Summary of Vector Analysis in Calculus V

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$$\text{0forms} \underset{\text{grad}}{\overset{\text{d}}{\longrightarrow}} \text{1forms} \underset{\text{curl}}{\overset{\text{d}}{\longrightarrow}} \text{2forms} \underset{\text{div}}{\overset{\text{d}}{\longrightarrow}} \text{3forms}$$

Dealing with vector analysis, formulas sometimes get very long and cumbersome. To help us some, we start by shortening the notation. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function, then the three components of the argument of f will be called x_1, x_2, and x_3. The partial derivatives with respect to, say, x_2 at the point (a_1, a_2, a_3) is thus given by:

$$\frac{\partial f(a_1, a_2, a_3)}{\partial x_2}.$$

This will be simplified if we replace (a_1, a_2, a_3) by $(\mathbf{a})$ and for each $i \in \{1, 2, 3\}$ we write $\frac{\partial f}{\partial x_i}$ by ∂_i. The above expression can now be shortened to:

$$\partial_2 f(\mathbf{a}).$$

Similarly, the components of a smooth vector field $\mathbf{F}$ will be denoted by F_i, where i is equal to 1, 2, or 3. Finally, we will also leave out the **bold** notation, since, hopefully there will be no confusion between points in $\mathbb{R}^3$ and components of the coordinates of those points.

First, let us review the basic notion of a differential form. We will suppose all functions we deal with are sufficiently smooth (usually continuous second derivatives suffices). Start with a function $f : \mathbb{R}^3 \to \mathbb{R}$. Such a function is also called a 0-form. The gradient of a function f, or a 0-form, is a vector field, or a 1-form. "Differentiating" f in this setting is called "calculating df" or, with more sophistication, calculating or taking its exterior derivative. (See the formula given at the heading of these notes.)

The Gradient.

The gradient ([3] [Section 16.1]) is given by:

$$df = \partial_1 f \, dx_1 + \partial_2 f \, dx_2 + \partial_3 f \, dx_3 \, . \tag{0.1}$$

In vector-form this is sometimes written as

$$\nabla f = \begin{pmatrix} \partial_1 f \\ \partial_2 f \\ \partial_3 f \end{pmatrix} \, .$$

To deal with integrals, it is more convenient to write a gradient as a 1-form. The 1-form is what is needed in a line integral: If $\int_c h$ is a line integral along the curve c, then h is a 1-form.

How to Compute the Exterior Derivative.

Now suppose we are given a vector field $g = (g_1, g_2, g_3)$. Notice that each of the three components are like functions on $\mathbb{R}^3$. It will be easier if we think of this as a 1-form $g = g_1 \, dx_1 + g_2 \, dx_2 + g_3 \, dx_3$. The differential or 'exterior derivative' of this 1-form will be a 2-form. So,

$$dg = dg_1 \, dx_1 + dg_2 \, dx_2 + dg_3 \, dx_3 \, .$$

Calculating as before, for the first component we have

$$dg_1 = \partial_1 g_1 \, dx_1 + \partial_2 g_1 \, dx_2 + \partial_3 g_1 \, dx_3 \, .$$

We substitute this and similar expressions for the second and third components of g back into the equation. Just doing it for the first component gives:

$$dg = (\partial_1 g_1 \, dx_1 + \partial_2 g_1 \, dx_2 + \partial_3 g_1 \, dx_3) dx_1 + dg_2 \, dx_2 + dg_3 \, dx_3 \, .$$

So now we have worked out dg_1, but the terms dg_2 and dg_3 remain as they were. Let's first see what happens to the terms that we worked out so far.

Rules for Exterior Differentiation.

To proceed we need a few rules. We calculate using the rules of normal differentiation plus a few others, that we will mention in passing.

Rule 1: The product of $dx_1 dx_1$ is zero. The same holds for $dx_i dx_i$ for any i.

Rule 2: Interchanging dx_i and dx_j results in multiplication by -1. So, $dx_2 dx_1 = -dx_1 dx_2$.

