Examples of Linear Programming Problems

Formulate each of the following problems as a linear programming problem by writing down the objective function and the constraints.

1. *Incinerators and Pollution Control.* Burtonville burns 3000 tons of trash per day in three elderly incinerators. All three have antipollution devices that are less than satisfactory. Their emission profiles differ, as is shown in Table 11-2. At present all three incinerators are operating at full capacity. The remainder of the city's trash, another 1500 tons per day, is dumped in a sanitary landfill area. This is a much more expensive method of disposal. The state's Environmental Quality Commission has brought suit against the city; the Superior Court has issued a temporary restraining order under which sulfur dioxide emissions must be limited to 400,000 units per day and particulate emissions to 50,000 units per day. What is the most economical way to make the necessary cutbacks?

Table 11-2

Incinerator	Capacity in tons/day	Emissions per ton burned: Units of sulfur dioxide	Emissions per ton burned: Units of particulate
A	1,200	250	20
B	800	150	30
C	1,000	220	24

2. *Police Shifts.* Burtonville has minimum requirements for the number of patrolmen on duty during each 4-hour period, as shown in Table 11-3. There are no part-time patrolmen, and union regulations prohibit split shifts. Hence each policeman works eight consecutive hours. Work out a daily schedule that employs the fewest policemen.

Table 11-3

Time of day	
12 noon to 4 P.M.	100
4 P.M. to 8 P.M.	250
8 P.M. to 12 P.M.	400
12 P.M. to 4 A.M.	500
4 A.M. to 8 A.M.	200
8 A.M. to 12 noon	150

3. *Assignments to Hospitals.* The director of the Burtonville Civil Defense Agency has been ordered to draw up a disaster plan for assigning casualties to hospitals in the event of a serious earthquake. For simplicity, we will assume that casualties will occur at two points in the city and will be transported to three hospitals. It is estimated that there will be 300 casualties at point *A* and 200 at point *B*. Travel times from point *A* to hospitals 1, 2, and 3 are 25, 15, and 10 minutes, respectively; from point *B* they are 20, 5, and 15 minutes. Hospital capacities for emergency cases are 250, 150, and 150 patients. How should the victims be assigned to minimize the total time lost in transporting them?
4. *Electricity Generation and Pollution Control.* The Burtonville Municipal Power Company must produce 2000 megawatt-hours (mwh) of electricity each hour. Air pollution ordinances require it to keep emissions of pollutants below 2800 pounds per hour. (For purposes of this exercise we ignore interesting phenomena like weather and load fluctuation.) The company must decide how to do this at least cost. It can switch to low-sulfur fuel, use stack filters and either high- or low-sulfur fuel, or import power from elsewhere. The relevant characteristics of these options (as well as the company's present method) are given in Table 11-4. The company can import only 200 mwh per hour. What should the company do?

Table 11-4

Method	Results	Cost/mwh
Use present fuel	Pollution = 10 lbs/mwh	\$3.50
Use low-sulfur fuel	Pollution = 1.2 lb/mwh	\$5.00
Use stack filters	Pollution reduced by 90%	\$.80
Import power	No pollution on Burtonville	\$4.00

Example Linear Programming Set-Up Problem

You are the head of a building department. Your department is making final inspections of two types of new commercial buildings--gas stations and restaurants. Final inspections involve the work of three separate inspectors: a plumbing inspector, an electrical inspector, and a building inspector. The plumbing inspector requires 4 hours to inspect a gas station and 2 hours to inspect a restaurant. The electrical inspector requires 2 hours to inspect a gas station and 6 hours to inspect a restaurant. The building inspector requires 4 hours to inspect a gas station and 6 hours to inspect a restaurant. Considering the time required for other duties, the plumbing inspector has 28 hours a week available for inspections; the electrical inspector has 30 hours available; and the building inspector has 36 hours available. You want to inspect as many commercial buildings as possible each week. Set up this problem as a linear programming problem.

a) Define the decision variables.
b) Define the objective function.
c) Write out the constraints.

Homework Problem

Stipak

EXAMPLE LINEAR PROGRAMMING PROBLEM

West Chester, a small eastern city of 15,000 people, requires an average of 300,000 gallons of water daily. The city is supplied from a central waterworks, where the water is purified by such conventional methods as filtration and chlorination. In addition, two different chemical compounds, softening chemical and health chemical, are needed for softening the water and for health purposes. The waterworks plans to purchase two popular brands that contain these chemicals. One unit of the Chemco Corporation's product gives two (2) pounds of the softening chemical and four (4) pounds of the health chemical. One unit of the American Chemical Company's product contains three (3) pounds and one (1) pound per unit, respectively, for the same purposes.

To maintain the water at a minimum level of softness and to meet a minimum in health protection, experts have decided that a minimum of sixty (60) and forty (40) pounds of the two chemicals that make up each product must be added to the water daily. At a cost of \$3 and \$6 per unit, respectively, for Chemico's and American Chemical's products, what is the optimal quantity of each product that should be used to meet the minimum level of softness and a minimum health standard?

Set up the problem and solve it.

Example of Linear Programming Graphical Solution

A municipality has two incinerators for burning trash. Incinerator A costs \$3.80 per ton of trash to operate, and has a capacity of 28 tons per day. Incinerator B costs \$4.25 per ton to operate, and has a capacity of 30 tons per day. The municipality produces over 100 tons of trash per day, and all trash not burned in the incinerators must be buried in a land fill at a cost of \$5.00 per ton. The city manager wants to minimize costs by burning as much trash as possible. However, the city must conform to environmental regulations limiting production of pollutants from burning in the incinerators to 180 pounds of hydrocarbons and 640 pounds of particulates a day. Incinerator A produces 3 pounds of hydrocarbons and 20 pounds of particulates for every ton of trash burned, and incinerator B produces 5 pounds of hydrocarbons and 10 pounds of particulates for every ton of trash. Determine the optimum amount of trash to burn in each incinerator.

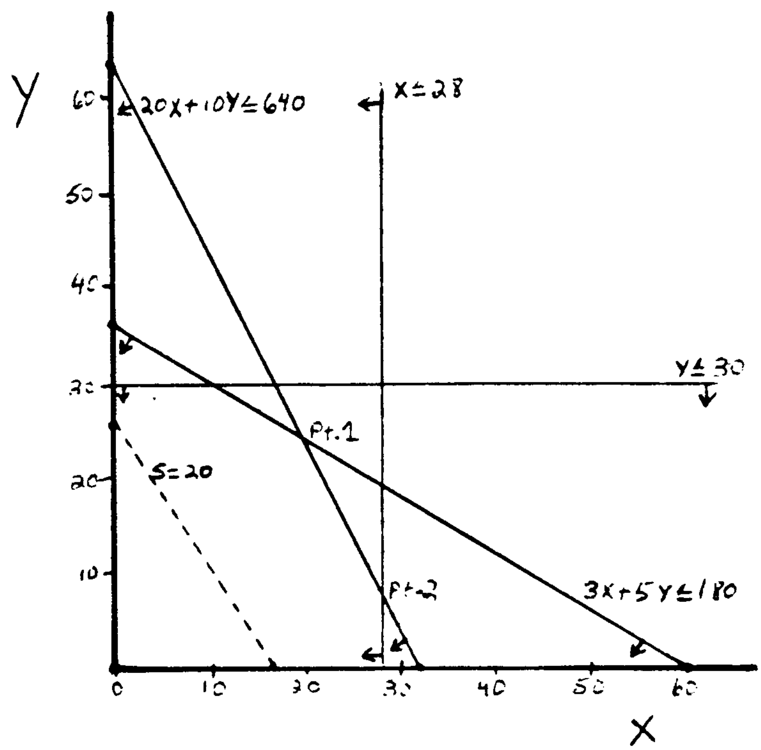
Let X = number of tons burned in incinerator A per day
Y = number of tons burned in incinerator B per day

Objective function: Maximize $S = 1.2X + .75Y$

Constraints:

$$X \leq 28$$
$$Y \leq 30$$
$$3X + 5Y \leq 180$$
$$20X + 10Y \leq 640$$
$$X \geq 0$$
$$Y \geq 0$$

Graph the constraints to identify the feasible solution set:



Identifying the optimal point:

Graph the objective function for some value of S, e.g. S=20. Careful drawing will reveal that the optimum value occurs when a parallel line is tangent to Pt. 1.

Alternatively, whether Pt. 1 or Pt. 2 is optimum can be determined by computing S for Pt. 1 and Pt. 2. For Pt. 1 X=20, Y=24, S=42. For Pt. 2 X=28, Y=8, S=39.6.

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EXAMPLE LINEAR PROGRAMMING PROBLEM SETUP, Spreadsheet Program

For New Versions of Quattro Pro (4.0 & up) with \Tools|Optimizer, and for other spreadsheet programs with similar optimization capabilities.

[Note: Quattro Pro terminology is given below in parentheses.]

[Note: This page shows the formulas, not the numeric values.]

Objective Function (Solution Cell):

1.2*X+0.75*Y	[You enter this formula and define this cell as the solution cell. X and Y are range names that refer to the cells below for the decision variables.]

Linear Inequalities (Constraints):

+X +Y 3*X+5*Y 20*X+10*Y	[You enter these formulas and define them as constraints, and indicate the type as <=, >=, or =, and give the right hand value.]

Decision Variables (Variable Cells):

X = Y =	[You define these empty cells as the variable cells, and when you tell Optimizer GO the optimum values for the decision variables will go here.]

