Stipak

REPLACEMENT ANALYSIS, BASIC APPROACH

A. Without Discounting

Construct a table to show for each time period t:

1) operating cost, C
2) total operating costs up to and including that time period, ΣC
3) salvage value, S
4) average cost per time period:

$$(\text{purchase cost} - S + \Sigma C)/t$$

Replace equipment after time period that results in minimum average cost per time period.

B. With Discounting

For costs incurred over a long time interval, a realistic analysis requires that future costs be discounted, based on an assumed interest rate i.

Construct a table to show for each time period t:

1) operating cost, C
2) present value of operating cost, PV(C)

$$PV(C) = \frac{C}{(1+i)^t}$$

(Note this assumes for simplicity that all operating costs for period t are incurred at end of period t.)

3) total present value of operating costs up to and including that time period, $\Sigma PV(C)$
4) Salvage value, S
5) present value of salvage value, PV(S)

$$PV(S) = \frac{S}{(1+i)^t}$$

6) net present value of all costs over the time period

$$PV(\text{all costs}) = \text{purchase cost} - PV(S) + \Sigma PV(C)$$

7) uniform annual equivalent cost

Use PMT for →
To compute the uniform annual cost, multiply PV(all costs) by a capital recovery (amortization) factor for a uniform annual series. Equivalently, divide PV(all costs) by an annuity discount factor.

Replace equipment after time period that results in minimum uniform annual equivalent cost.

In Excel:

PMT (rate, years, pv)

INTEREST TABLES

TABLE P

8% Compound Interest Factors

	1	2	3	4	5	6	
	Single Payment		Uniform Annual Series				
	Compound Amount Factor	Present Worth Factor	Sinking Fund Factor	Capital Recovery Factor	Compound Amount Factor	Present Worth Factor	
n	Given P To find S $(1+i)^n$	Given S To find P $\frac{1}{(1+i)^n}$	Given S To find R $\frac{i}{(1+i)^n-1}$	Given P To find R $\frac{i(1+i)^n}{(1+i)^n-1}$	Given R To find S $\frac{(1+i)^n-1}{i}$	Given R To find P $\frac{(1+i)^n-1}{i(1+i)^n}$	n
1	1.080	0.9259	1.00000	1.08000	1.000	0.926	1
2	1.166	0.8573	0.48077	0.56077	2.080	1.783	2
3	1.260	0.7938	0.30803	0.38803	3.246	2.577	3
4	1.360	0.7350	0.22192	0.30192	4.506	3.312	4
5	1.469	0.6806	0.17046	0.25046	5.867	3.993	5
6	1.587	0.6302	0.13632	0.21632	7.336	4.623	6
7	1.714	0.5835	0.11207	0.19207	8.923	5.206	7
8	1.851	0.5403	0.09401	0.17401	10.637	5.747	8
9	1.999	0.5002	0.08008	0.16008	12.488	6.247	9
10	2.159	0.4632	0.06903	0.14903	14.487	6.710	10
11	2.332	0.4289	0.06008	0.14008	16.645	7.139	11
12	2.518	0.3971	0.05270	0.13270	18.977	7.536	12
13	2.720	0.3677	0.04652	0.12652	21.495	7.904	13
14	2.937	0.3405	0.04130	0.12130	24.215	8.244	14
15	3.172	0.3152	0.03683	0.11683	27.152	8.559	15
16	3.426	0.2919	0.03298	0.11298	30.324	8.851	16
17	3.700	0.2703	0.02963	0.10963	33.750	9.122	17
18	3.996	0.2502	0.02670	0.10670	37.450	9.372	18
19	4.316	0.2317	0.02413	0.10413	41.446	9.604	19
20	4.661	0.2145	0.02185	0.10185	45.762	9.818	20
21	5.034	0.1987	0.01983	0.09983	50.423	10.017	21
22	5.437	0.1839	0.01803	0.09803	55.457	10.201	22
23	5.871	0.1703	0.01642	0.09642	60.893	10.371	23
24	6.341	0.1577	0.01498	0.09498	66.765	10.529	24
25	6.848	0.1460	0.01368	0.09368	73.106	10.675	25
26	7.396	0.1352	0.01251	0.09251	79.954	10.810	26
27	7.988	0.1252	0.01145	0.09145	87.351	10.935	27
28	8.627	0.1159	0.01049	0.09049	95.339	11.051	28
29	9.317	0.1073	0.00962	0.08962	103.966	11.158	29
30	10.063	0.0994	0.00883	0.08883	113.283	11.258	30
31	10.868	0.0920	0.00811	0.08811	123.346	11.350	31
32	11.737	0.0852	0.00745	0.08745	134.214	11.435	32
33	12.676	0.0789	0.00685	0.08685	145.951	11.514	33
34	13.690	0.0730	0.00630	0.08630	158.627	11.587	34
35	14.785	0.0676	0.00580	0.08580	172.317	11.655	35
40	21.725	0.0460	0.00386	0.08386	259.057	11.925	40
45	31.920	0.0313	0.00259	0.08259	386.506	12.108	45
50	46.902	0.0213	0.00174	0.08174	573.770	12.233	50
55	68.914	0.0145	0.00118	0.08118	848.923	12.319	55
60	101.257	0.0099	0.00080	0.08080	1253.213	12.377	60
65	148.780	0.0067	0.00054	0.08054	1847.248	12.416	65
70	218.606	0.0046	0.00037	0.08037	2720.080	12.443	70
75	321.205	0.0031	0.00025	0.08025	4002.557	12.461	75
80	471.955	0.0021	0.00017	0.08017	5886.935	12.474	80
85	693.456	0.0014	0.00012	0.08012	8655.706	12.482	85
90	1018.915	0.0010	0.00008	0.08008	12723.939	12.488	90
95	1497.121	0.0007	0.00005	0.08005	18701.507	12.492	95
100	2199.761	0.0005	0.00004	0.08004	27484.516	12.494	100