Both rules are very reasonable. In $\mathbb{R}^3$, one can think of dx_i as an infinitesimal vector or a tangent vector, as depicted in Figure 0.1. Expressions like $dx_1 dx_2$ or $dx_1 dx_2 dx_3$ are then to be thought of as determinants whose columns are given by the vectors dx_i, that is: little oriented

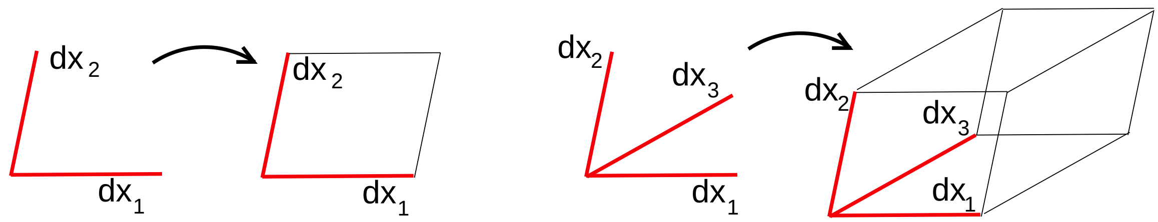

Figure 0.1: *Left: two tangent vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ and the oriented surface area they represent. Right: the same for three tangent vectors and the oriented volume they represent.*

volumes. See Figure 0.1. Thus $dx_1\, dx_2$ forms a little surface area and $dx_1dx_2dx_3$ a volume. In linear algebra, oriented n-dimensional volumes are represented by determinants. And the same holds for expressions like $dx_i \cdots dx_j$ in $\mathbb{R}^n$. As in linear algebra, the determinant of matrix with 2 identical columns is zero, which explains rule 1.

Similarly, swapping two columns in a determinant causes its determinant to be multiplied by -1. This explains rule number 2. Looking at it that way, we see that $f(x)\, dx_i\, dx_i = -f(x)\, dx_i\, dx_i$ by rule 2, and so $f(x)\, dx_i\, dx_i$ must be zero. Hence, rule 2, together with the idea that these expressions behave like determinants, implies rule 1.

Back to our Computation.

The above rules give that $\partial_1 g_1\, dx_1dx_1 = 0$ and $\partial_2 g_1\, dx_2dx_1 = -\partial_2 g_1\, dx_1dx_2$ and so

$$dg = (-\partial_2 g_1\, dx_1\, dx_2 + \partial_3 g_1\, dx_3\, dx_1) + dg_2\, dx_2 + dg_3\, dx_3\,.$$

We treat the remaining terms $dg_2\, dx_2$ and $dg_3\, dx_3$ in exactly the same way.

Exercise 1. *Complete this calculation and show that it gives:*

$$dg = (\partial_2 g_3 - \partial_3 g_2)dx_2\, dx_3 + (\partial_3 g_1 - \partial_1 g_3)dx_3\, dx_1 + (\partial_1 g_2 - \partial_2 g_1)dx_1\, dx_2\,. \tag{0.2}$$

What we obtained in (0.2) is a 2-form because it is appropriate for calculating surface integrals. Such a 2-form has the following general form:

$$h = h_1\, dx_2\, dx_3 + h_2\, dx_3\, dx_1 + h_3\, dx_1\, dx_2\,.$$

We can interpret (h_1, h_2, h_3) as a vector field. Thus h_1 is orthogonal to the surface element dx_2dx_3, h_2 is orthogonal to dx_3dx_1, and h_3 to dx_1dx_2. Thus, we can summarize the expression as $h \cdot dS$ where dS is the vector orthogonal to the surface.

There is one more thing. Note that we had to make choices. Do we list the first term in (0.2) as $(\partial_2 g_3 - \partial_3 g_2)dx_2\, dx_3$ or as $-(\partial_2 g_3 - \partial_3 g_2)dx_3\, dx_2$? Answer: the first. Here is why. In $\mathbb{R}^3$, we always use the convention known as the right-hand rule [3] [Sections 12.2 and 12.4]. If

h_1 is positive, it must represent a vector in the direction of $dx_2 \times dx_3$ where $\times$ represents the usual cross-product ([3] [Section 12.4]).

Note that the first term ($h_1\, dx_2\, dx_3$) consists of three factors, the first of whose indices is a 1. The first index in the second term is a 2, and so forth. The successive indices in one term are obtained by counting *forward* and setting 4=1. For example, $h_{\underline{\mathbf{3}}}\, dx_{\underline{\mathbf{1}}} dx_{\underline{\mathbf{2}}}$. Thus the groups of indices in the different terms are *cyclic* permutations of $(1, 2, 3)$.

The Curl.

