1 + n_2)^2 - s^2} + \epsilon_2\sqrt{(n_1 - n_2)^2 - s^2}\right) \\ a_2 &= qs \\ a_3 &= \tfrac{q}{2}\left(\epsilon_1\sqrt{(n_1 + n_2)^2 - s^2} - \epsilon_2\sqrt{(n_1 - n_2)^2 - s^2}\right) \end{aligned}$$

where ϵ_1 and ϵ_2 in $\{-1, +1\}$, s is an arbitrary real, and q is an arbitrary non-zero real. In this case the the eigenvalues relate to another as $(\pm n_1, \pm n_2)$ and the period T is given by $\frac{2\pi}{q}$.

Proof: From Equation 3.2 we see that the system is commensurate if and only if there are 2 non-negative integers n_i with GCD equal to 1 such that there is a positive T with

$$\begin{aligned} \tfrac{T}{2}\sqrt{2x^2 + 2\sqrt{x^4 - 4u^4}} &= 2\pi n_1 \\ \tfrac{T}{2}\sqrt{2x^2 - 2\sqrt{x^4 - 4u^4}} &= 2\pi n_2 \end{aligned}$$

We now solve for x^2 and u^2. The first is easy. The solution for x^2 is obtained by first squaring the last two equations and then adding them. The solution for u^2 is obtained by first squaring the last two equations and then subtracting them, and then squaring the result again. After that we need to use the solution for x^2 to find the solution for u^2:

$$\begin{aligned} x^2 &= q^2(n_1^2 + n_2^2) \\ u^2 &= q^2 n_1 n_2 \end{aligned}$$

where $q \equiv 2\pi/T$. Now we just apply Lemma 3.1 to obtain the result. ∎

Examples: If $n_1 = 3$ and $n_2 = 2$, we get (among other solutions) that $a_1 = 0.5q\left(\sqrt{25 - s^2} - \sqrt{1 - s^2}\right)$ and $a_3 = 0.5q\left(\sqrt{25 - s^2} + \sqrt{1 - s^2}\right)$. Not all values of s will give real values for the a_i. However all will result in giving eigenvalues with the ratios $\{\pm 2, \pm 3\}$. Notice that the period $T = 2\pi/q$. All this can be verified using MAPLE.

Another interesting example occurs when we choose $n_1 = n_2 = 1$. We are forced to choose $s = 0$. We obtain from the Theorem that $a_1 = a_3 = a$. Substituting this into the matrix indeed gives eigenvalues $\pm a$ (each with multiplicity 2).

We turn to the commensurability of $A_5(a_1, a_2, a_3, a_4)$. Its eigenvalues are given by:

$$0$$

$$\frac{1}{2}\sqrt{2{a_4}^2 + 2{a_3}^2 + 2{a_2}^2 + 2{a_1}^2 + 2\sqrt{{a_4}^4 + 2{a_4}^2{a_3}^2 - 2{a_4}^2{a_2}^2 - 2{a_4}^2{a_1}^2 + {a_3}^4 + 2{a_3}^2{a_2}^2 - 2{a_3}^2{a_1}^2 + {a_2}^4 + 2{a_2}^2{a_1}^2 + {a_1}^4}}$$

$$-\frac{1}{2}\sqrt{2{a_4}^2 + 2{a_3}^2 + 2{a_2}^2 + 2{a_1}^2 + 2\sqrt{{a_4}^4 + 2{a_4}^2{a_3}^2 - 2{a_4}^2{a_2}^2 - 2{a_4}^2{a_1}^2 + {a_3}^4 + 2{a_3}^2{a_2}^2 - 2{a_3}^2{a_1}^2 + {a_2}^4 + 2{a_2}^2{a_1}^2 + {a_1}^4}}$$

$$\frac{1}{2}\sqrt{2{a_4}^2 + 2{a_3}^2 + 2{a_2}^2 + 2{a_1}^2 - 2\sqrt{{a_4}^4 + 2{a_4}^2{a_3}^2 - 2{a_4}^2{a_2}^2 - 2{a_4}^2{a_1}^2 + {a_3}^4 + 2{a_3}^2{a_2}^2 - 2{a_3}^2{a_1}^2 + {a_2}^4 + 2{a_2}^2{a_1}^2 + {a_1}^4}}$$

$$-\frac{1}{2}\sqrt{2{a_4}^2 + 2{a_3}^2 + 2{a_2}^2 + 2{a_1}^2 - 2\sqrt{{a_4}^4 + 2{a_4}^2{a_3}^2 - 2{a_4}^2{a_2}^2 - 2{a_4}^2{a_1}^2 + {a_3}^4 + 2{a_3}^2{a_2}^2 - 2{a_3}^2{a_1}^2 + {a_2}^4 + 2{a_2}^2{a_1}^2 + {a_1}^4}}$$

