Pescara, Italy, July 2019

DIGRAPHS III
Applications: Pagerank, Contagion, Ford-Fulkerson

Based on various sources.

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SUMMARY:

* This is a review of three important applications of graph theory presented in a way that is consistent with the earlier lectures on the theory of digraphs.

* We discuss the pagerank algorithm and give a treatment that is dual to the usual one, namely cast in terms of consensus (and not random walk).

* We discuss contagion on a graph and give some elementary results about the probability that the invading species ‘takes over’.

* We discuss how to optimize transport on digraphs where each edge has a maximum capacity. This is known as the Ford Fulkerson algorithm and the max-flow is min-cut theorem.
OUTLINE:
The headings of this talk are color-coded as follows:

The Pagerank Algorithm
Teleporting and Pagerank
Contagion and Evolution
The Probability that the Invader Wins
The Ford Fulkerson Algorithm
When Ford Fulkerson Fails
PAGERANK
Recall of Definitions

We recall some definitions.

**Definition:** The combinatorial adjacency matrix $Q$ of the graph $G$ is defined as:

$Q_{ij} = 1$ if there is an edge $ji$ (if “$i$ sees $j$”) and 0 otherwise. If vertex $i$ has no incoming edges, set $Q_{ii} = 1$ (create a loop).

**Remark:** Instead of creating a loop, sometimes all elements of the $i$th row are given the value $1/n$. This is called Teleporting! The matrix is denoted by $\bar{Q}$.

**Definition:** The in-degree matrix $D$ is a diagonal matrix whose $i$ diagonal entry equals the number of (directed, incoming) edges $xi, x \in V$.

**Definition:** The matrices $S \equiv D^{-1}Q$ and $\bar{S} \equiv D^{-1}\bar{Q}$ are called the normalized adjacency matrices. By construction, they are row-stochastic (non-negative, every row adds to 1).

**Definition:** The pagerank adjacency matrices are given by $S_p = \beta S + \frac{1-\beta}{n} J$, where $S$ may be replaced by $\bar{S}$ (“with teleporting”). $J$ is the all ones matrix, $\beta \in (0, 1)$. 
Recall: consensus flows \textit{with} the arrows, random walk goes \textit{against} them.

The original pagerank algorithm by Page and Brin (as discussed in [5]). Our dual treatment mostly follows [1].

\textbf{Definition (Pagerank):} Let $J$ be the $n \times n$ all ones matrix. Define, for $\beta = 0.85$, say,

$$S_p \equiv \beta S + \frac{1-\beta}{n} J$$

The pagerank of $i$, or $\wp(i)$, is defined as the unique invariant probability measure given by

$$\wp = \wp S_p .$$
Crash Course Pagerank

\[ S_p \equiv \beta S + \frac{1 - \beta}{n} J \]

\( S_p \) strictly positive (every vertex “sees” every other vertex).

**Exercise:** Show that \( \varphi \) is unique.

**Hint:** show there is one reach and then use thms 3, 4, 5 of *Digraphs II*.

**Exercise:** Show that \( S \) and \( J \) are simultaneously diagonalizable.

Denote the *all ones* vector by \( \mathbf{1} \).

**Leading eigenpair:** eval 1 with evec \( \mathbf{1} \) (for \( S \) and \( J \)).

**Other evecs:** eval with abs. value less than \( \beta \approx 0.85 \) for \( S \) and 0 for \( J \).

Very fast convergence: \( 0.85^{57} \approx 10^{-4} \).

Can formulate the whole thing without using matrices.

**Observation:** Original algorithm uses \( \tilde{S} \) instead of \( S \).

[1] shows that the two rankings are trivially related.
Dual Approach to Pagerank 1

Recall Thm 8 of Digraphs II: Displacements in consensus caused by initial displacement $x_0$:

$$ \dot{x} = -Lx \implies \lim_{t \to \infty} x(t) = \Gamma x(0) $$

Left multiplying by $\frac{1}{n}1^T$ has the effect of taking an average of these displacements.

**Definition:** The influence $I(i)$ of the vertex $i$ is average of the displacements caused by unit displacement $e_i$:

$$ I(i) \equiv \frac{1}{n}1^T \Gamma e_i = \frac{1}{n}1^T \left( \sum_{m=1}^{k} \gamma_m \otimes \bar{\gamma}_m \right) e_i $$

$1$ is the all ones vector.