\Tools Advanced_Math Optimization

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Old Version of Quattro Pro

EXAMPLE LINEAR PROGRAMMING PROBLEM SETUP, Quattro Pro

Linear Inequalities:

X1	X2	Type	RHS
1	0	<=	28
0	1	<=	30
3	5	<=	180
20	10	<=	640
1	0	>=	0
0	1	>=	0

Linear constraint coefficients (X1, X2 columns)

Inequality/Equality Relations (Type column)

Constant constraint terms (RHS column)

Note: non-negativity constraints are not required, since Quattro Pro assumes them by default.

Objective Function:

X1	X2
1.2	0.75

objective function

Solution:

X1	X2	Obj Fn
20	24	42

Variables (X1, X2)

Solution (Obj Fn)

Example Linear Programming Output from CMMS Program

-=*=- INFORMATION ENTERED -=*=-

```
NUMBER OF VARIABLES          :  2
NUMBER OF <= CONSTRAINTS     :  4
NUMBER OF  = CONSTRAINTS     :  0
NUMBER OF >= CONSTRAINTS     :  0

MAX s =        1.2  x  +   .75 y

SUBJECT TO:

   1   x  +   0   y                          <=   28

   0   x  +   1   y                          <=   30

   3   x  +   5   y                          <=  180

  20   x  +  10   y                          <=  640
```

Here are the optimum values for the decision variables.

-=*=- RESULTS -=*=-

VARIABLE	VARIABLE VALUE	ORIGINAL COEFFICIENT	COEFFICIENT SENSITIVITY
x	20	1.2	0
y	24	.75	0

CONSTRAINT NUMBER	ORIGINAL RIGHT-HAND VALUE	SLACK OR SURPLUS	SHADOW PRICE
1	28	8	0
2	30	6	0
3	180	0	.043
4	640	0	.054

OBJECTIVE FUNCTION VALUE: 42 <--Here is the value of the objective function

Christopher K. McKenna, Quantitative Methods for Public Decision Making (New York: McGraw-Hill, 1980)

CHAPTER
EIGHT

INTRODUCTION TO LINEAR PROGRAMMING: FORMULATION AND GRAPHIC SOLUTION

The difficulties that people encounter often provide them with the opportunity to reestablish a wholesome relationship with their environment. A fuel crisis or an extremely cold winter, or both together, encourage us to reconsider our use of energy sources. In such a situation, it would not seem unrealistic or unreasonable for a municipality, an energy commission, or a power company to determine the level of wasted energy. The municipality, commission, or company will use its resources, in this case mostly manpower, to gather information about energy wasted in homes and commercial plants. The manpower available is most assuredly not unbounded, any more than the fuel itself. Within the bounds of available manpower, the municipality, commission, or company would want to gather as much information as possible. This problem typifies the setting for which linear programming may be appropriate.

THE LINEAR PROGRAMMING MODEL

Before examining the characteristics of the model, and its underlying assumptions, we will consider how it relates to the multiple-objective situation. After noting the assumptions of a linear programming model, we will relate it to our decision-making paradigm. Next we will focus on the formulation of a model, present the graphic solution to a few models, and then consider applications to a policy analysis of a national health insurance program and a school busing problem.

Multiple Objectives and Linear Programming

We have seen two approaches to multiple-objective decision making (Chapter 6). In one, preferences were traded off so that alternatives could be compared; in the other, objectives were assigned relative utilities, and alternatives were rated for their efficacy in achieving the objectives. Two other approaches are appropriate for certain kinds of problems.

The first of these approaches considers all but one of the objectives as "musts" rather than "shoulds," or as requirements rather than aims. The single remaining objective is treated as the only entity to be optimized. If certain quantitative conditions are also satisfied, this problem is suitably addressed by linear programming. This chapter introduces the linear programming model, the formulation procedure, and a graphic method of solving simple problems; Chapter 9 presents sensitivity analysis in linear programming; and Chapter 10 presents a computational procedure for solving the model.

The second approach considers none, one, or some of the objectives as constraints. The remaining objectives (more than one) are prioritized and sought after according to their priority level. If the same certain quantitative conditions are satisfied, then goal programming would be a suitable approach. Chapter 11 discusses the goal programming model and some applications.

Underlying Assumptions

Linear programming is a technique that provides the decision maker with a way of optimizing his objective within resource requirements and other constraints provided that the following basic assumptions apply:

1. The objective can be represented by a linear function.
2. Each constraint can be represented by a linear inequality.
3. Each variable is nonnegative (either positive or zero).

As the term implies, the graph of a linear inequality or equation is related to a straight line. The following are examples of linear functions:

$$Z = 2X + 3Y - 2Z$$

$$N = 4A - B + .5C - 6D$$

$$W = 2X_1 + 8X_2 - \tfrac{1}{2}X_3 + 1.7X_4$$

The following are not linear functions:

$$Z = X^2 + 2Y$$

$$M = 4A - 3B + \sqrt{D}$$

$$C = 3X - XY + Z$$

$$D = 2X_1 - \frac{6X_2}{X_3}$$

A linear inequality has a linear expression that is related to a constant by either "≤" (less than or equal to), "≥" (greater than or equal to), or "=." The following are linear inequalities:

$$3X + 2Y - Z \leq 8$$
$$4A - 3B + .5C - D = 14$$
$$X_1 - 2X_2 + 6X_3 \geq 0$$
$$X_1 \geq 0$$

The last inequality above is a nonnegativity constraint, requiring X_1 to be zero or positive. Most variables of decision-making interest are nonnegative; so this assumption poses no problem.

These basic assumptions have implications that we will consider a bit later.

An example of the basic form of a linear program is as follows:

Maximize (or minimize) the objective function

$$Z = 2w + 3x + 4y$$

subject to the constraints

$$3w + 2x - y \leq 20$$
$$w + 3x + 5y \geq 40$$
$$w + x + y = 18$$
$$w \geq 0$$
$$x \geq 0$$
$$y \geq 0$$

Linear Programming and Decision Making

The basic concept of linear programming is characterized by the decision maker who is attempting to accomplish something; he has limited resources with which to accomplish it. His immediate concern becomes using wisely the resources that are available, so that he can attain as much as possible of his objective. Aside from limited resources, there may be legal regulations, organizational restrictions, and accepted standards that in some way constrain the decision maker.

Regardless of the complexity of the model or the number of constraints, the essential steps in using linear programming stay the same: Formulate the problem as a linear program if appropriate, seek a solution, and interpret the solution in terms of the original problem statement. Insofar as linear programming yields a solution to the problem as formulated, it is classified as a prescriptive model. Alternatives do not have to be found outside the model itself; the

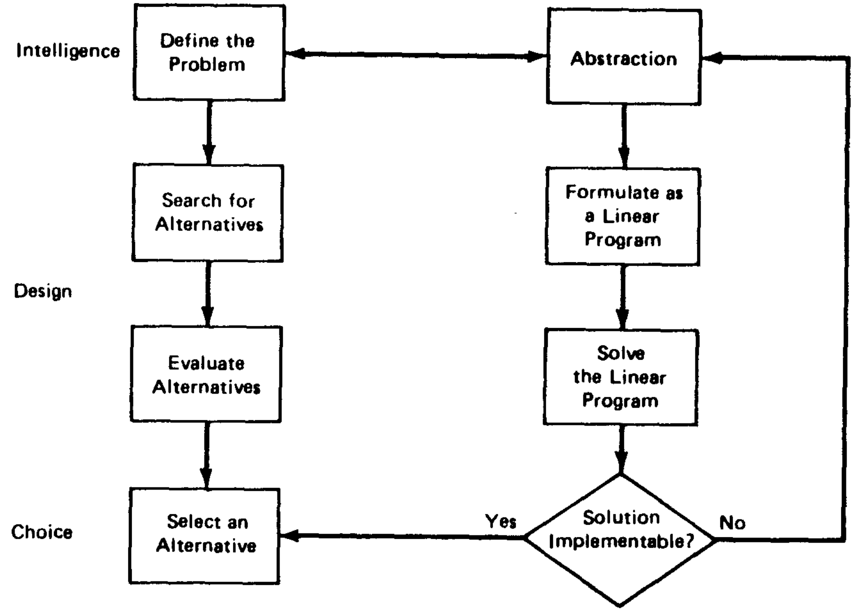

Figure 8-1 Linear programming and the decision-making process.

model generates allowable alternatives and identifies the best one relative to the formulated objective. Figure 8-1 depicts the role of a linear programming model in the decision-making process. We will explain these steps in using linear programming, first in simple examples and later in more complex and more realistic ones. Simple linear programs can be solved graphically; others require more sophisticated methods.

FORMULATION OF A LINEAR PROGRAM

Before becoming concerned with the solution to a linear program, it is useful to identify the salient ingredients of a linear program and to learn how to translate a problem situation into a linear program. Thus identifying and translating, we will also be better prepared to identify which problems are appropriately analyzed by this technique. A detailed illustration will serve as the vehicle for formulation and solution.

Assume that the Public Power Commission is undertaking a sample survey to estimate the extent of power loss in homes and industrial plants within its jurisdiction. Rather than being able to choose freely the number of homes and plants to inspect, the project has been assigned one person from each of the three relevant inspection categories for the duration of the project. These three people will inspect the insulation, the electrical wiring and circuitry, and the heating apparatus of the sampled homes and plants. In order for the information to be useful, a complete inspection must include all three types. The commission will try to have as many complete inspections as possible, either of homes or of plants.