↑ amortization factor

↖ Annuity factor

P = Present Value

S = Future Value

R = Uniform Annual Equivalent

Homework Problem, Replacement Analysis

As a result of a newspaper report describing the new car purchases by the department head of one of the city's departments, the city council of city X has before it a proposed ordinance that would require that city vehicles not be replaced until they were used at least ten years. The city manager is concerned how such an ordinance would affect how economically city vehicles could be operated, since the manager knows that operating costs increase with age of the vehicle. The manager therefore hires you as a consultant to analyze the potential economic impact of this proposed ordinance. You decide to undertake two analyses:

a) Analyze the effect of length of time the vehicles are kept on average total cost per year. Determine how long vehicles should be kept to minimize average per year cost. Show the effect that a policy of keeping vehicles for ten years would have on average per year cost.

b) Analyze the effect of length of time the vehicles are kept on uniform annual equivalent costs, assuming a discount rate of eight percent. Determine how long vehicles should be kept to minimize the uniform annual equivalent cost. Show the effect that a policy of keeping vehicles for ten years would have on the uniform annual equivalent cost.

Before beginning your analysis, you obtain the following information. The purchase cost of new vehicles of the type used in the city's motor pool is $6000. Total operation and maintenance costs for the first year of use are $500, and for subsequent years total operation and maintenace costs increase about $200 per year. The resale value of the vehicles after t years is given below:

t	resale value
1	$3000
2	$2500
3	$2200
4	$2000
5	$1800
6	$1500
7	$1200
8	$ 900
9	$ 600
10	$ 300

Jack Byrd, [redacted] Operations Research Models
for Public Administration
(Lexington Books: Toronto, 1975)

11 Replacement Models in Public Administration

Determining when the replacement of certain items should occur is a familiar problem to many public agencies. As with supply and service systems, this determination is a matter of trade-offs among various factors. In the case of replacement models, the two conflicting factors are the cost of keeping the item for another year (e.g., increased maintenance and operation costs) versus the cost of replacing the item.

There are essentially two reasons for replacing items. First, maintenance and operation result in increasing costs, and replacement with a new item may be considered desirable to keep these costs at a reasonable level. Second, the replacement may be stimulated by a failure of the item or an impending failure.

