On (Semi-)Conjugacies

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October 20, 2020

1) (See also *Chaos,...* by Yorke e.a., Springer, 1996, section 3.3.) Let $S^1$ be the set $\{z \in \mathbb{C} \mid |z| = 1\}$ in the complex plane. Denote by $I$ the interval $[-1, 1]$ and $J$ is the interval $[0, 1]$. Consider the commuting diagram below:

$$
\begin{array}{cccc}
\mathbb{R}/\mathbb{Z} & \overset{\text{phase}}{\longleftarrow} & S^1 & \overset{\mathbb{R}}{\longrightarrow} & [-1, 1] & \overset{h}{\longrightarrow} & [0, 1] & \overset{C}{\longleftarrow} & [0, 1] \\
p & \uparrow & q & \uparrow & s & \uparrow & f & \uparrow & t \\
\mathbb{R}/\mathbb{Z} & \overset{\text{phase}}{\longleftarrow} & S^1 & \overset{\mathbb{R}}{\longrightarrow} & [-1, 1] & \overset{h}{\longrightarrow} & [0, 1] & \overset{C}{\longleftarrow} & [0, 1]
\end{array}
$$

Now define the following maps on the domains specified in that diagram:

- $\text{phase}(e^{2\pi i \phi}) = \phi$
- $q(z) = z^2$
- $\mathbb{R}(z)$ is the real part of $z$
- $h : I \to J$ is affine, orientation reversing, and onto
- $C(x) = \frac{1}{2}(1 - \cos \pi x)$
- $T(x) = \begin{cases} 
2x & \text{if } x \in [0, 1/2] \\
2(1 - x) & \text{if } x \in [1/2, 1]
\end{cases}$

a) Calculate $p(\phi)$. (*Hint: the answer is phase-doubling.*)
b) Calculate $g(x)$. (*Hint: write $z = x + iy$ in the complex plane and thus the real part of $z^2$ is $x^2 - y^2$. So if $z = e^{it}$ then $Re(z^2) = \cos^2 t - \sin^2 t$.*)
c) Calculate $f(x)$.
d) Show that $t$ is the tent map $T$. (*Hint: use hint in b) cleverly.*)

2) Note that the projection from the unit circle onto the x-axis given by map $z \to \mathbb{R}(z)$ is 2 to 1. Thus even though all the other coordinate changes are invertible, $p$ and $t$ are not conjugate.

a) Check that reasoning.
b) Also verify it by counting the fixed points of each map.
c) Show that two maps that are topologically conjugate *must* have the same number of fixed points.

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