You can easily recognize the formula for the curl ([3] [Section 16.1]) of a vector field in (0.2). In vector form the curl of g is usually written as

$$\nabla \times g = \begin{pmatrix} \partial_2 g_3 - \partial_3 g_2 \\ \partial_3 g_1 - \partial_1 g_3 \\ \partial_1 g_2 - \partial_2 g_1 \end{pmatrix} .$$

The Divergence.

Now let us start with the above 2-form and differentiate it, just as we did with the other ones, we arrive at a 3-form. We obtain a 3-form which is called the divergence ([3] [Section 16.1])of h:

$$dh = (\partial_1 h_1 + \partial_2 h_2 + \partial_3 h_3)\, dx_1\, dx_2\, dx_3\, . \tag{0.3}$$

In vector notation this becomes:

$$\nabla \cdot h = \partial_1 h_1 + \partial_2 h_2 + \partial_3 h_3\, .$$

k-cells.

With all of this in mind, many complicated theorems can be handily summarized. They just

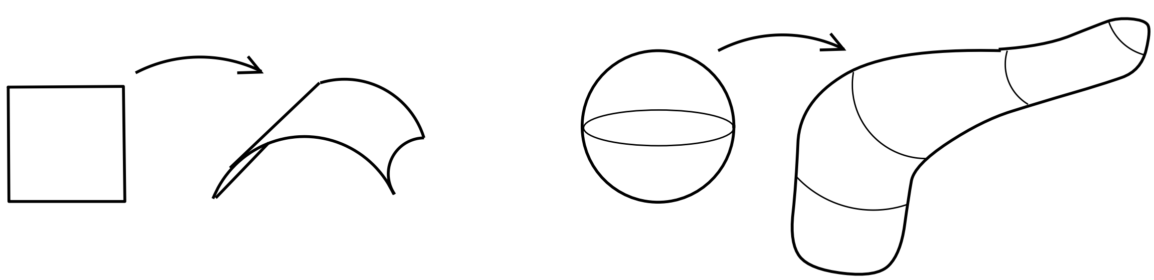

Figure 0.2: *Right: a nice map from the square to a surface is a 2-cell. Left: a nice map from a solid ball to a volume is a 3-cell. See Definition 0.1.*

boil down to 3 theorems. We state them a little bit more generally than needed in the course, just in case you run into these things in the future. You should think of n and ℓ as typically, but not necessarily, being 2 or 3. We first need another definition.

Definition 0.1 *A set U in R^n is a k-cell if it is the image of a "nice"*[1] *map from a k-dimensional ball or cube into $\mathbb{R}^n$. Intuitively, a k-cell is a k-dimensional shape that has no holes of any kind.*

Any ball or cube (their interiors included) in $\mathbb{R}^n$ is an n-cell. In $\mathbb{R}^n$, the complement of a ball is not an n-cell. In $\mathbb{R}^3$, the surface of a doughnut (or torus) is not a 2-cell, and neither is the sphere is.

Exercise 2. *Show that $\mathbb{R}^2$ (or $\mathbb{R}^3$) itself can be considered an k-cell. (Just map the point (r, ϕ) in the unit disk with polar coordinates to $(r/(1-r), \phi)$. This maps the open unit disk in $\mathbb{R}^2$ to all of $\mathbb{R}^2$. Do the same for $\mathbb{R}^3$.)*

$d^2 = 0$ Always.

At first, this heading seems very strange. But it turns out it is related to the fact that for any "regular" surface or volume in $\mathbb{R}^3$, the boundary of the boundary is empty. In fact, the same is true for a regular volume in any dimension. We give two examples in Figure 0.3.

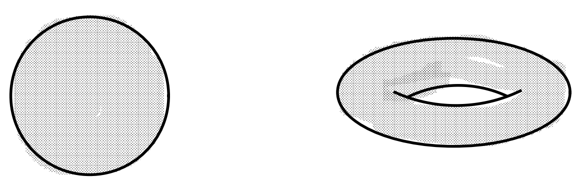

Figure 0.3: *Left: the boundary of the disk D is a circle $C = \partial D$, but the boundary of C is empty. Right: the same for the solid torus whose boundary is a torus (surface), but the boundary of the torus is empty.*

A 3-form in $\mathbb{R}^3$ has the following general form:

$$k = k\, dx_1\, dx_2\, dx_3\,.$$

In $\mathbb{R}^3$ we cannot get a 4-form, since any 4-form must have four distinct dx'es which is impossible in $\mathbb{R}^3$. (But it is possible in dimension greater than three!) So for any 2- or 3-form h in $\mathbb{R}^3$, we certainly have that $d^2h = 0$. More generally, we have the following result.