**Problem:**
By associativity, non-zero only if $\bar{\gamma}_m e_i \neq 0$ for some $m$.
Thus $I(i) > 0$ only if $i$ is in a cabal (by defn $\bar{\gamma}_m$)!!

**Definition:** The extended graph $G_\alpha$. for every vertex $v$ in $V$, attach a new vertex $b_v$ and an edge $b_v v$ with strength $\alpha$.

Think of $b_v$ as the boss/owner/administrator of the page $v$. 
$G_\alpha$ has $n$ leaders $b_i$. Each of these has a non-zero influence $\tilde{I}(b_i)$. The tilde ($\tilde{}$) indicates extended graph.

**Theorem 1 (Pagerank Theorem) [1]:** If we choose $\alpha = \frac{1-\beta}{\beta}$, then the pagerank $\varphi(i)$ of $i$ equals $2\tilde{I}(b_i) - \frac{1}{n}$.

The factor 2 is because the pagerank in $G_\alpha$ is averaged over $2n$ vertices. We have to subtract $\frac{1}{n}$ because we do not want to count the displacement of the “virtual” page $b_i$. 
The extended Laplacians are:

\[ \tilde{L} = \begin{pmatrix} 0 & 0 \\ -\alpha I & \alpha I + \mathcal{L} \end{pmatrix} \quad \text{and} \quad \tilde{\mathcal{L}} = \frac{1}{1 + \alpha} \begin{pmatrix} 0 & 0 \\ -\alpha I & \alpha I + \mathcal{L} \end{pmatrix} \]

Theorem 4 (in D II) says that the kernel of \( \tilde{\mathcal{L}} \) has basis \( \begin{pmatrix} e_m \\ \eta_m \end{pmatrix} \) where \( m \in \{1, \ldots, n\} \). Substituting gives:

\[ \eta_m = (I + \alpha^{-1} \mathcal{L})^{-1} e_m \]

Thus the influence of \( b_m \) on the “rest” (non-leaders) is

\[ I(m) = \frac{1}{n} \mathbf{1}^T (I + \alpha^{-1} \mathcal{L})^{-1} e_m \]

Theorem 10 (D II) implies* that \( \sum_m I(m) = 1 \) and so

**Proposition:** The (row) vector of influences \( p \) is given by

\[ p = \frac{1}{n} \mathbf{1}^T (I + \alpha^{-1} \mathcal{L})^{-1} \]

and is a probability measure.

**Exercise:** Show that for \( \alpha > 0 \), the above inverse is well-defined.

*Alternatively:* If all leaders move 1 unit, all others eventually do the same.
**Sketch of Proof Continued**

**Exercise:** $J$ is the all ones matrix. Show that

$$\beta S + \frac{1 - \beta}{n} J = I + \frac{\alpha}{1 + \alpha} \left( \frac{1}{n} J - (I + \alpha^{-1} \mathcal{L}) \right)$$

*Hint:* set $\alpha = \frac{1-\beta}{\beta}$ or $\beta = \frac{1}{1+\alpha}$.

**Exercise:** For any probability vector $p$, we have $pJ = 1^T$.

**Exercise:** Show that

$$\left( \frac{1}{n} 1^T (I + \alpha^{-1} \mathcal{L})^{-1} \right) \left( \frac{1}{n} J - (I + \alpha^{-1} \mathcal{L}) \right) = 0$$

*(Hint: see proposition previous page.)*

**Exercise:** Combine the last two exercises to show that the invariant probability measure $p$ satisfies

$$p = p \left( \beta S + \frac{1 - \beta}{n} J \right)$$

**Exercise:** Conclude that thus $p$ equals the pagerank $\varphi$.

**Exercise:** Relate this to the influence of $b_m$ in the extended graph. *(Hint: the extended graph has $2n$ vertices and the initial condition $x_{b_n} = 1$ moves none of the leaders except $b_n$ itself.)*
PAGERANK WITH TELEPORTING OR WITHOUT?
The Two Cases

So, to find the pagerank, we find the unique solution of:

\[
\varphi = \varphi \left( \beta S + \frac{1 - \beta}{n} J \right) \implies \varphi(I - \beta S) = \frac{1 - \beta}{n} \mathbf{1}^T
\]

There are two cases:
- **Case I:** no teleporting.
- **Case II:** with teleporting, marked by an overbar (\(\bar{S}\)).