The insulation inspector estimates that it will take 4 hours to fully inspect a home and only 2 hours to inspect a plant. She claims that insulation is usually more accessible in a plant than in a home. The electrical inspector estimates that it will take 2 hours to inspect a home and 6 hours to inspect a plant. The heating inspector estimates it will take 4 hours to inspect a home and 6 hours to inspect a plant. As might be expected, the more complex electrical and heating systems characteristic of industrial plants require more time to inspect them than the home systems. Not including the reports that must be completed and filed, it is estimated that the insulation inspector has 28 hours per week available for actual inspections; the electrical inspector has 30 hours per week available; and the heating inspector has 36 hours per week available. How should the commission assign the inspectors in order to complete as many inspections as possible?

The Objective Function

The commission realizes that although it wants to complete as many inspections as possible, it is restricted by the amount of time available to each inspector each week. It therefore attempts to assign the inspectors in such a way as to make the number of inspections per week as high as possible. In other words, it hopes to maximize the quantity

$$\text{homes} + \text{plants}$$

inspected each week. To simplify any further quantities or expressions, let

$$x_1 = \text{number of homes inspected each week}$$

$$x_2 = \text{number of plants inspected each week}$$

Then the objective is to maximize

$$N = x_1 + x_2$$

This expression is referred to as the *objective function* of the linear program. Here the aim is to maximize the objective function. If the objective function represented an expression of cost, then the objective would be to minimize it. The general term *optimize* applies to either maximize or minimize.

The Constraints

The constraints of the problem are represented by expressions not unlike the one that represents the objective function. The insulation inspection of a home requires 4 hours, and the number of homes inspected is x_1; so the number of hours spent inspecting homes is $4x_1$. Similarly, the insulation inspection of a plant requires 2 hours, and the number of plants inspected is x_2; so the number of hours spent inspecting plants is $2x_2$. The total number of hours the insulation specialist spends inspecting homes and plants is, therefore,

$$4x_1 + 2x_2$$

Since the insulation inspector has only 28 hours available, the number of hours she spends must be less than or equal to 28, that is,

$$4x_1 + 2x_2 \leq 28$$

Likewise, the electrical inspection of a home requires 2 hours, a plant requires 6 hours, and the electrical inspector has only 30 hours available; so

$$2x_1 + 6x_2 \leq 30$$

Finally, the heating inspection of a home requires 4 hours, a plant requires 6 hours, and the heating inspector has only 36 hours available; so

$$4x_1 + 6x_2 \leq 36$$

All of the constraints and the objective function satisfy the linearity assumptions.

Since it makes sense to consider assigning an inspector to inspect 2 homes or 5 homes or no homes, or 2 plants or 6 plants or no plants, but not to inspect -2 homes or -3 plants, the nonnegativity assumptions are applicable. Thus

$$x_1 \geq 0 \quad \text{and} \quad x_2 \geq 0$$

The full problem statement has now been translated into algebraic expressions; that is, we have formulated the linear program. In its complete form the linear program appears as:

Maximize the objective function

$$N = x_1 + x_2$$

subject to the constraints

$$4x_1 + 2x_2 \leq 28$$
$$2x_1 + 6x_2 \leq 30$$
$$4x_1 + 6x_2 \leq 36$$
$$x_1, x_2 \geq 0$$

A problem statement that is much more involved will yield a linear program that is considerably longer, with perhaps different kinds of constraints. The process of formulating the linear program would be essentially the same, however. After developing a solution to the present example, we shall examine other more involved problem statements and their associated linear programs.

GRAPHIC SOLUTION

A simple example such as ours can be solved graphically. The more complex problems could not even be represented graphically, much less solved graphically. Fortunately, there are techniques that can be applied quite directly to the larger problems. The graphic solution will hopefully provide an intuitive grasp that will be helpful in understanding if not solving the larger linear programs.

The basic approach of the graphic method of solution includes identifying solutions that are allowable or that are within the bounds of the constraints, and then choosing from among them the particular solution that provides the best value for the objective function. The specific steps of the method are:

1. Graph each constraint.
2. Identify the feasible region.
3. Graph the objective function for at least one value of N.
4. Consider lines parallel to the graphed objective function to find the one with the optimum feasible solution.

Graphs of the Constraints

The graph of each constraint is considered separately; then the graph of all the constraints taken together will be determined.

Recall that meaningful solutions require that

$$x_1 \geq 0 \quad \text{and} \quad x_2 \geq 0$$

These constraints are depicted in Figure 8-2. Each point in the graphed region

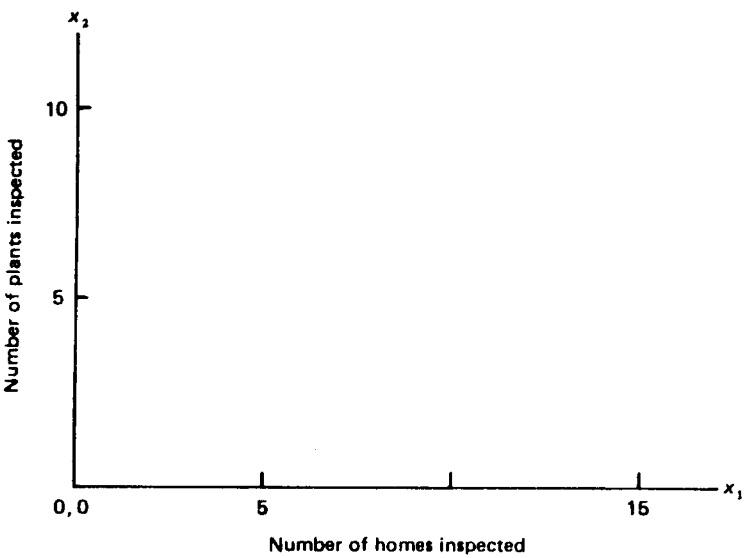

Figure 8-2 Region satisfying $x_1 \geq 0$ and $x_2 \geq 0$.

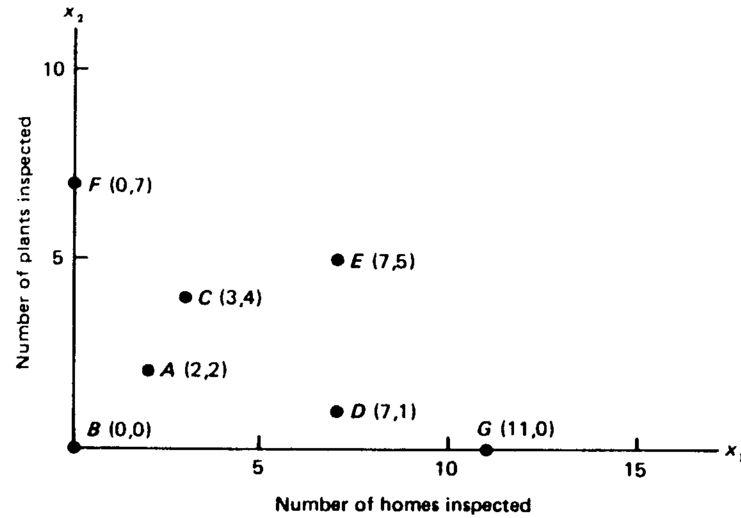

Figure 8-3 Points representing numbers of homes and plants inspected.

represents a particular number of homes inspected and a particular number of plants inspected.

Some examples of points and the numbers they represent are provided in Figure 8-3. The numbers represented by the points are usually referred to as *coordinates*, the horizontal and vertical distances of the point from the origin, the (0, 0) point.

By inserting the coordinates of the points in the three constraints, one can see that points A, B, and C satisfy all three constraints. This means that assigning each inspector to inspect (A) 2 homes and 2 plants, or (B) 0 homes and 0 plants, or (C) 3 homes and 4 plants would be within the time restrictions of each inspector. Such alternatives are said to be *feasible;* they satisfy all the constraints. On the other hand, point D is not feasible because 7 homes and 1 plant require 30 hours for the insulation inspection, more than the 28 hours available. Likewise point E is not feasible, since 7 homes and 5 plants require more time than any inspector has available; point F is not feasible since 0 homes and 7 plants require more time than is available for electrical and heating inspections; and point G is not feasible since 11 homes and 0 plants require more time than is available to the insulation and heating inspectors. Thus some of the points in the graphed region represent allowable alternatives and some do not.

The restriction on time available to the insulation inspector is given by the constraint

$$4x_1 + 2x_2 \leq 28$$

In order to represent this constraint on the graph, we will first find the graph of the related equation

$$4x_1 + 2x_2 = 28$$

An easy way to graph an equation in this form is to find the points at which the line will cross the x_1 axis and the x_2 axis. Such points are referred to as the x_1 *intercept* and the x_2 *intercept*. We know that the value of x_2 on the x_1 axis is always 0 (for example, point G in Figure 8-3). To find the x_1 intercept for an equation, simply determine the value of x_1 when $x_2 = 0$.

In $4x_1 + 2x_2 = 28$ if $x_2 = 0$, then $x_1 = 7$, and the x_1 intercept is 7. Similarly, if $x_1 = 0$, then $x_2 = 14$, and the x_2 intercept is 14. The two intercepts (or any two points, for that matter) completely determine the line. Figure 8-4 presents the line. Note that it is not being suggested that either intercept represents a feasible solution; the intercepts were found only to find the graph of the linear equation.

The constraint in the linear program is not the linear equation but rather the linear inequality

$$4x_1 + 2x_2 \leq 28$$

The graph of the linear equation was found as an aid to finding the graph of the inequality. In fact the graph of the inequality includes the line and the region below it. As you might have already inferred, if the constraint had a "$\geq$" instead of the "$\leq$," then the region above rather than below the line would be included in the graph. Figure 8-5 presents the graph of the constraint as it appears in the linear program.