Theorem 0.2 *If h is a smooth ℓ-form in $\mathbb{R}^n$. Then $d^2h = 0$.*

[1] By "nice", I mean differentiable and with differentiable inverse. But the point of these notes is more to get the intuition across than to convey the precise definitions. So "nice" will do for these notes.

This theorem can be easily checked for the cases discussed here (in $\mathbb{R}^3$). As noted above, we only need to check it for 0-forms and 1-forms. Starting with the forme, the gradient or exterior derivative of a 0-form $f(x)$ is

$$\partial_1 f\, dx_1 + \partial_2 f\, dx_2 + \partial_3 f\, dx_3\,.$$

and exterior derivative (curl) of this is

$$(\partial_2\partial_3 f - \partial_3\partial_2 f)dx_2\, dx_3 + (\partial_3\partial_1 f - \partial_1\partial_3 f)dx_3\, dx_1 + (\partial_1\partial_2 f - \partial_2\partial_1 f)dx_1\, dx_2\,.$$

You can see easily that then the whole expression is zero since, by Clairaut's Theorem, $\partial_i\partial_j f = \partial_j\partial_i f$ ([3] [Section 14.3]).

On the other hand, if you start with a 1-form $g_1\, dx_1 + g_2\, dx_2 + g_3\, dx_3$, you take the curl first, and then you add up the ∂_i of the ith components of the curl, you get

$$[\partial_1(\partial_2 g_3 - \partial_3 g_2) + \partial_2(\partial_3 g_1 - \partial_1 g_3) + \partial_3(\partial_1 g_2 - \partial_2 g_1)]\, dx_1 dx_2 dx_3\,.$$

Using Clairaut again, we see that this also vanishes.

Exercise 2. *For a smooth k-form h in $\mathbb{R}^3$, show that $d^2 h = 0$. First for 2- and 3-forms, then do the above computation explicitly for 0- and 1-forms.*

The Two Main Results.

The general proof of these main results are more complicated and we will not give it here. We just remark that [3] gives separate proofs for each individual case that can occur in $\mathbb{R}^3$. However, with a little more background *one* proof suffices; not just for all cases in $\mathbb{R}^3$ but in any dimension.

Theorem 0.3 *Suppose that h is a smooth ℓ-form defined on an ℓ-cell S in $\mathbb{R}^n$ for non-negative integers $\ell < n$. If $dh = 0$, then h is the exterior derivative of an $(\ell - 1)$-form f, that is: there exists an f on S such that $df = h$.*

Theorem 0.4 (Stokes' Formula) *(Gauss-Green-Kelvin-Ostrogradski-Stokes) Suppose h is an ℓ-form on $\mathbb{R}^n$ and the $(\ell + 1)$-form dh is its exterior derivative. Suppose further that S is a closed ℓ-cell in $\mathbb{R}^n$ with boundary ∂S, then we have*

$$\int_S dh = \int_{\partial S} h\,.$$

These are the three principles discussed in Calculus V. The theorems are often used in conjunction, as we will now discuss by way of examples.

Integrals in $\mathbb{R}^1$.

Suppose that g is a smooth function on an interval $I = [a, b]$. We consider the 1-form $g(x)\,dx$ in $\mathbb{R}^1$. Since $dx\,dx = 0$, the derivative of that form is zero, and so, by Theorem 0.3, there is another function f so that

$$df = g(x)\,dx\,.$$

Supposing we can find that function f, then by Stokes (Theorem 0.4),

$$\int_a^b g(x)\,dx = f(b) - f(a)\,. \tag{0.4}$$

The left side of this equation is the 1-form $df = g\,dx$ integrated over the interval $I = [a, b]$. The right hand side is the "integral" over the 0-form f over the boundary ∂I of I. The boundary of I, of course, consists of the points a and b. The "integral" of a 0-form f over ∂I is defined as $f(b) - f(a)$.

Simple Line integrals in $\mathbb{R}^n$.