Partition vertices into leaders \(L\) and rest \(R\).

The \(i\)th rows of the \(S\)’s differ only if \(i \in L\).

\[
(\varphi_L, \varphi_R) \left[ \begin{pmatrix} I_{LL} & 0 \\ 0 & I_{RR} \end{pmatrix} - \beta \begin{pmatrix} S_{LL} & S_{LR} \\ S_{RL} & S_{RR} \end{pmatrix} \right] = \frac{1 - \beta}{n} \left( \mathbf{1}_L^T, \mathbf{1}_R^T \right)
\]

**Case I:**

\[
\begin{pmatrix} S_{LL} & S_{LR} \\ S_{RL} & S_{RR} \end{pmatrix} = \begin{pmatrix} I_{LL} & \mathbf{0} \\ S_{RL} & S_{RR} \end{pmatrix}
\]

**Case II:**

\[
\begin{pmatrix} \bar{S}_{LL} & \bar{S}_{LR} \\ \bar{S}_{RL} & \bar{S}_{RR} \end{pmatrix} = \begin{pmatrix} \frac{1}{n} J_{LL} & \frac{1}{n} J_{LR} \\ S_{RL} & S_{RR} \end{pmatrix}
\]
The Two Cases

Exercise: Write out green equation for the two cases. Show that \( \wp_L, \wp_R, \) and \( \bar{\wp}_L \) can be expressed in terms of \( \wp_R \).

\( \text{Hint: use that } pJ = 1^T \text{ for any prob. measure } p. \)

Definition: Use \( \pi \) for probability that walker is in \( L \):

\[
\pi := \wp_L \mathbf{1}_L \quad \text{and} \quad \bar{\pi} := \bar{\wp}_L \mathbf{1}_L
\]

Exercise: Exercises plus definition imply the following.

Theorem 2 [1]: We have

\[
\bar{\wp}_L = \wp_L - \beta (1 - \bar{\pi}) \wp_L \\
\bar{\wp}_R = \wp_R + \frac{\beta}{1 - \beta} \bar{\pi} \wp_R
\]

Upon “teleporting”, leaders go down a bit, “rest” goes up. Like a card shuffle. The two subsets maintain relative rankings within them.
To complete the picture, need to express $\bar{\pi}$ in terms of “un-teleported” quantities.

**Exercise:** Sum the components of the first equation of Theorem 2 to show:

**Corollary:** $\bar{\pi} = \frac{(1 - \beta)\pi}{(1 - \beta\pi)}$.

**Exercise:** Substitute this into Theorem 2 to show:

**Corollary:**

$$\bar{\phi}_L = \left( \frac{1 - \beta}{1 - \beta\pi} \right) \phi_L$$

$$\bar{\phi}_R = \left( \frac{1}{1 - \beta\pi} \right) \phi_R$$

Thus pagerank with teleporting can be trivially expressed in terms of pagerank without teleporting.
Pagerank as function of $\beta$:

$$\varphi = 7^{-1}1^T (I + \alpha^{-1}L)^{-1} = 7^{-1}1^T \left( I + \frac{\beta}{1 - \beta} \mathcal{L} \right)^{-1}$$

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<th>$\varphi(0.10)$</th>
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<table>
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<th>$\bar{\varphi}(0.10)$</th>
<th>(0.151, 0.131, 0.152, 0.145, 0.146, 0.138, 0.138)</th>
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<td>$\bar{\varphi}(0.40)$</td>
<td>(0.156, 0.095, 0.183, 0.162, 0.168, 0.118, 0.118)</td>
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<td>$\bar{\varphi}(0.60)$</td>
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<td>$\bar{\varphi}(0.90)$</td>
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</tr>
</tbody>
</table>
CONTAGION OR EVOLUTION IN DIGRAPHS
$G$ initially has blue vertices. Color 1 vertex red (the ‘seed’).

**Definition:** Fitness is the probability (a priori likelihood) of procreating. How many kids are you likely to have? More precisely: anyone of “your” population group.