The other constraints are handled in the same manner. In order to graph the constraint

$$2x_1 + 6x_2 \leq 30$$

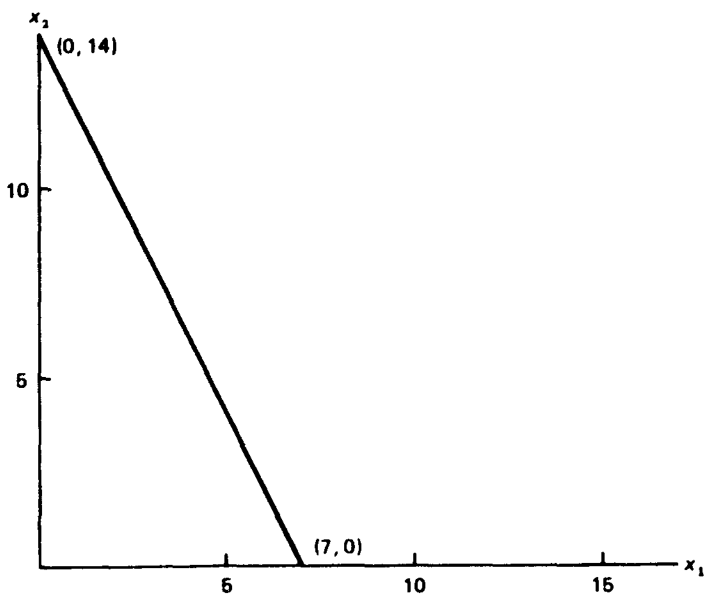

Figure 8-4 The graph of the linear equation $4x_1 + 2x_2 = 28$.

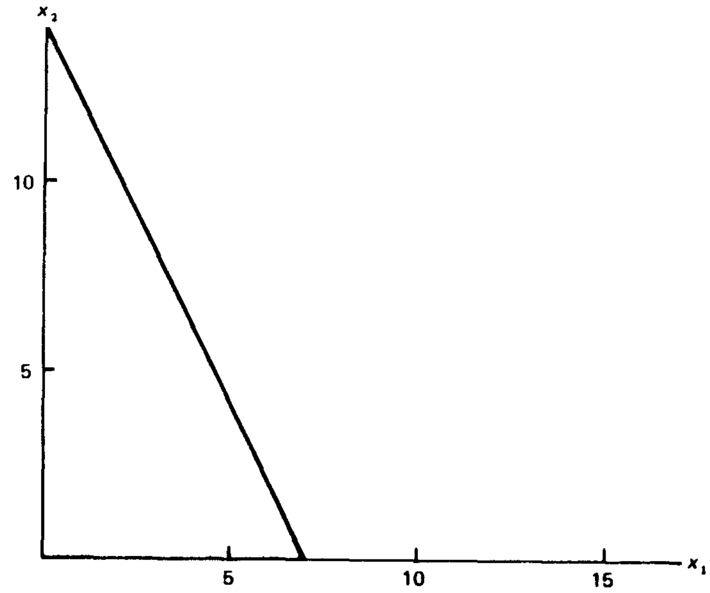

Figure 8-5 The graph of the constraint $4x_1 + 2x_2 \leq 28$.

first graph the equation

$$2x_1 + 6x_2 = 30$$

In this equation the x_1 intercept is 15 and the x_2 intercept is 5. The line and the region below it form the graph of the inequality constraint as in Figure 8-6.

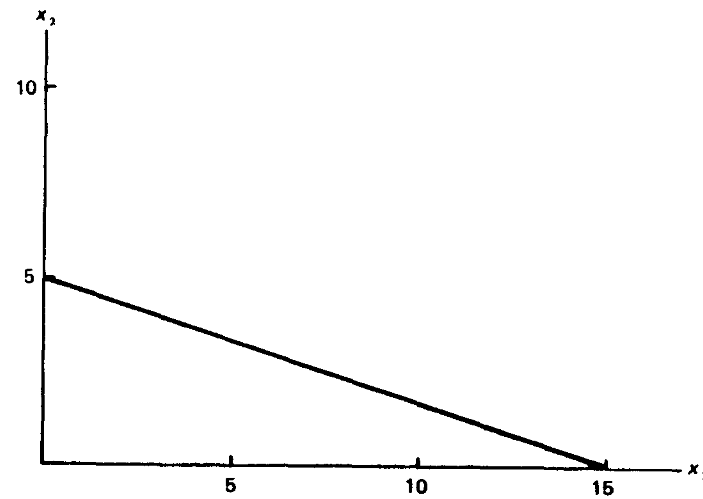

Figure 8-6 The graph of the constraint $2x_1 + 6x_2 \leq 30$.

Finally to graph the constraint

$$4x_1 + 6x_2 \leq 36$$

first graph the equation

$$4x_1 + 6x_2 = 36$$

In this equation the x_1 intercept is 9 and the x_2 intercept is 6. The line and the region below it form the graph of the inequality constraint as presented in Figure 8-7.

The Feasible Region

The graphs of the three constraints have been presented separately. Recall, however, that the number of homes and plants to be inspected must satisfy all three constraints. The region that satisfies all three constraints is that portion of the graph that is on or below all three lines. That region is presented in Figure 8-8. The region that satisfies all the constraints is called the *feasible region*.

Any point in the feasible region will satisfy all the constraints; each point represents a feasible alternative for the number of homes and plants to inspect. For example, 1 home and 4 plants, 5 homes and 2 plants, 4 homes and 3 plants, and 0 home and 5 plants all satisfy the three constraints. The Public Power Commission must choose the combination of homes and plants within the feasible region that yields the maximum number of complete inspections. That is, the objective function must now be considered in light of the constraints.

Figure 8-7 The graph of the constraint $4x_1 + 6x_2 \leq 36$.

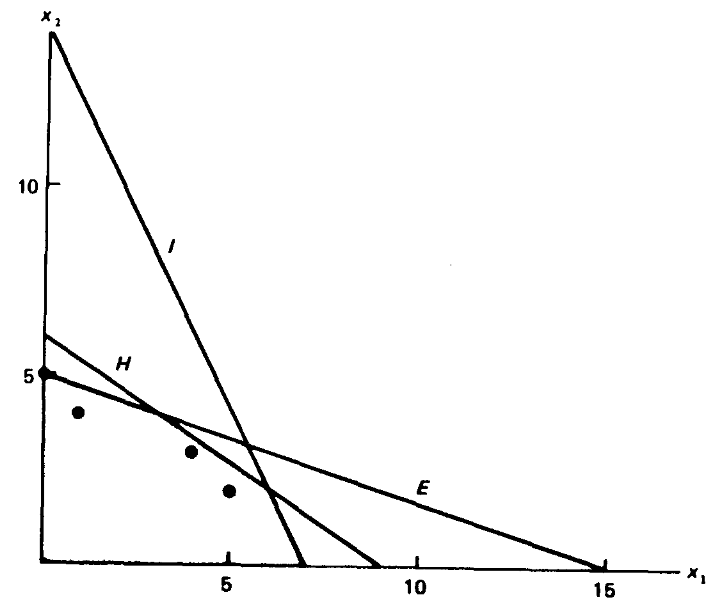

Figure 8-8 The graph of the feasible region satisfying $4x_1 + 2x_2 \leq 28$; $2x_1 + 6x_2 \leq 30$; $4x_1 + 6x_2 \leq 36$.

Graph of the Objective Function

The objective function

$$N = x_1 + x_2$$

has a different graph for each possible value of N. If $N = 8$, the objective function becomes

$$8 = x_1 + x_2$$

The graph of that equation can be found by finding the x_1 and x_2 intercepts; for this equation the x_1 intercept $= 8$ and the x_2 intercept $= 8$. Figure 8-9 presents the graphs of the objective function for various values of N, namely, $N = 2, 3, 5, 8$, and 12. Such a set of parallel lines is called a *family*.

Note that any point on the graph of

$$8 = x_1 + x_2$$

represents a number of homes and plants that together equal 8. For example, the point (5, 3) is on the line, and $5 + 3 = 8$. Similarly, any point on the graph of $5 = x_1 + x_2$ represents a number of homes and plants that together equal 5. By noting these points in Figure 8-8, one can see that the point (5, 3) is outside the feasible region; it corresponds to a number of homes and plants that is not allowable. The point (3, 2) is within the feasible region; it corresponds to a number of homes and plants that is allowable. The alternative (3, 2) is not the

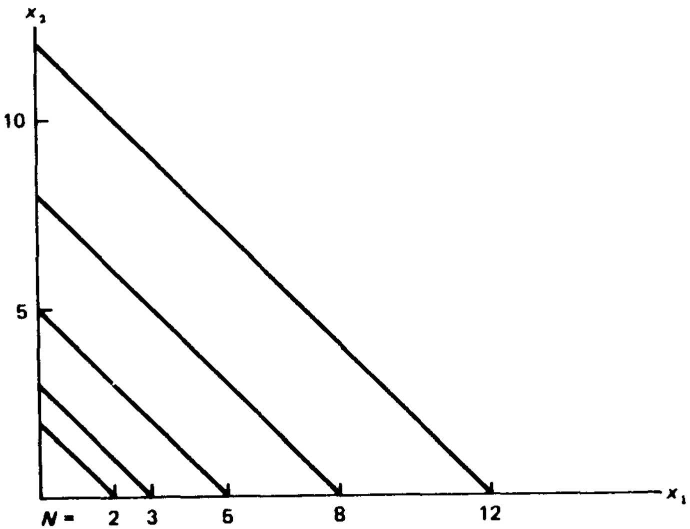

Figure 8-9 The objective function $N = x_1 + x_2$, for N = 2, 3, 5, 8, and 12.

solution that gives the maximum number of inspections, however, as there are feasible points beyond the line $5 = x_1 + x_2$.