While it may seem that replacement models are not appropriate in non-capital-intensive government operations, this is not the case. There are many situations in which replacement models are appropriate in public organizations. In the traditional sense, replacement models can be used for making decisions about such things as car pools, sanitation equipment, hospital equipment, lighting supplies, and public transportation vehicles. In addition, the replacement of public office buildings, airport facilities, and school buildings can be viewed as replacement problems, although these are more complex than traditional replacement applications.

A Replacement Model with Increasing Operation and Maintenance Costs

Suppose a community has a sanitation vehicle that initially cost $20,000. The cost of operating and maintaining the equipment is estimated to be $1000 the first year and is estimated to rise at a rate of $200 a year. The community would like to know the optimal time to replace the vehicle. Assuming that the replacement decision will be made only at the end of yearly intervals, the analysis of this problem can be developed in a tabular form.

Table 11-1 shows a summary of the costs incurred by the sanitation vehicle during each year of its operation. As the analysis in the first row of Table 11-1 shows, the average yearly cost of the vehicle if it were replaced after the first year of operation would be $1000 since there has only been

Table 11-1
Cost Summary for Sanitation Vehicle Replacement

End of Year	Average Capital Cost/ Year ($)	Operation and Maintenance Costs for Year ($)	Total Operation and Maintenance Costs ($)	Average Operation and Maintenance Costs ($)	Average Total Costs ($)
1	20,000	1,000	1,000	1,000	21,000
2	10,000	1,200	2,200	1,100	11,100
3	6,667	1,400	3,600	1,200	7,867
4	5,000	1,600	5,200	1,300	6,300
5	4,000	1,800	7,000	1,400	5,400
6	3,333	2,000	9,000	1,500	4,833
7	2,857	2,200	11,200	1,600	4,457
8	2,500	2,400	13,600	1,700	4,200
9	2,222	2,600	16,200	1,800	4,022
10	2,000	2,800	19,000	1,900	3,900
11	1,818	3,000	22,000	2,000	3,818
12	1,667	3,200	25,200	2,100	3,767
13	1,538	3,400	28,600	2,200	3,738
14	1,429	3,600	32,200	2,300	3,729
15	1,333	3,800	36,000	2,400	3,733

one year of operation. Thus the total average yearly costs would be $21,000.

The same process is repeated for the case in which the vehicle is not replaced until after its second year of operation. The average yearly cost of the vehicle itself would now be $10,000 since the $20,000 cost of the vehicle is spread over two years. The costs of operation and maintenance are $1200 for the second year, a $200 increase over year 1. The total operation and maintenance cost for the first two years is $2200, giving an average yearly cost of $1100. The total average yearly costs for replacing the vehicle after the second year are $11,100. Since the average yearly cost has decreased from a first year replacement, it is desirable to keep the vehicle at least for two years.

This process is repeated for each year until the total average yearly costs reach a minimum. As shown in Table 11-1, the minimum average yearly costs can be achieved by replacing the sanitation vehicle at the end of the fourteenth year.

While this procedure is straightforward, it fails to take into account the value of the money tied up in the capital investment. It may be inappropriate to consider costs incurred in separate years without converting these to equivalent annual costs, which take into account the value of the money invested. In order to examine how the value of money invested will change the replacement analysis, a diversion must be made to examine the concepts of interest calculations.

Economic Analysis Concepts

The use of money costs money. When someone purchases a home, he pays an interest charge for the money that he is lent to pay for the home. In government organizations, money may not be borrowed to pay for purchases, such as the sanitation vehicle, but the investment in the purchase will still cost money. The city, in purchasing the sanitation vehicle, is devoting $20,000 to this project that could be invested elsewhere. In a sense, the city experiences an *opportunity cost* in that it has lost the opportunity to invest the funds in some other project.

The cost of invested money is generally expressed as an interest rate and is calculated as the ratio between the interest paid on an investment at the end of some period of time and the money invested initially. Thus, if an investment of $1000 returned $80 in interest, the interest rate would be:

$$i = 80/1000 = 0.08$$

or

$$i = 8\%$$

In almost all cases the interest rate is compounded; the interest is computed not only on the original investment, but also on the accumulation of the interest on the investment. If a city had invested $100,000 of its revenue sharing funds in a bank paying 6 percent interest compounded annually, the city would have $100,000 + 0.06($100,000) = $106,000 at the end of the first year. In the second year, the investment would be worth $106,000 + 0.06($106,000) = $112,360.