The reasoning here is almost identical to that in the case of $A_4(a_1, a_2, a_3)$. We start by simplifying the eigenvalues as follows

$$\begin{aligned} x^2 &\equiv a_1^2 + a_2^2 + a_3^2 + a_4^2 \\ u^2 &\equiv \sqrt{a_1^2 a_3^2 + a_1^2 a_4^2 + a_2^2 a_4^2} \end{aligned} \tag{3.3}$$

And this gives rise to the eigenvalues:

$$\left[\ 0, \pm\tfrac{1}{2}\sqrt{2x^2 \pm 2\sqrt{x^4 - 4u^4}}\ \right] \tag{3.4}$$

The inversion in this case is a little trickier.

Lemma 3.3 *Suppose $a_1^2 - a_4^2 \neq 0$. The a_2 and a_3 can be calculated from a_1, a_4, x, and u as follows:*

$$\begin{aligned} a_2^2 &= \frac{(u^4 - a_1^2 a_4^2) - (x^2 - (a_1^2 + a_4^2))a_1^2}{a_4^2 - a_1^2} \\ a_3^2 &= \frac{-(u^4 - a_1^2 a_4^2) + (x^2 - (a_1^2 + a_4^2))a_4^2}{a_4^2 - a_1^2} \end{aligned}$$

Proof: Suppose $a_1^2 - a_4^2 \neq 0$. Rewrite Equation 3.3 as follows:

$$\begin{aligned} a_2^2 + a_3^2 &= x^2 - a_1^2 - a_4^2 \\ a_2^2 a_4^2 + a_3^2 a_1^2 &= u^4 - a_1^2 a_4^2 \end{aligned}$$

By multiplying the first equation by a_4^2, subtracting the two equations, and dividing both sides by $a_4^2 - a_1^2$, we obtain the equation for a_3^2. In a similar way we can get the equation for a_2^2. ■

Theorem 3.4 *The system given by $A_5(a_1, a_2, a_3, a_4)$ (a_i real) is commensurate if and only if:*
<u>If $a_1^2 - a_4^2 \neq 0$:</u> there are integers n_1, n_2 with GCD equal to 1 and real numbers s and t, such the a_i satisfy:

$$\begin{aligned} a_1^2 &= q^2 s^2 \\ a_2^2 &= q^2 \left(\frac{(n_1^2 n_2^2 - s^2 t^2) - (n_1^2 + n_2^2 - (s^2 + t^2)) s^2}{t^2 - s^2} \right) \\ a_3^2 &= q^2 \left(\frac{-(n_1^2 n_2^2 - s^2 t^2) + (n_1^2 + n_2^2 - (s^2 + t^2)) t^2}{t^2 - s^2} \right) \\ a_4^2 &= q^2 t^2 \end{aligned}$$

where s and t are arbitrary reals, and q is an arbitrary non-zero real. In this case the the eigenvalues relate to another as $(\pm n_1, \pm n_2)$ and the period T is given by $\frac{2\pi}{q}$.
<u>If $a_1^2 - a_4^2 = 0$:</u> there are integers n_1, n_2 with GCD equal to 1 such that the a_i satisfy:

$$\begin{aligned} a_1^2 &= q^2 n_1^2 \\ \begin{pmatrix} a_2 \\ a_3 \end{pmatrix} &= \sqrt{q^2 (n_2^2 - n_1^2)}\, R_\phi \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

where q is an arbitrary non-zero real and R_ϕ is a rotation by an arbitrary angle ϕ. In this case the the eigenvalues relate to another as $(\pm n_1, \pm n_2)$ and the period T is given by $\frac{2\pi}{q}$.

