On (Semi-)Conjugacies

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1) (See also *Chaos,...* by Yorke e.a., Springer, 1996, section 3.3.) Let S^1 be the set $\{z \in \mathbb{C} \,|\, |z| = 1\}$ in the complex plane. Denote by I is the interval $[-1, 1]$ and J is the interval $[0, 1]$. Consider the commuting diagram below:

$$\begin{array}{ccccccccc}
\mathbb{R}/\mathbb{Z} & \xleftarrow{phase} & S^1 & \xrightarrow{\Re} & [-1,1] & \xrightarrow{h} & [0,1] & \xleftarrow{C} & [0,1] \\
p\uparrow & & q\uparrow & & g\uparrow & & f\uparrow & & \uparrow t \\
\mathbb{R}/\mathbb{Z} & \xleftarrow[phase]{} & S^1 & \xrightarrow[\Re]{} & [-1,1] & \xrightarrow[h]{} & [0,1] & \xleftarrow[C]{} & [0,1]
\end{array}$$

Now define the following maps on the domains specified in that diagram:

$$\begin{aligned}
&\text{phase}(e^{2\pi i\phi}) = \phi \\
&q(z) = z^2 \\
&\Re(z) \quad \text{is the real part of } z \\
&h : I \to J \quad \text{is affine, orientation reversing, and onto} \\
&C(x) = \tfrac{1}{2}(1 - \cos \pi x) \\
&T(x) = \begin{cases} 2x & \text{if} \quad x \in [0, 1/2] \\ 2(1-x) & \text{if} \quad x \in [1/2, 1] \end{cases}
\end{aligned}$$

a) Calculate $p(\phi)$. (*Hint: the answer is phase-doubling.*)
b) Calculate $g(x)$. (*Hint: write* $z = x + iy$ *in the complex plane and thus the real part of* z^2 *is* $x^2 - y^2$*. So if* $z = e^{it}$ *then* $Re(z^2) = \cos^2 t - sin^2 t$*.*)
c) Calculate $f(x)$.
d) Show that t is the tent map T. (*Hint: use hint in b) cleverly.*)

2) Note that the projection from the unit circle onto the x-axis given by map $z \to \Re(z)$ is 2 to 1. Thus even though all the other coordinate changes are invertible, p and t are *not* conjugate.
a) Check that reasoning.
b) Also verify it by counting the fixed points of each map.
c) Show that two maps that are topologically conjugate *must* have the same number of fixed points.

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