The Optimum Feasible Solution

To find the *maximum* solution, we consider lines parallel to $5 = x_1 + x_2$ until we reach the *highest* line with a point in the feasible region. Figure 8-10 presents such a line. This line has only one point in the feasible region, namely, (6, 2). Any point beyond $8 = x_1 + x_2$ is outside the feasible region and therefore does not represent a feasible solution. Any point below $8 = x_1 + x_2$ yields fewer inspections. Hence the optimum solution is (6, 2), corresponding to 6 homes and 2 plants, or a total of 8 inspections per week.

It can be shown mathematically that the optimum solution to a linear programming problem always occurs at an extreme point, that is, a corner point. If it should happen that the optimum solution occurs at two extreme points, then every point on the line segment between those two points is also a solution. This means that an alternate method of finding the optimum solution is to graph the feasible region (by graphing all the constraints), identify the feasible points of intersection of the constraints (the extreme points of the feasible region), and evaluate the objective function for each extreme point. The point that yields the highest value in the objective function is the optimum solution. It should be emphasized that the graphic method of solution is appropriate only when there are only two variables in the linear program. Most real problems have more than two variables; so the graphic method is not used in practice. Our main purpose in using it here is to develop an intuitive understanding of a linear program,

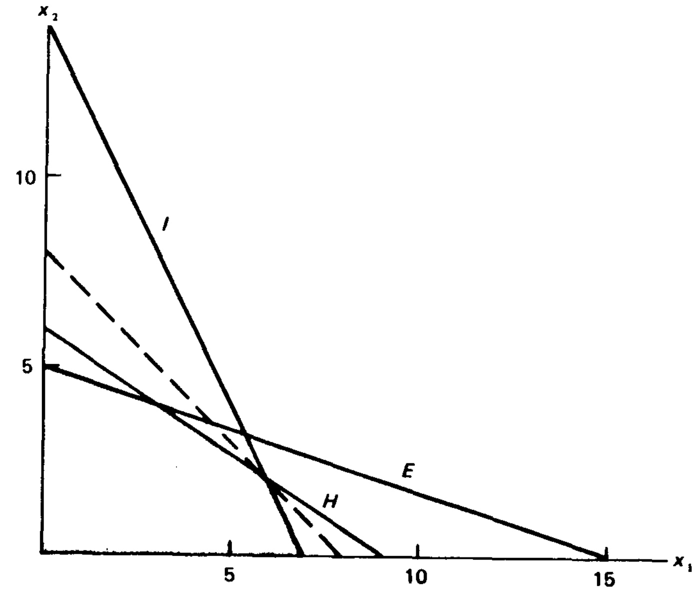

Figure 8-10 Finding the optimum solution to the energy inspection problem.

specifically of the relationships between the constraints of a problem, the ensuing feasible region, and the objective function. The basic relationships remain valid for linear programs of any size.

INTERPRETATION

In general, the solution provided by the linear programming model must be interpreted in terms of the original problem, not just with reference to the abstracted version. Conditions that were not quantifiable in the first place are necessarily omitted from the analysis. A qualitative analysis may indicate that the apparent optimum solution is not feasible for political or organizational reasons. Reconsideration of the solution in light of the whole situation may prompt a restatement of the problem that will alter the objective function and/or some constraints.

The reaction to the optimum solution in the wasted-power problem might be something like this: "Since industrial plants use more energy than homes, information about plants is more valuable. Surely plants should comprise more than 25 percent of the inspections." Consequently, the objective function should be altered to reflect this belief, which had not crystallized when the problem was originally stated. Such alterations are analyzed through a reformulation and new solution, or through sensitivity analysis of the old solution.

All linear programming models have certain common characteristics that are implications of the underlying assumptions, and that indicate limitations on the use of the model.

IMPLICATIONS AND LIMITATIONS

The basic assumptions of linear programming are:

1. The objective function is linear.
2. The constraints are linear.
3. The variables are nonnegative.

There are implications of these assumptions, which limit its use.

1. The model is *deterministic:* Each coefficient is assumed to be known with certainty. In practice, the coefficients are often estimated. Sensitivity analysis can be used to assess the consequences of fluctuations in the coefficients. If the coefficients are really random variables, however, then stochastic programming, a more advanced technique, should be used.
2. The model is *proportional:* The objective function and constraints change in proportion to each variable. For example, doubling the number of homes inspected x_1 doubles the time it takes each inspector to inspect them. In some situations, more or less of a resource may be used for higher levels of activity. For instance, each hour of a worker's time beyond a certain level may consume more dollars (time-and-a-half) than the basic amount. When proportionality holds over specific ranges of values, piecewise linear programming can be used; in other cases of nonproportionality, nonlinear programming may be needed.
3. The model is *additive:* The contribution of the various components are added to get the contribution of all of them together. A public health education program tries to reach as many people as possible through public relations-type television spot announcements (x_1 = how many) and through the distribution of informational brochures (x_2 = how many). It has found that each brochure is more effective if the recipient has seen the television message. The average effectiveness of a brochure depends on how many television spots there have been. Hence, in the objective function, the coefficient of x_2 would contain x_1, and the resulting term would not be linear. Linear programming could not be used for such a problem; nonlinear programming would be applicable.
4. The model is *divisible:* Fractional values for the variables are permissible. Some problems may require integer solutions. In such cases integer programming is the recommended approach, unless the linear program solution just happens to be in integers. An alternative is to round off the linear program solution to the nearest integer solution. Extreme caution must be exercised in rounding off because the result has three possibilities:

   a. The rounded solution is the best integer solution.
   b. The rounded solution is feasible but not the best.
   c. The rounded solution is not even a feasible alternative.

Stochastic, nonlinear, and integer programming are more advanced techniques and are beyond the scope of this book. We will reinforce the basic ideas of linear programming with the graphic solution of another problem and in so doing introduce a minimization problem.

A MINIMIZATION PROBLEM

The County of Hillandale operates a small game park for the benefit of its residents, and to attract tourists during the summer. Among its many inhabitants is a pride of rare minilions. The County Recreation Board, and especially the curator of the game park, want to be sure that the animals' basic nutritional requirements are satisfied, but at the same time they do not want to spend more for their food than is necessary. Their necessary vitamins are provided in a supplement. Their daily diet must provide at least 500 units of protein and 960 units of mineral fiber. To keep the minilions from feeling hungry—and hence unhappy—they must be given at least 6 kilograms of food daily. There are also certain ceilings on the amounts of protein, mineral fiber, and total food provided, but we will assume that the objective of minimizing the total cost of feeding the minilions will keep the diet within the maximum allowable levels. The foods that they will be given are prime meat and ground bone meal. A kilogram of the bone meal provides 50 units of protein and 240 units of mineral fiber, and costs \$1 per kilogram. A kilogram of prime meat provides 250 units of protein and 80 units of mineral fiber, and costs \$2 per kilogram. The combined weight of the bone meal and the meat must be at least 6 kilograms. How many kilograms of each food type should be included in the diet of the minilions?

It is sometimes helpful to summarize the salient features of the problem statement prior to formulating the linear program. The content and cost of a kilogram of each food type is presented in Table 8-1.

Each of the minimum requirements generates a constraint. Each kilogram of bone meal provides 50 units of protein; so x_1 kilograms provides $50x_1$ units of protein. Similarly, each kilogram of meat provides 250 units of protein; so x_2 kilograms of meat provides $250x_2$ units of protein. Therefore the total amount of protein provided is $50x_1 + 250x_2$. Since this total must be at least 500, the protein constraint is

$$50x_1 + 250x_2 \geq 500$$

In a similar way the mineral fiber constraint is found to be

$$240x_1 + 80x_2 \geq 960$$

Table 8-1 Contents and cost of food types in minilion diet problem

	Food type		
	Bone meal	Meat	Minimum requirement
Protein units per kg	50	250	500 units
Mineral fiber units per kg	240	80	960 units
Weight, kg	1	1	6 kg
Cost per kg	1	2	dollars

and the weight constraint is

$$x_1 + x_2 \geq 6$$

The cost of providing the x_1 kilograms of bone meal is $1x_1$, and the cost of providing the x_2 kilograms of meat is $2x_2$; so the total cost in dollars of providing the daily diet is

$$x_1 + 2x_2$$

The cost is the quantity that the County Recreation Board wishes to minimize. The complete linear program is as follows:

Minimize $\qquad C = x_1 + 2x_2$

subject to
$$50x_1 + 250x_2 \geq 500$$
$$240x_1 + 80x_2 \geq 960$$
$$x_1 + x_2 \geq 6$$
$$x_1, x_2 \geq 0$$

The last constraints are the nonnegative constraints; it is senseless to consider providing a negative quantity of bone meal or meat!

To find the solution graphically, we first find the graphs of the constraints. The line associated with each constraint is found by finding the intercepts and the line that joins them. The graph of the constraint is the line and the region above the line. Figure 8-11 contains the graphs of the constraints. The feasible region satisfies all the constraints and is the shaded region.