Proof: First assume that $a_1^2 - a_4^2 \neq 0$. The same reasoning as in Theorem 3.2 immediately leads to the observation that the system is commensurate if and only if there are 2 non-negative integers n_i with GCD equal to 1 such that there is a positive T with

$$\begin{aligned} x^2 &= q^2 (n_1^2 + n_2^2) \\ u^2 &= q^2 n_1 n_2 \end{aligned}$$

where $q \equiv 2\pi/T$. Now we apply Lemma 3.3 to get the result.

If $a_1^2 = a_4^2 = a^2$ we see that the matrix $A_5(a_1, a_2, a_3, a_1)$ has eigenvalues $\{0, \pm a_1, \pm\sqrt{a_1^2 + a_2^2 + a_3^2}$. A similar reasoning to the above (but simpler), gives the second result. ■

Examples: Note there are other two cases reduce to simpler cases, even though they fall under the first case of the Theorem. If $a_1^2 + a_4^2 = 0$, then of course the matrix is block diagonal (two 0's and a copy of $A_3(a_2, a_3)$), and so Lemma 2.2 applies. If $a_2^2 + a_3^2 = 0$, then the matrix is block diagonal with $A_2(a_1)$, 0, and $A_2(a_4)$ on the diagonal. It is easy to see that the eigenvalues are $\{0(3\times), a_1, a_4\}$.

We conclude with a numerical example: $n_1 = 13$, $n_2 = 5$, $s = 12/10$, $t = 733/100$. If we enter the numbers as quotients of integers in MAPLE, then MAPLE will actually perform integer arithmetic to calculate the eigenvalues. It indeed verifies that the eigenvalues are $\{0, \pm 5, \pm 13\}$.

4 Commensurability of M_7 and M_9

Now we turn our attention to the real, symmetric, nearest neighbor interaction with left-right symmetry acting on $\mathbb{R}^7$ and $\mathbb{R}^7$. It is my understanding that these cases are new to the literature. We will see in the calculations below that the sign of the a_i is irrelevant. So we will assume without loss of generality that $a_i \geq 0$ from hereon out.

We start with M_7:

$$M_7(a_1, a_2, a_3) \equiv \begin{bmatrix} 0 & a_1 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & a_2 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & a_3 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 & a_2 & 0 & a_1 \\ 0 & 0 & 0 & 0 & 0 & a_1 & 0 \end{bmatrix}$$

We calculate the eigenvalues using MAPLE and obtain:

$$\begin{bmatrix} 0 \\ \sqrt{a_2{}^2 + a_1{}^2} \\ -\sqrt{a_2{}^2 + a_1{}^2} \\ \frac{1}{2}\sqrt{4a_3{}^2 + 2a_2{}^2 + 2a_1{}^2 + 2\sqrt{4a_3{}^4 + 4a_3{}^2a_2{}^2 - 4a_3{}^2a_1{}^2 + a_2{}^4 + 2a_2{}^2a_1{}^2 + a_1{}^4}} \\ -\frac{1}{2}\sqrt{4a_3{}^2 + 2a_2{}^2 + 2a_1{}^2 + 2\sqrt{4a_3{}^4 + 4a_3{}^2a_2{}^2 - 4a_3{}^2a_1{}^2 + a_2{}^4 + 2a_2{}^2a_1{}^2 + a_1{}^4}} \\ \frac{1}{2}\sqrt{4a_3{}^2 + 2a_2{}^2 + 2a_1{}^2 - 2\sqrt{4a_3{}^4 + 4a_3{}^2a_2{}^2 - 4a_3{}^2a_1{}^2 + a_2{}^4 + 2a_2{}^2a_1{}^2 + a_1{}^4}} \\ -\frac{1}{2}\sqrt{4a_3{}^2 + 2a_2{}^2 + 2a_1{}^2 - 2\sqrt{4a_3{}^4 + 4a_3{}^2a_2{}^2 - 4a_3{}^2a_1{}^2 + a_2{}^4 + 2a_2{}^2a_1{}^2 + a_1{}^4}} \end{bmatrix}$$

We can simplify this a bit by writing

$$\begin{array}{rcc} x^2 & \equiv & a_1^2 + a_2^2 \\ y^2 & \equiv & a_1^2 + a_2^2 + 2a_3^2 \\ u^2 & \equiv & a_1 a_3 \end{array} \tag{4.1}$$

And this gives for the eigenvalues:

$$\begin{bmatrix} 0 \\ \pm x \\ \pm\frac{1}{2}\sqrt{2y^2 \pm 2\sqrt{y^4 - 8u^4}} \end{bmatrix} \tag{4.2}$$

Lemma 4.1 *If $a_3 \neq 0$, then the a_i^2 can be calculated from x^2, y^2, and u^2 as follows:*

$$\begin{aligned} a_1^2 &= \frac{2u^4}{y^2 - x^2} \\ a_2^2 &= x^2 - \frac{2u^4}{y^2 - x^2} \\ a_3^2 &= \frac{y^2 - x^2}{2} \end{aligned}$$

If $a_3 = 0$ then $u = 0$ and $x^2 = y^2$.