When compounded interest is used, it is often desirable to determine the future worth of an initial investment. If the future worth of the investment is designated as F and the initial investment designated as P, the value of F can be found using the relationship:

$$F = P(1 + i)^n$$

where n is the number of investment periods. If the city would like to know the worth of its $100,000 investment at the end of 5 years using an interest rate of 6 percent, the value of the investment can be found as:

$$F = 100{,}000\ (1 + 0.06)^5$$

$$= \$133{,}823\,.$$

Thus, after five years the city will have $133,823 in its investment account.

While this procedure is relatively straightforward, the calculation of $(1 + i)^n$ can be burdensome. Therefore, tables have been developed which give these values. In the Appendix, this value and others are provided and can be found in the table for i = 6 percent (0.0600 Compound Interest Factors) under the column labelled F/P. The notation F/P indicates that the factor for the calculation of the future worth F is desired given that the present worth P is known. In mathematical notation, this would be read as *F given P*. Once the appropriate column is found, the factor value for the appropriate value of n is found. For $n = 5$, the factor is 1.3382. The calculation of F is easily found as:

$$F = 100{,}000\ (1.3382)$$
$$= 133{,}820.$$

The only difference between this value and the previous value is the four-place accuracy of the table. From this point, the tables will be used extensively and the original formula will be rewritten as:

$$F = P(F/P\text{—}i\text{—}n)$$

where $(F/P\text{—}i\text{—}n)$ indicates the factor value for finding F when P is given with an interest rate of i compounded for n investment periods.

The reverse operation is also possible. Suppose the city has an obligation of $250,000 to meet in 10 years and would like to know how much must be invested now to meet this obligation. The interest rate is 8 percent and is compounded semiannually. The appropriate formula in this case will be

$$P = F(P/F\text{—}i\text{—}n)$$
$$= 250{,}000\ (P/F\text{—}0.04\text{—}20).$$

Notice that $n = 20$ in the above formula. Since the interest rate is compounded semiannually, there are essentially 2 investment periods per year, or 20 investment periods for 10 years. The interest rate, since it is compounded semiannually, is generally divided in half. The factor $(P/F\text{—}0.04\text{—}20)$ is found in the table as 0.4564. Thus, the needed investment will be

$$P = 250{,}000\ (0.4564)$$
$$= \$114{,}100.$$

Suppose the city would like to meet its $250,000 obligation by putting away money each year for 10 years rather than putting one amount away at the beginning of the entire 10-year investment period. These periodic investments are generally assumed to be of the same amount and will be designated as A. The same general principle applies in this case as before. The future worth will be found from the relationship

$$A = F(A/F\text{—}i\text{—}n).$$

If the interest rate is 8 percent compounded semiannually, the factor will be based upon a 4 percent interest rate for 20 investment periods. Thus, the

factor $(A/F\text{—}0.04\text{—}20)$ can be found from the tables as 0.0336. The annual payment would thus be:

$$A = 250{,}000\ (0.0336)$$
$$= \$8400.$$

Other calculations are also possible. Suppose a city issues a $100,000 bond at 6 percent interest to be compounded annually. The city would like to know what annual payments it must make to pay off the bond with interest in 20 years. In this case, the initial investment is known to be $100,000 and the annual payments are to be found. Thus the relationship

$$A = P(A/P\text{—}i\text{—}n)$$

will be used. The factor $(A/P\text{—}0.06\text{—}20)$ is found from the tables to be 0.0872. Thus, the annual payments will be

$$A = 100{,}000\ (0.0872)$$
$$= \$8720.$$

As is evident from the Appendix, any of the three investment values can be identified if one of the other investment values is known. This brief section cannot do justice to the general concepts of finance, and the reader is encouraged to pursue this general topic in more detail by examining one of the numerous textbooks in the area. Given this brief exposition, attention will now be returned to replacement models.