**Definition:** Assume from now on that

$$\text{fitness}(\text{red}) = r \cdot \text{fitness}(\text{blue})$$

Contagion/procreation occurs along a directed graph. Gene flow is information flow, so it follows the arrows.
First Results

Definition: Fixation probability $P$ is the probability that 1 red takes over the entire graph by contagion.

Gene flow follows the arrows. So in essence we look for influence vectors (see DII).

Corollary: Given a digraph $G$.

a) Red cannot take all ($P = 0$) if $G$ has more than 1 reach.

b) Red dies out ($P = 0$) if the seed is not in a cabal.

Proposition: Given a digraph $G$ with $n$ vertices. If red conquers cabal $m$, then red will average a proportion $\frac{1}{n}1\gamma_m$ of the population.

Idea of Proof: By DII, Thm 6: $\gamma_m(j)$ is the probability that $j$’s information comes from cabal $m$.

Thus the relevant question becomes:
Investigate $P$ for Strongly Connected Components (SCC’s).
Contagion on SCC’s

From now on, assume \( G \) is SCC.

**Definition:** Probability measure \( \mu \) on outgoing edges:
- Assign blue vertices a probability \( b \) (normalization).
- Assign red vertices a probability \( r \cdot b \).
- Assign each of the outgoing edges at a vertex equal probability whose sum is the probability of that vertex.

From now on \( x^{(n)}(i) \) is the **color** of vertex \( i \) at time \( n \).

\[
x^{(n)}(i) = 0 \text{ if uninfected} ; \quad x^{(n)}(i) = 1 \text{ if infected}
\]

An “evolutionary” dynamical system \( F : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2^n \):
At time step \( n \), choose a \( \mu \)-random (outgoing) edge \( v \rightarrow w \).
Then \( v \) ‘spreads’ to \( w \), or \( w \) assumes the color of \( v \):

\[
x^{(n+1)}(7) := x^{(n)}(6)
\]

Now denote by \( m \) the numbers of **infected**, and by \( n - m \) the number of **uninfected**.
As in DII, set $Q_{ij} = 1$ if there is edge $ji$ and 0 otherwise. There are no loops ($Q_{ii} = 0$).

**Definition:** Normalized out-degree adjacency matrix
This time the average is over **outbound edges**, and so $W \equiv QD^{-1}$ where $D$ is the diagonal matrix of **column** sums.

Thus the time-dependent prob. to select the edge $ji$ equals

$$\Pr(ji) = \frac{W_{ij}}{n - m + rm}$$

if $j$ is **uninfected**, and $r$ times that if $j$ is **infected**.

$\pi_{m,m+1}$ (resp. $\pi_{m,m-1}$) is the probability that in next time step the system goes from $m$ to $m+1$ (resp. $m-1$) infected.

**Lemma:** For $m \in \{1, \cdots n - 1\}$ we have

$$\pi_{m,m+1} = \frac{r \sum_{ij} W_{ij} (1 - x(i)) x(j)}{n - m + rm}$$

$$\pi_{m,m-1} = \frac{\sum_{ij} W_{ij} x(i)(1 - x(j))}{n - m + rm}$$

**Exercise:** Compute $\pi_{m,m}$.

**Exercise:** Use that $W$ is **column stochastic** to verify that $\pi_{m,m+1} + \pi_{m,m-1} + \pi_{m,m} = 1$. 

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The Associated Graph

**Definition:** The associated graph $A$ is a graph on $n + 1$ vertices. The vertex $i$ stands for the total number of infected in $G$. The dynamical system $F$ induces a **random walk** $R$ on $A$ with transition probabilities $\pi_{i,i\pm 1}$ (see figure).

\[
p_{i,0} \quad p_{i,1} \quad \cdots \quad p_{i,n} = p_{i,n+1}
\]

**Definition:** Let $S$ be the rw adjacency matrix on $A$. Thus

\[S_{ij} = \pi_{i,j} \text{ with row-sum 1}\]

**Important:** $S$ flips the arrows in the graph. Random walk becomes

\[p^{(n+1)} = p^{(n)} S\]

The **problem** is that the transition probabilities $\pi_{i,i\pm 1}$ depend on which $i$ vertices are infected.

Reversing the arrows, we see.....
Reversing the arrows, we see.....

Now the rw moves against the arrows, as per the conventions in DII.