Proof: This proof consists of simply substituting these relations back into Equation 4.1. ■

Theorem 4.2 *The system given by $M_7(a_1, a_2, a_3)$ (a_i real) is commensurate if and only if:*
<u>If $a_3 \neq 0$</u>: there are integers n_1, n_2, and n_3 with GCD equal to 1 such that the a_i satisfy:

$$a_1^2 = q^2 \left(\frac{n_2^2 n_3^2}{n_2^2 + n_3^2 - n_1^2} \right)$$
$$a_2^2 = q^2 \left(n_1^2 - \frac{n_2^2 n_3^2}{n_2^2 + n_3^2 - n_1^2} \right)$$
$$a_3^2 = q^2 \left(\frac{n_2^2 + n_3^2 - n_1^2}{2} \right)$$

where q is an arbitrary (strictly) positive real. In this case the the eigenvalues relate to another as $(0, \pm n_1, \pm n_2, \pm n_3)$ and the period T is given by $\frac{2\pi}{q}$.
<u>If $a_3 = 0$</u>: In this case the system is always periodic with period $T = \frac{2\pi}{\sqrt{a_1^2 + a_2^2}}$.

Proof: We assume that not all coefficients are zero.

The system is commensurate if and only if there are 3 non-negative integers n_i with GCD equal to 1 such that there is a positive T with

$$\begin{aligned} Tx &= 2\pi n_1 \\ \tfrac{T}{2}\sqrt{2y^2 + 2\sqrt{y^4 - 8u^4}} &= 2\pi n_2 \\ \tfrac{T}{2}\sqrt{2y^2 - 2\sqrt{y^4 - 8u^4}} &= 2\pi n_3 \end{aligned}$$

We now solve for x^2, y^2, and u^2. The first is easy. The solution for y^2 is obtained by first squaring the last two equations and then adding them. Finally the solution for u^2 is obtained by first squaring the last two equations and then subtracting them, and then squaring the result again. After that we need to use the solution for y^2 to find the solution for u^2:

$$\begin{aligned} x^2 &= q^2 n_1^2 \\ y^2 &= q^2 (n_2^2 + n_3^2) \\ u^2 &= q^2 \frac{n_2 n_3}{\sqrt{2}} \end{aligned}$$

where $q \equiv 2\pi/T$. Now if $a_3 \neq 0$ we just apply Lemma 4.1 to obtain the result.

If $a_3 = 0$ the matrix M_7 is in fact block diagonal, with two blocks equal to $A_3(a_1, a_2)$ and one block equal to 0. It follows from Lemma 2.1 that the eigenvalues are 0 (with multiplicity 3), $\pm\sqrt{a_1^2 + a_2^2}$ (each with multiplicity 2). The conclusion follows immediately. ■

Remark: The method we employ (squaring repeatedly) does not work for M_{11}. Those eigenvalues have cubic roots in them. But it just might work for M_9 with a little more effort.

Examples: One of the eigenvalues of M_7 must be zero (because its determinant is zero). We choose $\{n_1, n_2, n_3\} = \{1, 2, 3\}$. This means that the absolute values of the eigenvalues have ratios $0 : 1 : 2 : 3$. To make sure that all a_i^2 in the theorem are positive we must choose $n_1 = 2$. The expressions are invariant under $n_2 \leftrightarrow n_3$, so we choose $(n_1, n_2, n_3) = (2, 1, 3)$. This gives $(a_1^2, a_2^2, a_3^2) = q^2(\frac{3}{2}, \frac{5}{2}, 3)$. Direct verification (using MAPLE) indeed shows that the eigenvalues of the resulting matrix have the required ratio. Notice that the period is given by $2\pi/q$.