A Replacement Model with Increasing Operation and Maintenance Costs with Interest

In the previous replacement example, the interest rate was assumed to be zero. However, the consideration of the value of money should be incorporated into the analysis. In doing this, the respective costs must be put on an equivalent basis. The easiest way to do this is to convert all of the costs to an equivalent annual cost. In other words, the different costs, although they were incurred in each year of equipment operation, will be converted to an equivalent annual cost as if an equal cost were incurred in each year.

The calculation of the equivalent annual costs is more difficult than the previous example and will be analyzed with the use of Table 11-2. The first cost to convert to an annual basis is the initial cost of the sanitation vehicle. This is a relatively straightforward procedure in which the present worth P is given, and the equivalent annual cost A is desired. The factor $(A/P\text{—}0.06\text{—}n)$ given in column (2) of the table is multiplied by the $20,000 initial investment to give the equivalent annual costs of the original cost of

Table 11-2
Cost Summary of Sanitary Vehicle Replacement with Interest Considerations

Year (1)	A/P—0.06—n (2)	Equivalent Annual Cost of Initial Investment ($) (3)	P/F—0.06—n (4)	Maintenance and Operation Costs ($) (5)	Present Worth of Maintenance Operations Cost ($) (6)	Total Present Worth, Maintenance and Operation Cost ($) (7)	Equivalent Annual Costs of Maintenance and Operations ($) (8)	Equivalent Annual Costs ($) (9)
1	1.06000	21,200	0.9434	1,000	943	943	1,000	22,200
2	0.54544	10,909	0.8900	1,200	1,068	2,011	1,097	12,006
3	0.37411	7,482	0.8396	1,400	1,175	3,186	1,192	8,674
4	0.28859	5,772	0.7921	1,600	1,267	4,453	1,285	7,057
5	0.23740	4,748	0.7473	1,800	1,345	5,798	1,376	6,124
6	0.20336	4,067	0.7050	2,000	1,410	7,208	1,466	5,533
7	0.17914	3,583	0.6651	2,200	1,463	8,671	1,553	5,136
8	0.16104	3,221	0.6274	2,400	1,506	10,177	1,639	4,860
9	0.14702	2,940	0.5919	2,600	1,539	11,716	1,722	4,662
10	0.13587	2,717	0.5584	2,800	1,564	13,280	1,804	4,521
11	0.12679	2,536	0.5268	3,000	1,580	14,860	1,884	4,420
12	0.11928	2,386	0.4970	3,200	1,590	16,450	1,962	4,348
13	0.11296	2,259	0.4688	3,400	1,594	18,044	2,038	4,297
14	0.10758	2,152	0.4423	3,600	1,592	19,636	2,112	4,264
15	0.10296	2,059	0.4173	3,800	1,586	21,222	2,185	4,244
16	0.09895	1,979	0.3936	4,000	1,574	22,796	2,256	4,235
17	0.09544	1,909	0.3714	4,200	1,560	24,356	2,325	4,234
18	0.09236	1,847	0.3503	4,400	1,541	25,897	2,392	4,239
Calculation	Table	20000 × (2)	Table	1000 + 200 (n − 1)	(4) × (5)	(6)	(2) × (7)	(3) + (8)

the vehicle, as shown in column (3). For a one-year replacement strategy the value of n is set equal to one: for a two-year replacement strategy, $n = 2$, etc.

The calculation of the equivalent annual costs of maintenance and operation are more difficult. The usual procedure is to compute the present worth of the maintenance and operation costs. This is performed by considering the maintenance and operation costs to be a future cost F in the year in which they are incurred and then converting these to an equivalent initial cost P. This calculation is performed in Table 11-2 by multiplication of column (4) by column (5), with the result shown in column (6). The total costs of maintenance and operation are then found by adding together the costs for each year of operation as shown in column (7). These total costs are then converted to equivalent annual costs by multiplication of columns (2) and (7), giving the value in column (8).

The two replacement costs components, columns (3) and (8), are then added together, giving the total equivalent annual costs of column (9). Searching column (9) for the minimum total cost identifies 17 years as being the optimal replacement interval.

