Simplified “Proof” of Singer’s Theorem

J. J. P. Veerman*

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1) First, prove the “chain rule for the Schwarzian”:

$$S(f \circ g)(x) = Sf|_{g(x)} \cdot Dg(x)^2 + Sg(x) \quad .$$

Conclude that if f and g have negative Schwarzian, so does $f \circ g$. In particular, if f has negative Schwarzian, then so do all iterates of f.

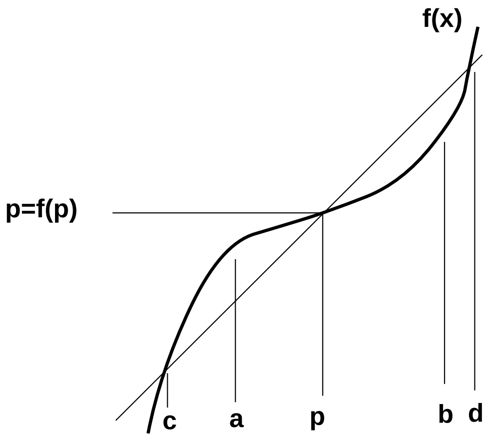
Remarks: The Schwarzian crops up in unexpected contexts. Check for example `http://en.wikipedia.org/wiki/Schwarzian`. The fact that only fractional linear transformations have Schwarzian 0 means that it is an important in hyperbolic geometry. If you calculate the Schwarzian of the unimodal map $g(x) = -|x|^\alpha + c$, where $\alpha \in \mathbb{R}^+$, you’ll find that the Schwarzian is negative for $\alpha > 1$ (when the function g is at least once differentiable at the singularity) and positive in the “pointy” case where $\alpha \in (0, 1)$ (where it’s not). However, the property of having a negative Schwarzian is *not* an invariant under smooth coordinate changes (=differentiable conjugacy): a good substitute that is invariant under coordinate change has, to my knowledge, not been found to date.

Theorem 0.1 *The map $f : \mathbb{R} \to \mathbb{R}$ is C^3 and has negative Schwarzian. Let p be a periodic point (possibly a fixed point). Then either:*
- p has an infinite basin of attraction,
- there is a critical point in p’s basin, or
- p is a repeller.

2) *Simplified “Proof” of Singer’s Theorem for fixed points.* Assume p is an attracting fixed point of a unimodal g where g has negative Schwarzian (except that it is undefined at the critical point). Note that if $g(x) \equiv ax(1 - x)$ has a finite attractor (ie: not counting ∞), then its basin must be finite.
Check the following facts (you can use the figures below).
a) The attractive fixed point of g must be a fixed point of $f \equiv g^2$ with derivative in $(0, 1)$. (If it’s 0, then p is a critical point and we are done.
b) Assume there is no critical point in the basin of f, then f must cross the diagonal as in the figure at points c and d (see figure).
c) So (by the Intermediate Value Theorem) there are points a and b in the BOA or Basin Of Attraction, where the derivative of f equals 1. See figure. Thus f', being smooth, must have a local minimum in (a, b) (because $f'(p) \in (0, 1)$).
d) That local minimum is called m (see figure). By assumption m is not critical, and so given the fact that Sg is negative (previous exercise), and that $f''(m) = 0$ (calculus), $f'''(m)$ must be positive.
e) Using that $Sf < 0$, we see that $f'(m) < 0$.
f) Using the Intermediate Value Theorem (again), there is a point c between m and p (see second figure) where the derivative of f equals 0. This point c is in the basin of attraction.
g) The fact that $f'(c) = \frac{d}{dx} g^2(c) = 0$ means that (use the chain rule for derivatives) that the critical point of f is attracted to the fixed point p.

3) Now make sure you understand this works for periodic orbits as well. You need to use, of course, item 1 above.

*e-mail: veerman@pdx.edu



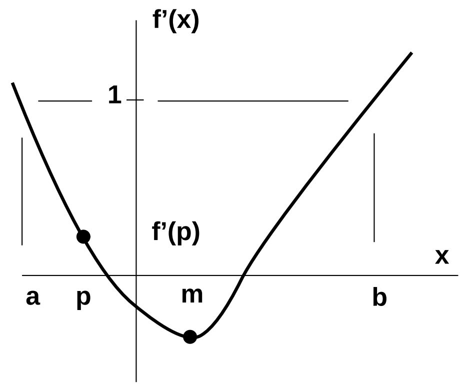


Figure 0.1: *The first figure depicts $f = g^2$ with the attracting fixed point p, which must have positive derivative smaller than 1, together with the diagonal $y = x$. The second is a rendering of $f'(x)$. The minimum m of f' and the fixed point p of f are indicated by solid dots.*

Remarks: The generalization of this for complex analytic maps in the complex plane also works, but makes use of some more advanced complex analysis. I am talking about the map in the complex plane $z \longrightarrow -z^2 + c$. The pictures you have seen of Julia Sets and the Mandelbrot Set are really based on these ideas, as I explained in class. Excellent references are Devaney's *A First Course in Chaotic Dynamical Systems* and his slightly more advanced *An Introduction to Chaotic Dynamical Systems*. He also has nice computer programs an beautiful pictures of the Mandelbrot and Julia Sets in these books. If you want to look at whacky pictures and a serious effort to generalize these ideas to non-analytic maps, please go to our paper:
`http://web.pdx.edu/~veerman/mandelbug.pdf` . A completely different direction was taken by people who tried tried extend "complex analytic dynamics" to several variables. Ask me for references case you are interested.