We try another example, namely all ratios n_i are equal to 1. Now we get from the theorem that $(a_1^2, a_2^2, a_3^2) = q^2(1, 0, \frac{1}{2})$ and the period is $2\pi/q$. Again checking independently by MAPLE bears this out.

Here is an unusual example. We choose the eigenvalue ratios $(n_1, n_2, n_3) = (10, 1, 100)$. The theorem gives that we have to set $(a_1^2, a_2^2, a_3^2) = q^2(\frac{10000}{9901}, 100 - \frac{10000}{9901}, \frac{9901}{2})$ to get these ratios. We now set $q = 10$. The theorem also gives that the period for these values of the period T is $2\pi/10$. Again, a quick MAPLE calculation confirms both conclusions.

Let us look at the real, symmetric, nearest neighbor interaction with left-right symmetry acting on $\mathbb{R}^9$:

$$M_9(a_1, a_2, a_3, a_4) \equiv \begin{bmatrix} 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & a_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & a_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_4 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_3 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 & a_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 \end{bmatrix}$$

We need some notation to write the eigenvalues is a meaningful way:

$$\begin{array}{rcl} x^2 & \equiv & a_1^2 + a_2^2 + a_3^2 \\ y^2 & \equiv & a_1^2 + a_2^2 + a_3^2 + 2a_4^2 \\ u^4 & \equiv & a_1^2 a_3^2 \\ v^4 & \equiv & a_1^2 a_3^2 + 2a_1^2 a_4^2 + 2a_2^2 a_4^2 \end{array} \tag{4.3}$$

We will need the following lemma.

Lemma 4.3 *If a_4 and a_3 are not zero, then the a_i^2 can be calculated from x^2, y^2, u^2, and v^2 as follows:*

$$\begin{aligned}
a_1^2 &= \frac{u^4(y^2-x^2)}{x^2(y^2-x^2)-(v^4-u^4)} \\
a_2^2 &= \frac{v^4-u^4}{y^2-x^2} - \frac{u^4(y^2-x^2)}{x^2(y^2-x^2)-(v^4-u^4)} \\
a_3^2 &= \frac{x^2(y^2-x^2)-(v^4-u^4)}{(y^2-x^2)} \\
a_4^2 &= \frac{y^2-x^2}{2}
\end{aligned}$$

If $a_3 = 0$ then $u = 0$. On the other hand if $a_4 = 0$, then $x^2 = y^2$ and $u^2 = v^2$.

Proof: This proof consists mostly of simply substituting the relations back into Equation 4.3, which is easy to do. The first step is to obtain a_4 from the difference of y^2 and x^2. Then we express v^4 as $u^4 + 2(a_1^2 + a_2^2)a_4^2$. This gives a relation for $a_1^2 + a_2^2$ *provided* $a_4 \neq 0$. Using the relation for x^2 again, this gives an equation for a_3^2. Then a_1^2 is obtained from the relation for u^4 *provided* $a_4 \neq 0$. Subtracting this from the relation we obtained earlier for $a_1^2 + a_2^2$. Notice that this inversion works iff a_4 and a_3 are not zero (both). ∎

Using MAPLE one verifies that the 9 eigenvalues of M_9 are given by:

$$\begin{array}{c}
0 \\
\pm\frac{1}{2}\sqrt{2x^2 \pm 2\sqrt{x^4-4u^4}} \\
\pm\frac{1}{2}\sqrt{2y^2 \pm 2\sqrt{y^4-4v^4}}
\end{array} \tag{4.4}$$

Theorem 4.4 *The system given by $M_9(a_1, a_2, a_3, a_4)$ (a_i real) is commensurate if and only if:*
1. If $a_3 \neq 0$ and $a_4 \neq 0$: there are integers n_1, n_2, n_3, and n_4 with GCD equal to 1 such that the a_i satisfy:

$$a_1^2 = q^2\left(\frac{n_1^2n_2^2(n_3^2+n_4^2-n_1^2-n_2^2)}{(n_1^2+n_2^2)(n_3^2+n_4^2-n_1^2-n_2^2)-(n_3^2n_4^2-n_1^2n_2^2)}\right)$$