The effect of interest considerations in the calculation of the optimal replacement interval is generally to extend the optimal life of the item under analysis. The specification of the interest rate is perhaps the most difficult aspect of the analysis. In many cases, the value used is the interest paid on government bonds. Where interest rate is a matter of considerable conjecture, various interest rates are used in a sensitivity analysis.

A Replacement Model for Items That Fail

In the previous replacement models, it was assumed that items would continue in service indefinitely and would not fail. While this assumption is obviously invalid for most cases, the useful life of many items is generally long enough to make this assumption appropriate. In other cases, item failures are an inherent part of the replacement problem. For items such as street lights, filters in sewage treatment plants, and the components of many larger places of equipment used in government service, failures can be troublesome and expensive, and policies for their control can be valuable.

Since the light bulb is a classic example of this type of replacement problem, the example to be discussed here will then be the analysis of a city's replacement of street lights. Suppose that the city has collected data on its replacement of street lights and observes their failure characteristics to be as shown in Table 11-3.

Given the number of survivors listed in the table, the failures during the month can be found quite easily as:

Table 11-3
Failure Data for Street Lights

Month	*Survivors at Beginning of Month*	*Failures during Month*	*Probability of Failure during Month*
1	5000	100	0.02
2	4900	400	0.08
3	4500	500	0.10
4	4000	400	0.08
5	3600	500	0.10
6	3100	600	0.12
7	2500	600	0.12
8	1900	700	0.14
9	1200	400	0.08
10	800	600	0.12
11	200	200	0.04
12	0	0	0.00

$$F_t = S_t - S_{t+1}$$

where F_t = the failures during time t

S_t = the number of survivors at the beginning of time t

The probability of failures during a month are also quite easily found as:

$$P_t = F_t/N$$

where P_t = the probability of failures during month t

N = the number of original items

In staffing its street light maintenance program, the city would like to know how many replacements will be made each month. In order to determine this, it will be assumed that all the lights begin at time (0) as completely new lights. The number of replacements to be made in time t will be designated as R_t. For the first month, the expected number of replacements will be:

$$R_1 = NP_1$$
$$= 5000(0.02)$$
$$= 100$$

During the second month, a given proportion of the original street lights are expected to fail. In addition some of those street lights replaced in the first month are also expected to fail. The total number of replacements can be found from the relationship:

$$R_2 = NP_2 + R_1P_1$$
$$= 5000(0.08) + 100(0.02)$$
$$= 402.$$

In successive months, the number of replacements will be found in a similar fashion. The relationships for months three and four will be:

$$R_3 = NP_3 + R_1P_2 + R_2P_1 = 516$$
$$R_4 = NP_4 + R_1P_3 + R_2P_2 + R_3P_1 = 452.$$

Replacement formulas for the remaining months are found in similar fashion but are not given here.

The number of replacements during each month are shown in Table 11-4 for the first 20 months of operation. As indicated in Table 11-4 the number of replacements each month varies considerably at first but eventually begins to reach steady-state conditions.

The expected number of replacements once the steady-state conditions have been reached are found from the relationship:

$$R_e = N/t_{AVG}$$

where R_e = the expected number of replacements under steady-state conditions

N = the number of items in use

t_{AVG} = the average life of the equipment.

Thus, for an average life of 6.34 months, the expected number of street light replacements will be:

$$R_e = 5000/6.34$$
$$= 788 \text{ replacements/month}$$

With this information, the city's maintenance department can be staffed accordingly.

One alternative to replacing all street lights as they fail is to replace the lights in a group before they fail. By replacing lights in a group, it is expected that the extra costs of individual replacement will be saved through the economies of scale of group replacements. Of course, any lights that fail before group replacements are made will continue to be replaced on an individual basis. The total costs will be the sum of the cost of group replacements C_1 and the cost of replacing individual items that fail before group replacements are made C_2.

The costs of group replacements for a given time period will be simply:

Table 11-4
Replacement Requirements

Month	Necessary Replacements	Month	Necessary Replacements	Month	Necessary Replacements
1	100	8	983	15	779
2	402	9	775	16	813
3	516	10	1074	17	810
4	452	11	763	18	824
5	599	12	655	19	790
6	742	13	725	20	795
7	801	14	747		

$$C_1 = NC_g/t$$

where C_1 = costs of group replacements

N = the total number of items replaced as a group

C_g = the cost per unit replaced when items are replaced in a group

t = the time interval between group replacement.