We see **two reaches with single leaders**! So by DII, the random walk has two equilibria: 0 infected or \( n \) infected.
THE FIXATION
PROBABILITY
Doubly Stochastic SCC’s

Doubly stochastic: row sum is 1 and column sum is 1. All elemts $\geq 0$.

**Theorem 3:** We have the following:

a) $G$ is SCC $\iff$ $A$ has reaches $\{0, \cdots n-1\}$ with 0 as leader and $\{1, \cdots n\}$ with $n$ as leader.

b) [4] $W$ is doubly stochastic $\iff$ $\pi_{m,m+1} = r \pi_{m,m-1}$.

(b) holds if $W$ symm. But there are interesting other examples.

![Graph](image)

**Example:** This graph has norm. outdegr. adj. matrix $W$

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \implies W = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Spectrum $\{-1, 0^{(2)}, 1\}$ with one 2-dimensional Jordan block.
Sketch of Proof of Theorem 3

Proof of (a).

0 and n are leaders. If there are 0 infected, no infections can occur. So $S_{0i} = 0$ for all $i$. Same for $n$.

Recall that the $\pi_{i,i+1}$ depend on which vertices are infected. Suppose that at any point in the process $A$ has reaches different from (a). This can happen if and only if one of $\pi_{i,i\pm1}$ is zero (see Figure).

But that means there is a non-trivial set $V$ of $i$ red (or blue) vertices that cannot infect (or de-infect) $V^c$. In turn that happens if and only if there are no arrows leading out of $V$. And that means that $G$ is not SCC.
Proof of (b). Suppose $W$ doubly stochastic. Recall

$$
(n - m + rm)\pi_{m,m+1} = r (1 - x)Wx = r (1Wx - xWx)
$$

$$
(n - m + rm)\pi_{m,m-1} = xW(1 - x) = xW1 - xWx
$$

Use double stochasticity of $W$ to see that $1Wx = xW1$. Then $\pi_{m,m+1}$ equals $r \pi_{m,m-1}$.

Vice versa, if $\pi_{m,m+1}$ equals $r \pi_{m,m-1}$, set $x = e_\ell$. The same computation now shows that then

$$
1We_\ell = e_\ell W1 \iff 1 = \sum_i W_{i\ell} = \sum_i W_{li}
$$

Thus $W$ is doubly stochastic.

Remark. It is possible that $\pi_{m,m\pm1} = 0$. This can happen, for example, if $G$ is not an SCC.

Exercise 9: Analyze the associated graph (and its reaches) for the graph in the figure. Indicate qualitatively how the $\pi$’s depend on who is infected.
Fixation Probability for Doubly Stoch.

Recall that infected vertices have relative fitness $r$. The fixation probability, is the probability that 1 red vertex takes over the entire graph.

**Theorem 4:** If $G$ is an SCC whose norm. out-degree adj. matrix is doubly stochastic, then $G$ has fixation probability equal to $\frac{1 - r^{-1}}{1 - r^{-n}}$.

The fixation probability as function of $r$ and $n$.

<table>
<thead>
<tr>
<th>n</th>
<th>r</th>
<th>4</th>
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<th>1</th>
<th>0.5</th>
<th>0.25</th>
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<td>4</td>
<td>0.753</td>
<td>0.53</td>
<td>1/4</td>
<td>1/16</td>
<td>1.18E-2</td>
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<tr>
<td>8</td>
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<td>32</td>
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<td>0.50</td>
<td>1/32</td>
<td>2.3E-10</td>
<td>1.6E-19</td>
<td></td>
</tr>
</tbody>
</table>

When $r = 1$, use L’Hôpital.
**Sketch of Proof of Theorem 4**

**Thm 3a):** The associated graph $A$ has reach $\{1, \cdots n\}$.

**DII, Thm 5):** $\text{Ker} \mathcal{L}$ contains $\gamma$ st $\gamma(n) = 1$ and $\gamma(0) = 0$.

**DII, Thm 6):** $\gamma(1)$ is the fixation probability.

**Thm 3b):** The rw adjacency of the assoc. graph $A$ is

$$S = \begin{pmatrix}
1 & 0 & \cdots & \\
p_{1,0} & p_{1,1} & r p_{1,0} & \cdots \\
\cdots & \cdots & \cdots & \\
p_{n-1,n-2} & p_{n-1,n-1} & r p_{n-1,n-2} & 0 & 1
\end{pmatrix}$$

with row-sum 1.