The objective function for two values of cost is presented in Figure 8-12. Lines parallel to these are considered until we find the lowest line containing at least one point in the feasible region. Such a line is identified in Figure 8-13. The minimum solution is therefore (5, 1), which represents 5 kilograms of bone meal and 1 kilogram of meat for the daily diet of the minilions. The cost of such a diet is 5 × 1 + 1 × 2 = $7 per day per minilion. This is the least expensive diet that will satisfy the nutrition requirements.

The minimization linear program differs from the maximization program es-

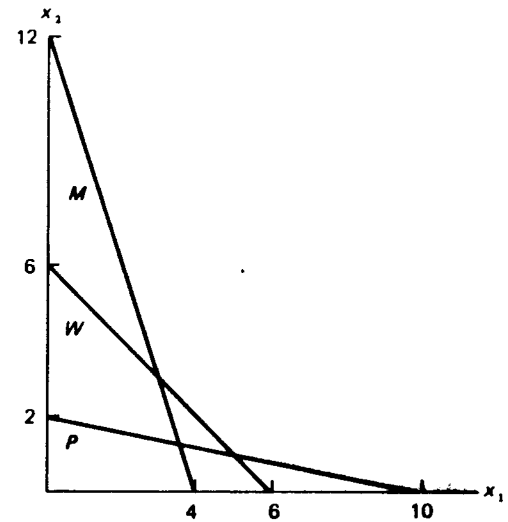

Figure 8-11 Graphs of the constraints: protein, $50x_1 + 250x_2 \geq 500$; mineral fiber, $240x_1 + 80x_2 \geq 960$; weight, $x_1 + x_2 \geq 6$.

sentially in its orientation. The specific steps in solving the two programs are similar, except for that difference in orientation.

The energy inspection linear program had constraints that are all of the "≤" variety; the minilion diet linear program had all "≥" constraints. In general, a linear program, whether it be a minimization or maximization problem, can have some constraints of each of the three types: "≤," "≥," and "=."

The next illustration is of such complexity that it does not permit a graphic solution. It contains constraints of all three types. Such a problem would undoubtedly be solved by the simplex method of solution using a computer.

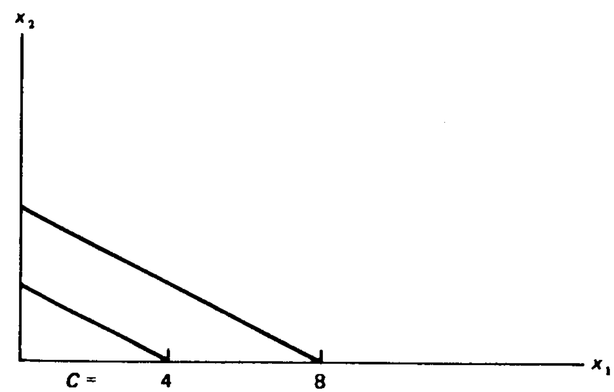

Figure 8-12 The objective function $C = x_1 + 2x_2$, for C = 4, 8.

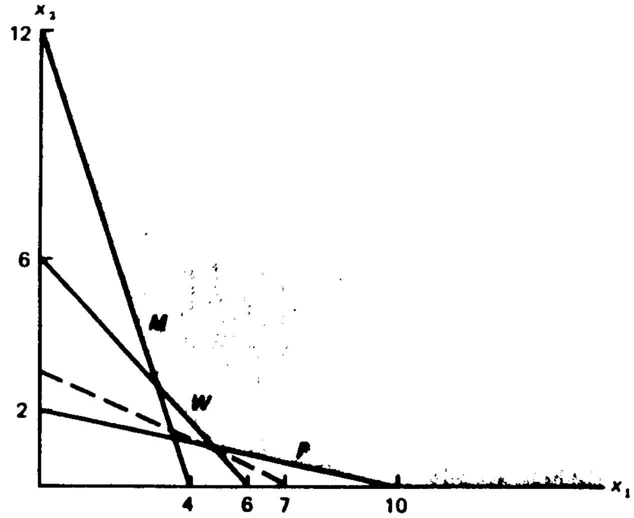

Figure 8-13 The optimum solution to the minilion diet problem.

The details of the simplex method, not essential for interpreting a solution, are presented in Chapter 10 for the reader interested in developing a deeper understanding of linear programming.

A POLICY ANALYSIS APPLICATION

Policy analysis is concerned primarily with examining the consequences of various alternatives. These alternatives may be different programs delivering different services to the public, for example, a reading readiness program for preschoolers and a dental clinic for preschoolers. They may be different strategies for delivering the same service, for example, to increase public safety by improving street lighting or by increasing the number of police officers assigned to foot patrol. Finally, they may be a single strategy's alternative levels of implementation. For example, a national health insurance program has many possible levels of funding by the federal government, contributions by individuals in the program, and benefits to the insured. The latter instance of policy analysis may well be profitably addressed by linear programming. At this point it should be emphasized that linear programming or any quantitative technique should neither be the sole basis for choosing public policy nor the prime reason for determining the specific levels of involvement of the various participants in the public program. The quantitative analysis can provide some information about the consequences of choosing particular levels of participation. This can help to provide an increased opportunity for rational decision making.

National health insurance, grants for education, and housing subsidies are some examples of attempts to redistribute income among income classes. Each such program consists of prescribed levels of participation and consequent impacts.[1] For the national health insurance program the prescribed levels of participation include (1) the deductible amount, or the initial expenditure that is paid by the individual; (2) the coinsurance rate, or the portion of each dollar of expenditure above the deductible that the individual must pay; and (3) the premium, the amount paid by each individual regardless of whether he or she ever needs the insurance. These prescribed levels are directly controlled by the government and as such may be referred to as the *program design variables*.

The impacts or effects of different levels of participation can be described in various terms, for example, the impact on the family expenditure within each income class or the value of the benefits received by each income class. These impacts are not directly controlled by the government; rather they are the effects of the specific prescribed levels of participation. These impacts may be referred to as the *distributional impact variables;* they are the measures of the redistribution of income that takes place as the result of the enactment of the program at the specified levels.

For the sake of simplicity we will assume that only lower- and middle-income classes are to be included in the program. However, in practice one would include in the same model many income classes defined as finely as is relevant and useful. Let all families with incomes under $10,000 be class I, and all those with incomes from $10,000 to $20,000 be class II. The objective of redistributing income through the subsidized health insurance plan will generate various constraints, which we now consider.

Suppose the direct benefits are required to decrease at the rate of at least $5 per $1000 in additional income. Assuming a $10,000 difference between the two income classes, the direct benefits to the average family in class I must be at least $50 more than to the average family in class II. This generates the constraint

$$B_1 - B_2 \geq 50$$

To assure that the greater benefits are not accompanied by greater premiums, the premium paid by a family in class I should not be more than the premium paid by a family in class II:

$$P_1 - P_2 \leq 0$$

It is believed by some that maintaining reasonably high deductible amounts and coinsurance rates will prevent the generating of overwhelming levels of health care; not permitting them to be too large will prevent any family from bearing too great a burden. Finally, the deductible amounts and the coin-

[1] This illustration is based on the article, "Distributional Constraints in Public Expenditure Planning," by Martin Feldstein and Harold Luft, appearing in *Management Science*, vol. 19, August 1973, pp. 1414–1422.

surance rates should not be larger for the lower-income group than for the middle-income group. These considerations lead to three constraints for both the deductible amounts and the coinsurance rates.

$$\begin{array}{ll} D_1 \geq \$100 & C_1 \geq .15 \\ D_2 \leq \$300 & C_2 \leq .40 \\ D_1 - D_2 \leq \$0 & C_1 - C_2 \leq 0 \end{array}$$

The constraint values \$100, \$300, .15, .40 and any other such values are called parameters. A *parameter* is a quantity that can vary from problem to problem but is constant for a specific problem. For example, the deductible amounts can vary according to the desires of the program designers, but once the choice is made they remain constant for all individuals in the same income class. Hence the choice of values is necessarily arbitrary, but hopefully not capricious. As we shall see shortly, the linear program will allow the decision makers to view the consequences of choosing any specific parameter values before the insurance program is implemented.

Since the program is voluntary, in order to enroll all the households, the program designers adopt the constraint that the premiums must not exceed 90 percent of the direct benefits at any income level. Hence

$$.90B_1 - P_1 \geq 0$$

$$.90B_2 - P_2 \geq 0$$

The final constraint relating to distribution of benefits comes from the amount of the subsidy. We assume the program is limited to the 4.8 million four-member (that is, two children) lower-income families, and the 2.4 million four-member middle-income families. In reality, of course, a program of this type would not be limited to such families. Including families of any size in the model would involve identifying the number of such families and perhaps adjusting the various program variables. Such is not difficult, but it would make our illustration more cumbersome than our purpose allows. If the limit on government subsidy is \$1 billion, then total benefits cannot exceed total premiums by more than that amount. Hence we have the constraint

$$4{,}800{,}000(B_1 - P_1) + 2{,}400{,}000(B_2 - P_2) \leq \$1{,}000{,}000{,}000$$

The relationship between the direct benefits and the deductible amounts and coinsurance rates is not one that is determined by setting program variable levels. Rather, it is a complicated one involving the stochastic nature of health expenditure. Hence an analytic form of the relationship can only be estimated through multiple regression based on available data. The equations that follow have been found to approximate the relationship rather well.

$$B_1 = 524 - .33D_1 - 300C_1$$

$$B_2 = 524 - .33D_2 - 300C_2$$

These equations complete the set of constraints for the problem statement. All that remains for the linear program is the objective function.