The cost of individual replacements of bulbs that fail between the scheduled time for group replacements will be estimated from the relationship:

$$C_2 = C_U \sum_{j=1}^{t-1} X_j \Big/ t$$

where C_2 = the total costs of replacing items that fail before group replacements

C_U = the cost for a replaced unit when replacements are made on an individual basis

X_j = the number of replacements expected during the jth time period

t = the time interval between group replacements.

The total cost of a particular replacement strategy will thus be the sum of the two costs or:

$$TC = C_1 + C_2.$$

Suppose the city accountant has estimated the replacement costs to be C_g = \$2/light and C_U = \$20/light. For group replacements every two months the costs will be:

$$\begin{aligned} C_1 &= NC_g/t \\ &= 5000(2)/2 \\ &= 5000 \text{ \$/month} \end{aligned}$$

$$\begin{aligned} C_2 &= C_U \sum_{j=1}^{t-1} X_j \Big/ t \\ &= C_U X_1/t \\ &= 20(100)/2 \\ &= 1000 \text{ \$/month} \end{aligned}$$

$$\begin{aligned} TC &= C_1 + C_2 \\ &= 5000 + 1000 \\ &= 6000 \text{ \$/month} \end{aligned}$$

The cost calculations are continued in a similar fashion until the minimum total cost is found. The results of these calculations are shown in Table 11-5. The minimum cost of 6000 \$/month is associated with the replacement of the entire group of street lights every two months. If the lights were replaced only upon failure, there would be 788 replacements expected every month at a cost of \$20 per replacement or a total cost of \$15,760. Thus it is clear that the replacement of the lights every 2 months would be a more economical policy. While cities may not follow replacement strategies as suggested here, the economics of the situation may indicate that an alteration in conventional replacement strategies may lead to savings.

The Role of Replacement Models in Public Administration

As with other probabilistic models, the range of models presented here is limited but gives some insight into the general approach of replacement models. Replacement situations are somewhat like supply systems in that they seem to receive little formal analysis in public organizations. One reason for this may be the practical realities of replacement financing within public budget restrictions. Replacements may be constrained to given years when funds are available regardless of the economics of the policy. Of course, a more comprehensive analysis of replacement needs should lead to a better system of replacement funding.

In general, replacement models are designed for operational level deci-

Table 11-5
Cost Summary for Different Replacement Periods

t	C_1	C_2	TC
1	10000	0	10000
2	5000	1000	6000
3	3333	3346	6679
4	2500	5090	7590
5	2000	5880	7880

sions. The singular objective of minimizing total cost is typical of most replacement models. Although most replacement models would have to be altered significantly to be effective at the operational level, most replacement decisions seem to be operational in nature. Thus replacement models appear to be consistent with the level of decision making at which they are used.

R. D. Eck, An Introduction to Quantitative Methods for Business Applications (Belmont: Wadsworth, 1979)

14

Synopsis

ESSENTIAL CONCEPTS:

- Definitions and types of business queuing systems
- The nature of waiting-line models
- Probability distributions for arrival and service processes
- The single-queue, single-server system
- The single-queue, double-server system

APPLICATIONS:

- Accounting systems
- Airport procedures
- Assembly lines
- Banking
- Computer center management
- Educational administration
- Equipment selection
- Income tax services
- Industrial sales
- Materials management
- Quality control
- Reservation systems
- Service organizations
- Warehousing

Waiting-Line Models and Applications

Basic Definitions

A waiting line is simply one or more items waiting for some type of service. Waiting lines are also called queues.

A *waiting line* is simply one or more items waiting for some type of service. Waiting lines are also called *queues*. Examples of waiting lines include automobiles waiting for fuel at a service station, food products waiting to be dispensed by an automatic vendor, income tax returns waiting to be processed by the Internal Revenue Service, patients waiting for medical facilities to become available, and parts on a production line waiting for a machine to perform an assembly operation.

A server, or service facility, provides service to the items in a waiting line.