Let us start with a line integral, and suppose we want to calculate $\int_c \mathbf{F}\cdot d\mathbf{r}$ where c is some curve connecting a and b. What we do is the following: Find out if $\nabla \times \mathbf{F}$ is zero. If it is, then, by Theorem 0.3, there is a function or 0-form f so that $\nabla f = F$. Namely $g = \mathbf{F}\cdot d\mathbf{r}$ is a 1-form. So dg corresponds to taking the curl. If it is zero, there is a 0-form f so that $df = g$. Since f is a 0-form, its 'd' corresponds to the gradient. Supposing we can find this function or 0-form f, the integral can be evaluated as before:

$$\int_c \mathbf{F}\cdot d\mathbf{r} = f(b) - f(a)\,.$$

(Recall that c goes from a to b.) Notice that the value of the integral does not depend on what path you took to get from a to b!

Boundary integrals in $\mathbb{R}^2$.

Suppose now that c is a curve enclosing a cell S, so that $\partial S = c$. See Figure 0.4. Suppose we wish to evaluate the integral $I = \int_c f_1\,dx_1 + f_2\,dx_2$. Denote the 1-form $f_1\,dx_1 + f_2\,dx_2$ by h, then by Theorem 0.4

$$\int_{\partial S} h = \int_S dh \qquad \text{or} \qquad \int_c f_1\,dx_1 + f_2\,dx_2 = \int_S (\partial_1 f_2 - \partial_2 f_1)\,dx_1 dx_2\,,$$

because from the rules given before, we derive

$$d(f_1\,dx_1 + f_2\,dx_2) = (\partial_1 f_2 - \partial_2 f_1)\,dx_1 dx_2\,.$$

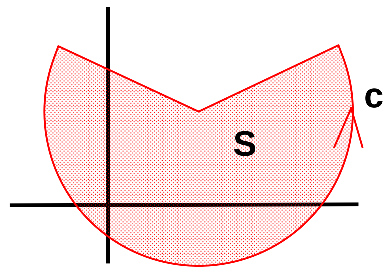

Figure 0.4: *The curve c encloses a 2-cell S in* $\mathbb{R}^2$.

Interlude.

These three results really tell the whole story. Nothing changes in higher dimension, except how you write out the details of differential forms and their derivatives in any given set of coordinates. So say you want to compute the exterior derivative of a 2-form k in $\mathbb{R}^3$. It can can be written as

$$k = f_1\, dx_2 dx_3 + f_2\, dx_3 dx_1 + f_3\, dx_1 dx_2\,.$$

Since it has 3 components, we naturally think of it as a vector field $f = (f_1, f_2, f_3)$. Computing the exterior derivative is the same as computing the curl of the vector field f. Any 3-form in $\mathbb{R}^3$, however, can be written as

$$k = f\, dx_1 dx_2 dx_3\,,$$

and so we can think of it as a function. From a physical perspective this may make sense. But mathematically, this is not as logical as you may think. For instance, in $\mathbb{R}^4$, the 2-forms form a $\binom{4}{2}$-dimensional space. This is because there are $\binom{4}{2} = 6$ combinations of $dx_i dx_j$ where i and j are distinct elements of $\{1, 2, 3, 4\}$. Thus in $\mathbb{R}^4$, the 2-forms do *not* correspond to vector fields in $\mathbb{R}^4$! It is easy to see that the largest dimensions in which all n-forms correspond to vector fields or functions (in that dimension) is 3!

The following diagram serves as a guide between forms and their derivatives and their corresponding vector fields/functions in $\mathbb{R}^3$.

$$\text{0-forms} \xrightarrow[\text{grad}]{\text{d}} \text{1-forms} \xrightarrow[\text{curl}]{\text{d}} \text{2-forms} \xrightarrow[\text{div}]{\text{d}} \text{3-forms}$$

Historically, however, the unified and abstract view I present here came long after most special cases cases were proved one by one. The first one was equation (0.4) in the 17th century (by Gregory, Barrow, and Newton). We now know this statement as the fundamental theorem of calculus. Most of the 2 and 3 dimensional cases followed in the 19th century (Stokes, Kelvin, and others). Especially in the science, these cases are still referred to to by their old individual names (divergence theorem, Green's theorem, et cetera). It was only in 1945 that a unified version was proved and written by Élie Cartan [1]. These proofs are much more advanced than we can present here. The most readable introduction that I know of is [2][Chapter 5].

We now give some other examples of integrals in $\mathbb{R}^3$.

An Integral in $\mathbb{R}^3$.