$$a_2^2 = q^2\left(\frac{n_3^2n_4^2-n_1^2n_2^2}{n_3^2+n_4^2-n_1^2-n_2^2} - \frac{n_1^2n_2^2(n_3^2+n_4^2-n_1^2-n_2^2)}{(n_1^2+n_2^2)(n_3^2+n_4^2-n_1^2-n_2^2)-(n_3^2n_4^2-n_1^2n_2^2)}\right)$$

$$a_3^2 = q^2\left(\frac{(n_1^2+n_2^2)(n_3^2+n_4^2-n_1^2-n_2^2)-(n_3^2n_4^2-n_1^2n_2^2)}{(n_3^2+n_4^2-n_1^2-n_2^2)}\right)$$

$$a_4^2 = q^2\left(\frac{n_3^2+n_4^2-n_1^2-n_2^2}{2}\right)$$

where q is an arbitrary (strictly) positive real. In this case the the eigenvalues relate to another as $(0, \pm n_1, \pm n_2, \pm n_3, \pm n_4)$ and the period T is given by $\frac{2\pi}{q}$.
2. If $a_4 = 0$: The matrix consists of three diagonal blocks: two are equal to $A_4(a_1, a_2, a_3)$ (see Theorem 3.2 and the third block is the number 0.
3. If $a_3 = 0$: Also here there are three diagonal blocks, namely $A_3(a_1, a_2)$, $A_3(a_4, a_4)$, and $A_3(a_2, a_1)$ (see Lemma 2.1). These are commensurate iff $\sqrt{a_1^2 + a_2^2}$ and $\sqrt{2a_4^2}$ are commensurate.

Proof: We assume that not all coefficients are zero.

As before, the system is commensurate if and only if there are 4 non-negative integers n_i with GCD equal to 1 such that there is a positive T with

$$\begin{aligned} \tfrac{T}{2}\sqrt{2x^2+2\sqrt{x^4-4u^4}} &= 2\pi n_1 \\ \tfrac{T}{2}\sqrt{2x^2-2\sqrt{x^4-4u^4}} &= 2\pi n_2 \\ \tfrac{T}{2}\sqrt{2y^2+2\sqrt{y^4-4v^4}} &= 2\pi n_3 \\ \tfrac{T}{2}\sqrt{2y^2-2\sqrt{y^4-4v^4}} &= 2\pi n_4 \end{aligned}$$

Using the same strategy as in the proof of Theorem 4.2, we can easily solve for x, y, u, and v:

$$\begin{aligned} x^2 &= q^2(n_1^2+n_2^2) \\ y^2 &= q^2(n_3^2+n_4^2) \\ u^2 &= q^2 n_1 n_2 \\ v^2 &= q^2 n_3 n_4 \end{aligned}$$

where q equals $2\pi/T$. If $a_3 \neq 0$ and $a_4 \neq 0$, the stament follows directly from substituting these relations into Lemma 4.3.

When $a_4 = 0$, M_9 is block-diagonal, and this case thus follows from the results in Section 3. In this case it has two blocks of the form A_4. Similarly when $a_3 = 0$, the matrix M_9 has three diagonal block of the form A_3 and again the results of Section 3 apply. ■

Examples: One of the eigenvalues of M_9 must be zero. Suppose for the others we desire the Fibonacci ratios $\{5, 8, 13, 21\}$. Substitute all permutations of these values into the Theorem until a permutation gives positive values for the a_i^2: $(n_1, n_2, n_3, n_4) = (5, 13, 8, 21)$. The values given by the theorem for a_i^2 are, respectively : $q^2(\frac{1555}{43}, \frac{548352}{13373}, \frac{36335}{311}, \frac{311}{2})$. Furthermore if we choose $q = 1$, the period T equals 2π. Both conclusions are of course easily verified using MAPLE.

Now we try $(n_1, n_2, n_3, n_4) = (1, 1, 1, 1)$. Interestingly we find that a_1, a_2, are a_3 undefined. The reason becomes clear as we calculate a_4: it is zero! So in this case the second part of the theorem applies. In the examples pertaining to $A_4((a_1, a_2, a_3)$ we see that it is indeed possible to get the eigenvalue ratios ± 1. So it turns out we need to choose $a_4 = a_2 = 0$ and $a_1 = a_3 \neq 0$ to obtain four eigenvalues equal in modulus.

References

[1] J. Petrovic, J. J. P. Veerman, *A New Method for Multi-Bit and Qudit Transfer Based on Commensurate Waveguide Arrays*, Submitted.