A dollar of direct net benefit to a lower-income family is considered to be worth more than a dollar of direct net benefit to a middle-income family. A way of expressing this consideration is to apply a greater weight to the class I benefits than to the class II benefits. For example, assuming the average income in the two classes to be \$5000 and \$15,000, the net benefits might be assigned the weights $\frac{1}{5}$ and $\frac{1}{15}$, respectively. If the objective is to maximize the total worth (not dollar value) of the program, then a suitable objective function would be

Maximize

$$W = \tfrac{1}{5} \cdot 4{,}800{,}000(B_1 - P_1) + \tfrac{1}{15} \cdot 2{,}400{,}000(B_2 - P_2)$$

Letting the weight be the reciprocal of the average income in each class quantifies the claim that the worth of a dollar decreases with increased income, an example of the so-called law of diminishing marginal utility. It also permits the easy extension of the model to any number of income classes, with each class covering any range of income.

Summarizing our translation from problem statement to linear program, we have the following:

Maximize

$$W = 960{,}000B_1 + 160{,}000B_2 - 960{,}000P_1 - 160{,}000P_2$$

subject to

$$\begin{array}{llllllllll}
B_1 & & & & -B_2 & & & & \geq & \$50 \\
 & P_1 & & & & -P_2 & & & \leq & \$0 \\
 & & D_1 & & & & & & \geq & \$100 \\
 & & & & & & D_2 & & \leq & \$300 \\
 & & D_1 & & & & -D_2 & & \leq & \$0 \\
 & & & C_1 & & & & & \geq & .15 \\
 & & & & & & & C_2 & \leq & .40 \\
 & & & C_1 & & & & -C_2 & \leq & 0 \\
.90B_1 & -P_1 & & & & & & & \geq & \$0 \\
 & & & & .90B_2 & -P_2 & & & \geq & \$0 \\
4.8B_1 & -4.8P_1 & & +2.4B_2 & & -2.4P_2 & & & \leq & 1{,}000 \\
B_1 & & +.33D_1 & +300C_1 & & & & & = & 524 \\
 & & & & B_2 & & +.33D_2 & +300C_2 & = & 524 \\
\end{array}$$

$$B_1, P_1, D_1, C_1, B_2, P_2, D_2, C_2 \geq 0$$

Table 8-2 Solutions to the national health insurance linear program

Income class	Variables	Solution A with $D_2 \leq \$300$	Solution B with $D_2 \leq \$200$
Lower:	Direct benefits	\$447	\$447
	Premiums	254	256
	Deductible	100	100
	Net benefits	193	191
	Coinsurance rate	.15	.15
Middle:	Direct benefits	\$306	\$339
	Premiums	275	305
	Deductible	300	200
	Net benefits	31	34
	Coinsurance rate	.40	.40
Objective function		190,880,000	189,810,000

We note that this linear program contains constraints that are not all of the same type; that is, some are "≤," some are "≥," and some are "=" constraints.

Needless to say, this linear program, much less the full-scale one from which this illustration was developed, could hardly be solved using graphic methods. Fortunately, there are methods that do not depend on the graphic representation, and such methods are available on most computers. The solution to our linear program was found by using a computer program; that solution is presented in solution A of Table 8-2. One of the main advantages of analyzing proposed program levels in this manner is to see the effects of altering some of the program variables. Solution B of Table 8-2 presents the solution to the linear program with one alteration. The limit on the deductible amount for the middle-income class has been decreased from \$300 to \$200; that is, the fourth constraint has been replaced by $D_2 \leq \$200$.

Interpretation

The interpretation of the objective function is best described in terms of aggregate worth to the families or, more technically, in terms of "uniformly distributed dollars." Changing the constraint on the middle-income-class deductible amount has little effect on the objective function and on the lower-income-class variables, but it does have an appreciable effect on the premiums and the direct benefits for the middle-income class. The small increase in their net benefits balances the decrease in net benefits to the lower-income class.

This model could be manipulated in many ways to determine the overall effect of altering one or more of the program variables. This type of "what if" analysis, referred to as "sensitivity analysis," is more fully treated in the next chapter. Aside from considering the effect of altering values, the model could be expanded to determine the effect of including other variables. The decision makers may wish to know the effect of including higher-income classes in the program, of accounting for regional differences in health care costs, of segmenting the population into narrower income classes or by other characteristics.

Linear programming has been used here in a simulation mode, in which the parameters can be modified to simulate program activity generated by various values for the program variables. Solving the program for each such modification shows the various impacts of such program variable values. The value of the specific program can then be assessed with reference to the nonquantitative factors that were not represented in the model.

Another application will conclude our discussion of linear programming.

AN APPLICATION TO SCHOOL BUSING

Busing students from one school to another to achieve racial balance has become a volatile social and political issue. In establishing policy there would undoubtedly be a number of higher-level goals (for example, to create the proper environment for the educational process) that would generate lower-level operational objectives. Among these might be to integrate schools according to some legally specified proportion, to transport children in order to satisfy the integration objective, and to avoid having the pupils spend an inordinate amount of time on the buses. Educational planners must be able to address the travel-time issue, while striving to satisfy other requirements. One approach to the problem is to consider racial balance, school capacity, and population of the school district as constraints, while striving to transport the children with as little inconvenience to them as possible. Suppose the Capital City School District has two elementary schools, with individual capacities as follows:

School	Capacity
I	600
II	300

For simplicity's sake, we assume there are three neighborhoods with the elementary school populations as summarized in Table 8-3. The distances in miles from the neighborhoods to the schools are:

	Neighborhood		
School	*A*	*B*	*C*
I	2.4	1.8	1.5
II	1.6	4.0	2.0

Table 8-3 Elementary school population by neighborhood and race

	Population		Percent	
Neighborhood	White	Nonwhite	White	Nonwhite
A	240	80	75	25
B	60	120	33.33	66.67
C	180	120	60	40
Total	480	320	60	40

The district has decided to integrate by school rather than by grade level. A school will be considered to be in racial balance when it deviates by no more than 10 percent from the district ratio of 60 percent white and 40 percent nonwhite. The district wishes to devise a busing plan that will transport pupils to any school in the same racial ratio as their neighborhood and will minimize the distance the youngsters must travel.

In order to formulate the linear program that would represent the problem, we let A_I stand for the number of pupils to be bused from neighborhood A to school I, A_{II} the number of pupils to be bused from neighborhood A to school II, and so forth.

The capacities of the schools are represented in two constraints:

$$A_I + B_I + C_I \leq 600$$

$$A_{II} + B_{II} + C_{II} \leq 300$$

The populations of the neighborhoods generate three constraints:

$$A_I + A_{II} = 320$$

$$B_I + B_{II} = 180$$

$$C_I + C_{II} = 300$$

Transporting pupils according to their neighborhood's racial ratio and the criterion for a school to be in racial balance generate two constraints for each school according to the following reasoning:

1. Racial balance requires that in school I white students comprise between 50 and 70 percent of the school population, that is,

   White students in school I $\leq$ .70(all students in school I)

   White students in school I $\geq$ .50(all students in school I)

2. Since 75 percent of the students in neighborhood A are white, racial ratio transportation requires that 75 percent of the students bused from neighborhood A to school I be white. Similarly, 33.3 percent of those bused from neighborhood B to school I must be white, and 60 percent of those bused from neighborhood C to school I must be white. Thus the number of white students being bused to school I is

$$.75A_I + .333B_I + .60C_I$$

3. The total number of students being bused to school I is

$$A_I + B_I + C_I$$

Putting the three relationships together yields the next two constraints. The same reasoning for school II yields the succeeding constraints.

$$.75A_I + .333B_I + .60C_I \leq .70(A_I + B_I + C_I)$$

$$.75A_I + .333B_I + .60C_I \geq .50(A_I + B_I + C_I)$$

$$.75A_{II} + .333B_{II} + .60C_{II} \leq .70(A_{II} + B_{II} + C_{II})$$

$$.75A_{II} + .333B_{II} + .60C_{II} \geq .50(A_{II} + B_{II} + C_{II})$$

Within these constraints, the objective is to minimize the total distance traveled; that is,

Minimize

$$D = 2.4A_I + 1.8B_I + 1.5C_I + 1.6A_{II} + 4.0B_{II} + 2.0C_{II}$$

The solution to the linear program, which merely indicates *the number of children* from each neighborhood to be assigned to each school, is as follows:

	Neighborhood		
School	A	B	C
I	56	144	300
II	264	36	0

In reality, not all youngsters from a given neighborhood are the same distance from a particular school. By defining neighborhoods small enough, a reasonable approximation to actual distances traveled is possible. However, if the pupils are widely dispersed over a large area, so that there are few pupils in any one "neighborhood" and very many neighborhoods, then linear programming would not be an appropriate tool for analysis. Other factors that would be present but which we have not considered are whether each grade within an integrated school should be integrated, different specific pupil needs that may be satisfied at one school but not at another, and costs of and resources available for transportation. These add to the complexity of the decision situation, the linear program (if, in fact, linear programming is considered appropriate), and the extent of the sensitivity analysis that would be conducted.

SUMMARY

Linear programming is particularly well suited as a decision-making aid in situations in which the decision makers have a specific linear objective function to be maximized or minimized, within fixed linear constraints. Linear programming is employed through problem formulation, solution, and interpretation. Simple programs may be solved by the graphic method; more complex programs require more involved methods, but the underlying concepts remain applicable. The linear constraints determine the feasible region, or set of allowable alternatives. The solution is a feasible alternative that optimizes the objective function. Applications vary from the operational level, as in aiding school bus assignments, to the policy level where the impacts of various program alternatives can be assessed through reformulation and re-solution.