Suppose you need to calculate an integral $\int_S k$ where k is, say, a 2-form in $\mathbb{R}^3$, and S a surface bounded by some curve $c = \partial S$ (see Figure 0.5).

$$\int_S k = \int_S f_1\, dx_2 dx_3 + f_2\, dx_3 dx_1 + f_3 dx_1 dx_2\,.$$

Suppose that $dk = 0$. This corresponds to the divergence of the associated vector field being zero, or

$$\partial_1 f_1 + \partial_2 f_2 + \partial_3 f_3 = 0\,.$$

Then there is a 1-form g such that $dg = k$. This means that k can be expressed as the curl of another field. By Stokes' Formula, the integral $\int_S k$ may now be evaluated as the line integral along the boundary of S of the 1-form g. Theorem 0.4 gives

$$\int_S dg = \int_{\partial S} g \qquad \text{or}$$
$$\int_S f_1\, dx_2 dx_3 + f_2\, dx_3 dx_1 + f_3\, dx_1 dx_2 = \int_c g_1\, dx_1 + g_2\, dx_2 + g_3\, dx_3\,,$$

where $\nabla \times g = f$.

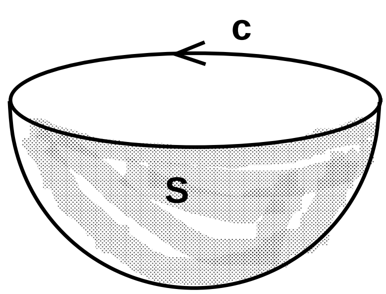

Figure 0.5: *The curve c is the boundary of a surface S in $\mathbb{R}^3$.*

One More Integral in $\mathbb{R}^3$.

Suppose you want to calculate $\int_V G\, dV$ where $G\, dV$ is a 3-form in $\mathbb{R}^3$ and V is a 3-cell in $\mathbb{R}^3$ whose boundary is the surface $S = \partial V$ (as in Figure 0.6). Note that the exterior derivative of a

3-form in $\mathbb{R}^3$ is always zero, so we can write $G\,dV$ as the "d" of a 2-form $F\,dS$: $G\,dV = d(F\,dS)$. In traditional vector analysis terms

$$G = \nabla \cdot F\,.$$

Therefore by Stokes' formula, $\int_V dF = \int_{\partial V} F$ or

$$\int_V G\,dV = \int_V \nabla \cdot F\,dV = \int_S F \cdot dS\,.$$

As mentioned before, this 2-form is interpreted as a vector field orthogonal to the surface.

As a simple example going in the other direction, consider the integral

$$\int_S x_1\,dx_2 dx_3\,.$$

At first glance, it might look like you're in for an afternoon of computations. That is, until you realize that the exterior derivative of the 2-form is the 3-form $dx_1 dx_2 dx_3$ or the usual volume. So the answer is: the volume of the cylinder (area of circle times height).

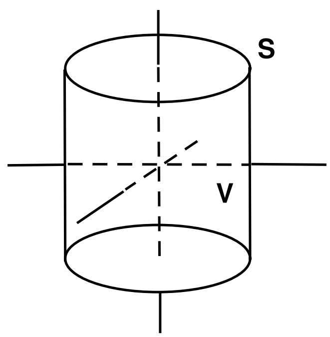

Figure 0.6: *The surface S is the boundary of the volume V in $\mathbb{R}^3$. V contains the origin.*

In Spite of That, Yet Another Integral in $\mathbb{R}^3$.

Suppose you want to compute the integral I of $F \cdot dr$ where F is the smooth vector field $(F_1, 0, 0)$ and $r(t)$ for $t \in [0,1]$ is the parametrized boundary of the surface S given by (see Figure 0.7)

$$S(x_1, x_2) = (x_1, x_2, f(x_1, x_2))\,,$$

for $x \in [0,3]$ and $y \in [0,2]$. Since F has only one component and $r(t) = (x_1(t), x_2(t), x_3(t))$, we may compute I as follows

$$I = \int_{\partial S} F_1\,dx_1 = \int_0^1 F_1(x_1(t), x_2(t), f(x_1(t), x_2(t)))\dot{x}_1(t)\,dt\,.$$