**Exercise:** Show that $(I - S)\gamma = 0$ plus row-sum=1 imply

$$(\gamma(i + 1) - \gamma(i)) = r^{-1}(\gamma(i) - \gamma(i - 1))$$

Furthermore,

$$\sum_{i=0}^{n-1} (\gamma(i + 1) - \gamma(i)) = \gamma(n) = 1$$

**Exercise:** Show that the previous exercise implies that

$$1 = \sum_{i=0}^{n-1} (\gamma(i + 1) - \gamma(i)) = \sum_{i=0}^{n-1} r^{-i} \gamma(1)$$

from which the fixation probability follows.
THE FORD FULKERSON ALGORITHM
Our treatment is mostly based on [2] and [6].

Here: edges correspond to physical conduits. Oil or water pipes (of differing diameters), transportation networks, nutrient networks in ecology, etc. So for now: arrows indicate direction of physical flow.

**Definition:** An **FF network** $N$ is a digraph with 1 leader (called **source** $s$) and 1 goose (called **sink** $t$) together with a flow satisfying **feasibility conditions**.

**Definition:** Every edge $e$ has a **capacity** $c(e) \geq 0$ and a **flow** $f(e)$. The **value** $\text{val}(f)$ of the flow is the output at $t$.

**Feasibility Conditions:**

1. $f(e) = -f(-e)$ \quad c(e) = -c(-e)$
2. $0 \leq f(e) \leq c(e)$ (or $c(e) \leq f(e) \leq 0$)
3. At every vertex, except $s$ and $t$: flow in = flow out.
4. flow out of $s$ = flow into $t$. The VALUE of the flow.

This amounts to (a) conservation of mass and (b) don’t exceed capacity.

**Remark:** Could be any digraph with input in cabals and output in gaggles. For example, can create one super source and one super sink.
Maximize Flow

Want to find the maximum flow.

Notation: \(a(b)\) means flow(capacity) along arrow (see figure).

A maximal flow: cannot increase flow on any edge (left).
A maximum flow: exists no flow with greater value (right).

Definition: An augmenting path is a continuous path from \(s\) to \(t\) with spare capacity.

Example: In left figure, \(\gamma = sabt\) has spare capacity of 1. Feasibility conditions require:

\[
f(sa) \in [0, 1], \ f(ab) \in [-1, 0], \ f(bt) \in [0, 1]
\]

Let \(f_\gamma\) be flow along \(\gamma\) with value 1. Flow of right figure:

\[
f' := f + f_\gamma
\]

is a feasible flow with a higher value than \(f\).
Max-Flow Min-Cut

**Definition:** An *st cut*, \([S, T]\), is a partition of the vertices into \(S\) containing \(s\) and \(T\) containing \(t\). Its *capacity*, \(\text{cap}[S, T]\), is the sum of the capacities of edges from \(S\) to \(T\).

**Theorem 5:** flow is maximum \(\iff\) no augm. path.

**Sketch of Proof:** Flow max \(\implies\) no augm. path is obvious. To prove: no augm. path \(\implies\) flow maximum.

\(S\) the set of vertices in augmenting **semi-paths** out of \(s\) (pink in the figure). \(T\) is its complement.

By defn of \(S\), in the absence of an augm. path, we have:
1. \(s \in S\) and \(t \in T\).
2. \(e \in [S, T] \implies f(e) = c(e)\) and \(e \in [T, S] \implies f(e) = 0\).

Thus \(\text{val}(f) = \text{cap}[S, T]\), and \(f\) must be **maximum**. QED

By mass conserv., no flow is greater than minimum of \(\text{cap}[S, T]\).

A flow with \(\text{val}(f) = \text{cap}[S, T]\) can be constructed. Thus:

**Theorem 6 (Max-flow min-cut theorem, FF 1956):**

\[
\max_{\text{feas. flows}} \text{val}(f) = \min_{\text{st cuts}} \text{cap}[S, T]
\]
**The Ford Fulkerson Algorithm**

**Definition:** At every step of the algorithm, the set of vertices is partitioned into the following sets. $S$ stands for searched, $R$ stands for reached, and $C$, the complement of $S \cup R$.