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EXERCISES

Extensions of chapter examples

8-1 Refer to the energy inspection example. (See p. 169.) If an additional 6 hours of the electrical inspector's time could be applied to the inspection project, the second constraint would become

$$2x_1 + 6x_2 \leq 36$$

The other constraints and the objective function would not be affected. Hence the new linear program would be:

Maximize the objective function

$$N = x_1 + x_2$$

subject to the constraints

$$4x_1 + 2x_2 \leq 28 \quad \text{I}$$

$$2x_1 + 6x_2 \leq 36 \quad \text{E}$$

$$4x_1 + 6x_2 \leq 36 \quad \text{H}$$

$$x_1, x_2 \geq 0$$

(*a*) Solve the new linear program graphically.

(*b*) How does this solution compare with the solution to the original program?

(*c*) If another 6 hours of the electrical inspector's time (or a total of 42 hours) could be applied to the project, what would then be the number of homes and plants inspected? Base your answer not on a new graphic solution but on your response to parts *a* and *b*.

8-2 Refer to the minimization example in the chapter. (See p. 180.)

(*a*) How will the minilions' diet be altered if the cost of meat goes up to $3 per kilogram?

(*b*) How will their diet change if the cost of bone meal goes up to $3 per kilogram (with the cost of meat at $2 per kilogram)?

(*c*) How will their diet change if the cost of bone meal goes to $2 per kilogram and meat goes to $4 per kilogram?

(*d*) What are the effects of the changes in part *c* on the objective function?

(*e*) Generalize the results of parts *c* and *d*.

8-3 Refer to the national health insurance policy analysis application. (See p. 184.) The equations for direct benefits are:

$$B_1 = 524 - .33D_1 - 300C_1$$

$$B_2 = 524 - .33D_2 - 300C_2$$

These are estimated through the multiple regression statistical technique. Comment on using such equations in linear programming, with particular reference to the implications of the underlying assumptions. How should such equations be treated in the linear programming model?

Other applications

8-4 Which of the following relationships might be found in a linear programming model?

(*a*) $2X_1 - 4X_2 + \frac{1}{2}X_3 \geq 100$

(*b*) $3X_1 + \frac{1}{X_2} \leq 50$

(*c*) $3X_2 - 6X_3^2 = 80$

(*d*) $X_1 - X_2 + X_3 \leq 40$

(*e*) $X_1X_2 - X_3 \geq 17$

8-5 (*a*) Solve graphically:

Maximize $Z = 4Y_1 + 10Y_2$

subject to

$$Y_1 + 4Y_2 \leq 20$$
$$8Y_1 + 2Y_2 \leq 40$$
$$Y_1 + Y_2 \leq 10$$
$$Y_1, Y_2 \geq 0$$

(*b*) Can any one of the constraints be eliminated without altering the feasible region?

8-6 Solve the following minimization problem graphically:

Minimize $C = 2X + Y$

subject to

$$Y \geq 6$$
$$X + Y \geq 10$$
$$X, Y \geq 0$$

8-7 Solve graphically:

Maximize $Z = 4X_1 + 6X_2$

subject to

$$X_1 + 2X_2 \leq 200$$
$$4X_1 + 3X_2 \leq 480$$
$$X_1 \geq 100$$
$$X_1, X_2 \geq 0$$

8-8 Refer to problem 8-7. Change the last constraint to

$$X_1 \geq 140$$

(*a*) What is the effect on the feasible region?
(*b*) What is the new solution?

8-9 Formulate and solve graphically: Food *A* costs \$2 per kilogram and food *B* costs \$3 per kilogram. A kilogram of *A* yields 2 units of vitamins, 10 units of starch, and 6 units of protein. A kilogram of *B* yields 6 units of vitamins, 2 units of starch, and 4 units of protein. The minimum requirements of each ingredient are 178, 200, and 240, respectively. What combination of *A* and *B* will give an adequate diet with least cost?

8-10 The American Safety Council has allocated \$220,000 to efforts to prevent automobile accidents. An assumed measure of the effectiveness of such efforts is the reduction in fatalities and property damage. The projects that have been suggested for funding and some relevant values are presented in Table 1. It is readily admitted that saving lives is the more important objective, but yet the decision makers are not willing to ignore the objective of averting property damage. Realizing

Table 1 Suggested safety projects and values

	Maximum allowable expenditure	Expected fatalities prevented per \$1000	Property damage averted per \$1000
Teen-age safety education	\$160,000	.22	\$20,000
Seat-belt advertising	130,000	.30	0
Lobbying for stiff DWI* laws	110,000	.28	15,000
Research in improved vehicle design	70,000	.16	40,000

* Driving while intoxicated.

that a human life cannot be equated to dollars, reference is made to other government agencies that use \$300,000 for the value of a human life for internal analysis purposes.

(*a*) Formulate a linear programming model for the optimal allocation of the \$220,000 based on the information given and the fatalities-prevention objective alone.

(*b*) Formulate another linear programming model that considers both lives and property and uses the assumed value of a life.

(*c*) Without solving either linear program, by comparing the two objective functions, how do you think the solutions will differ?

8-11 The demand for hospital services has been found to be quite seasonal. As a result, staffing the various services without frequent layoffs and rehirings requires some planning. The personnel director of Metropolitan Hospital is trying to decide how many orderlies to hire and train for the next 4 months. According to admissions forecasts, the demand for orderly-hours for the next 4 months is:

Month	Hours needed
Dec.	800
Jan.	1000
Feb.	900
Mar.	1200

The IOU (International Orderlies Union) has recently signed a contract calling for a guaranteed 37½-hour work week, with overtime prohibited to maintain a high employment level. We will take that to mean each employed orderly is paid for 150 hours a month. Hospital records indicate that 10 percent of the orderlies quit their jobs each month.

Orderlies are given a 1-month training period to become familiar with hospital layout, patient handling, and emergency codes; hence they must be hired a month before being assigned regular duty. They are paid \$400 a month while in training and \$800 a month afterward. It takes approximately 10 hours of regular orderly time for each trainee during the training period; that is, the number of hours available for regular service by orderlies is reduced by 10 for each trainee.

Since there are currently six orderlies available, the hospital has 100 hours in excess of its needs for December. No one will be laid off; each orderly will work a few hours less, but be paid for full employment.

Formulate a linear program to solve the personnel director's staffing problem, identifying each symbol used. Do not solve the formulated problem.

8-12 The State Employees Credit Union makes various kinds of loans to members and invests up to one-third of its funds in market securities. Not more than 20 percent of money lent can be in unsecured (signature) loans. Furniture and expansion loans cannot be more than half of all secured loans. Signature and expansion loans may not exceed the market securities investment. The investments have the following yields:

Investment	Yield, %
Securities	10
Signature loans	13
Secured loans:	
Automobile	9
Furniture	11
Expansion	12

The credit union wants to determine the proportion of its funds to allocate to each category in order to maximize interest earned. Formulate (but do not solve) the problem as a linear program.

8-13 The state's Department of Administrative Services maintains a few light planes for official use. $10,000 has been allocated for fuels, which are purchased on a competitive bid basis. By adjusting the planes' engines, varying proportions of regular, super, and fuel additive can be used. The department wants to determine how many gallons of each type to purchase from each of the only two bidders, Poly Oil Co. and Petros Corp. The bids' summary follows:

	Price per gallon		Maximum to be supplied (gallons)	
	Poly	Petros	Poly	Petros
Regular	$1.00	$ .50	12,000	6,000
Super	1.50	2.00	1,000	8,000
Additive	5.00	6.00	200	200

A summary of the department's experience follows:

	Miles per gallon	Minimum number of gallons needed
Regular	5	8000
Super	10	2000
Additive	20	240

The department's objective is to maximize the number of air miles provided for by the total purchase. Formulate (but do not solve) the problem as a linear program.

8-14 The Department of Agriculture of a developing nation is encouraging better crop planning. The central region consists of three provinces, each with its own land availability and water capacity. The province specifics are presented in Table 1:

Table 1 Province data for the central region

Province	Available land (acres)	Water capacity (gallons)
A	800	600,000
B	1200	800,000
C	600	375,000

The Department of Agriculture has established a maximum planting for each crop in each region. Each crop has fairly well-established levels of water consumption and an expected return per acre. The crop information is presented in Table 2.

Table 2 Crop data for the central region

Crop	Maximum (acres)	Water consumption (gallons per acre)	Revenue (dollars per acre)
Millet	650	1000	$200
Cane	1200	3000	800
Cotton	1000	2000	600

The three provinces of the central region have agreed to plant the same proportion of available land. They want to determine how many acres of each crop should be planted in each province in order to maximize the total revenue to the region.

Formulate (but do not solve) this problem as a linear program.

Project problems

8-15 Observe any governmental or service agency process that consumes limited resources and/or has other restrictions.

(*a*) Describe the process. If specific data are not readily available, make up reasonable data.

(*b*) Formulate the process as a linear program, noting any assumptions.

(*c*) What (if any) aspects of the problem cannot be included in the linear program formulation?

8-16 Refer to any linear programming application from a journal in your own field of interest.

(*a*) Describe the problem verbally, including the objective and restrictions.

(*b*) What are the objective function and constraints of the linear program formulation?

(*c*) Is the problem directly presented as a linear programming model, or are modifications or assumptions made to convert it into a linear programming model?

(*d*) Does the original problem suffer for the conversion?

(*e*) What is the interpretation of the solution with reference to the original problem?