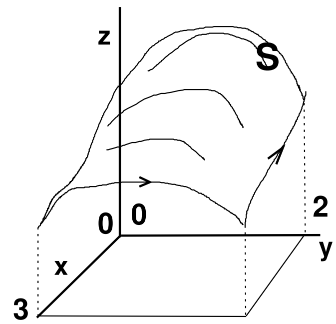

Figure 0.7: *The surface S with parametrized boundary $r(t)$ is a graph over the $x - y$ plane.*

On the other hand, by Stokes, we may also compute this another way, namely

$$I = \int_S d(F_1\, dx_1) = \int_S -\partial_2 F_1\, dx_1 dx_2 + \partial_3 F_1\, dx_3 dx_1\,.$$

But since $dx_3 = \partial_1 f dx_1 + \partial_2 f dx_2$ and $dx_1^2 = 0$, we get

$$I = \int_0^2 \int_0^3 \left(-\partial_2 F_1 - \partial_3 F_1 \partial_2 f\right)\, dx_1 dx_2\,.$$

A Simple Extension of the Theory.

Earlier we stipulated that the regions we consider need to be cells. It is possible to get around that if the region in question can tiled into regions that are cells.

Suppose we want to integrate $\int_S f\, dS$ where S is the shaded region in $\mathbb{R}^2$ in Figure 0.8. S is the union of S_1, S_2, S_3 and S_4, each of which is a cell. Clearly,

$$\int_S f\, dS = \int_{S_1} f\, dS + \int_{S_2} f\, dS + \int_{S_3} f\, dS + \int_{S_4} f\, dS\,.$$

Since the exterior derivative of $f\, dS$ is zero (there are no 3-forms in $\mathbb{R}^2$), there is a g such that $d(g_1\, dx_1 + g_2\, dx_2) = f\, dS$ in each of the four pieces. Thus for each i in $\{1, 2, 3, 4\}$:

$$\int_{S_i} f\, dS = \int_{\partial S_i} f\, dc\,.$$

Now if we sum up the four contributions, then the green line integrals in Figure 0.8 cancel. Thus we are left with

$$\int_S f\, dS = \int_c g\, ds\,.$$

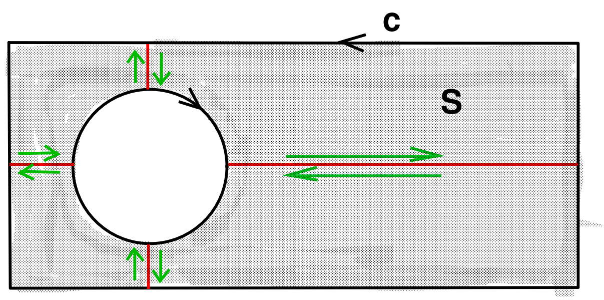

Figure 0.8: *The shaded region S is built up from cells. The line integrals along the green sides cancel.*

Note the opposite orientation on both circles.

As an example in $\mathbb{R}^3$, consider

$$\int_S \frac{e_r}{r^2} \cdot dS\,,$$

where e_r is the unit vector in the outward radial direction, r is the radius, and S is the cylinder surface of Figure 0.6. Again, this might seem like a painful computation. But in actual fact, the answer is fairly simple. Consider a little sphere S_ℓ around the origin inside V. The solid region W between the cylinder surface and the little sphere can be tiled by cells. The boundary of W is the union of S and $-S_\ell$. The negative is to indicate that (as before) the orientation of the two surfaces are opposite. By Stokes, we get

$$\int_S \frac{e_r}{r^2} \cdot dS - \int_{S_\ell} \frac{e_r}{r^2} \cdot dS = \int_W \operatorname{div}\left(\frac{e_r}{r^2}\right)\, dx_1 dx_2 dx_3\,.$$

But by example 4 [3] [Section 17.3], the divergence of that vector field is zero! But that means that the integrals over S and S_ℓ are equal. However, the latter is very easily computed, because the integrand $\frac{e_r}{r^2}$ is constant on the surface of a sphere. Since the surface area of a sphere of radius r is $4\pi r^2$, we obtain

$$\int_{S_\ell} \frac{e_r}{r^2} \cdot dS = \frac{4\pi r^2}{r^2} = 4\pi\,.$$

In fact we could have taken any surface that contains the origins and we would have gotten the same answer.

How about taking a similar surface that does *not* contain the origin? Then, of course, the divergence is well-defined everywhere inside that surface and equals zero. So this time, applying Stokes immediately returns the answer zero for the integral.

References

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