**Steps of the algorithm:**

<table>
<thead>
<tr>
<th>step</th>
<th>$S$</th>
<th>$R$</th>
<th>comment</th>
</tr>
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<td>start</td>
</tr>
<tr>
<td>2.</td>
<td>$s$</td>
<td>$u, x$</td>
<td>find spare cap. $su$</td>
</tr>
<tr>
<td>3.</td>
<td>$s, u$</td>
<td>$v$</td>
<td>find spare cap. $suv$</td>
</tr>
<tr>
<td>4.</td>
<td>$s, u, v$</td>
<td>$x, t$</td>
<td>find augm. path $suvt$</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
<td>:</td>
<td>start again at $s$ until no spare cap.</td>
</tr>
</tbody>
</table>

**Remark.** Note that when searching a vertex $v$, there is no strategy specified how to order edges incident to $v$. Improved formulations specify search strategy.
Exercise: Use the algorithm to find an augm. path.

Exercise (FF for Artists): Given FF network with many sources and many sinks. Each source inputs a specific color of paint. What is color mix of each output? (See below.)

Comment. Answer is not unique as figure below shows. Use flow adjacency matrix of the flow computed by FF.
**Remarks**

**Example:** In exercise pg 35, you should have found flow of value 2. There is cut of capacity 2. By Thm 5, max-flow=min-cut=2.

![Graph](image)

**Corollary:** If the capacities are rational, then FF converges to a maximum flow solution in finitely many steps.

**Proof:** Sufficient to do this for integers. Every augm. path has spare cap. at least 1. So FF terminates after finite steps.

**Remark:** The result of the FF algorithm depends of the search strategy. The outcome is not unique (see below).
In presence of irrational capacity, convergence can be beaten, but one has to be really clever to carefully craft a search strategy so that FF fails to converge to max flow [7].

All unmarked edges have capacity \( m \geq 2 \). Furthermore:

\[
\begin{align*}
  c(e_1) &= c(e_2) = 1 \\
  c(e_3) &= r := \frac{\sqrt{5} - 1}{2} \approx 0.618 \\
  p_0 &= (s, v_2, v_3, t) \\
  p_1 &= (s, v_4, v_3, v_2, v_1, t) \\
  p_2 &= (s, v_2, v_3, v_4, t) \\
  p_3 &= (s, v_1, v_2, v_3, t)
\end{align*}
\]

**Exercise 14:** Start with \( f = 0 \). Execute FF in such a way that the sequence of augm. paths is \( (p_0, p_1, p_2, p_1, p_3, p_1, p_2, \cdots) \).
Exercise 15: Check the listed flow and spare capacities in the following table. (*Hint: use that* $r^2 = 1 - r$.)

<table>
<thead>
<tr>
<th>step</th>
<th>augm. path</th>
<th>val(path)</th>
<th>sp. cap. $e_1$</th>
<th>sp. cap. $e_2$</th>
<th>sp. cap. $e_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.</td>
<td>$\emptyset$</td>
<td>0</td>
<td>$r^0$</td>
<td>$r^0$</td>
<td>$r^1$</td>
</tr>
<tr>
<td>1.</td>
<td>$p_0$</td>
<td>$r^0$</td>
<td>$r^0$</td>
<td>0</td>
<td>$r^1$</td>
</tr>
<tr>
<td>2.</td>
<td>$p_1$</td>
<td>$r^1$</td>
<td>$r^2$</td>
<td>$r^1$</td>
<td>0</td>
</tr>
<tr>
<td>3.</td>
<td>$p_2$</td>
<td>$r^1$</td>
<td>$r^2$</td>
<td>0</td>
<td>$r^1$</td>
</tr>
<tr>
<td>4.</td>
<td>$p_1$</td>
<td>$r^2$</td>
<td>0</td>
<td>$r^2$</td>
<td>$r^3$</td>
</tr>
<tr>
<td>5.</td>
<td>$p_3$</td>
<td>$r^2$</td>
<td>$r^2$</td>
<td>0</td>
<td>$r^3$</td>
</tr>
<tr>
<td></td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Exercise 15: Conclude that FF does not terminate, and that the value of the (total) flow converges to $1 + 2r \sum_{i \geq 0} r^i = r^{-3} \approx 4.24$.

Exercise 16: Exhibit a cut and a flow of value $2m + 1 \geq 5$. 
References