A *server, or service facility*, provides service to the items in a waiting line. A gasoline pump is a server in a service station. The mechanism that dispenses food is a server in an automatic vendor. The server that processes income tax returns might be a clerk or a computer. If a queue consists of requests for telephone installations, then the server is a telephone installation crew.

Note that items in a queue can come to the server, as when patients report to a physician's office, or the server can go to items waiting for service, as when a visiting nurse makes house calls.

A queuing system is composed of one or more waiting lines and one or more servers.

A *queuing system* is composed of one or more waiting lines and one or more servers. A diagram of several of many possible queuing systems is shown in Figure 14.1. In Figure 14.1, queuing systems are enclosed by the shaded boxes. An example of the single-queue, single-server queuing system is a branch bank that has only one loan officer. Customers arrive, wait in line until the loan officer is free, discuss their business, and depart. The single-queue, multiple-server system occurs, for example, in situations where customers take a ticket upon arrival to establish service order, and clerks (two or more servers) call a new number when they complete service to a customer. The multiple-queue, multiple-server system is found at checkout facilities in large supermarkets, where the customer is free to pick a queue at any one of several checkout stations.

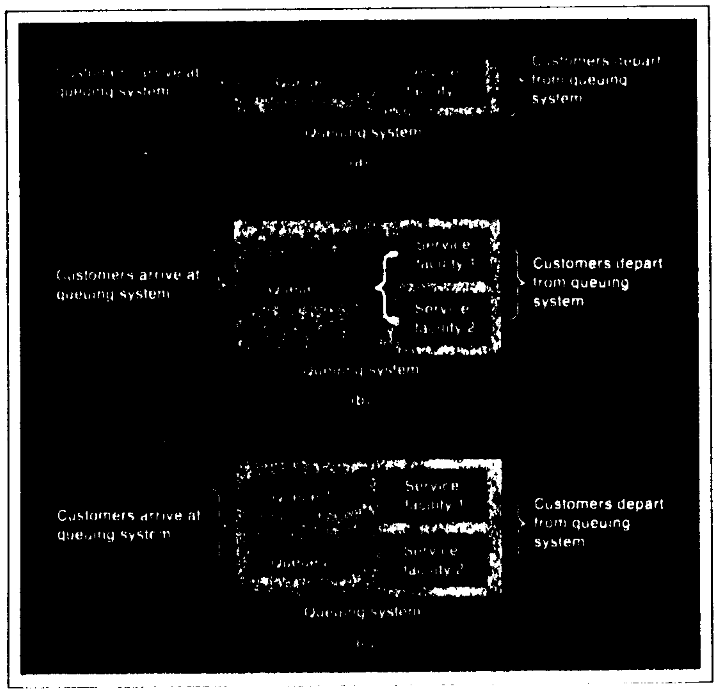

FIG 14.1 Examples of queuing systems

Queuing Systems in Business

Queuing systems pervade business. Consider the following examples:

Materials Management Queuing Systems Raw materials arrive at factory receiving docks and enter a queue to await further processing. Queuing models can provide insight into the logistics of materials management.

Accounting Department Queuing Systems Bills for goods and services arrive at the cashier's office and await processing. Queuing models can provide insight into the nature of an efficient accounting or disbursements office.

Production Line Queuing Systems An assembly line, with various "work" and "wait" stations, is a queuing system. The application of queuing models to production management is obvious.

Retailing Queuing Systems In retailing, cash registers constitute service facilities. Queuing models can be useful in the design of efficient stores.

Industrial Sales Queuing Systems In other marketing situations, such as industrial sales and real estate sales, customers require services from sales representatives. Queuing models can be useful in determining the type of sales force that is required.

Organizational Queuing Systems The employees in an office provide services for other offices and groups in the firm. If one thinks of an office as a service facility, the relevance of queuing models to problems of organization and staffing is obvious.

Service-Firm Queuing Systems The product of many firms is service. Furthermore, consulting firms, plumbing contractors, accounting firms, and lawn-mowing services take service facilities to the customer's home or business. Requests for services enter queues and wait until a plumber, an auditing team, or other personnel are available.

Transportation Queuing Systems Airliners arrive at airports and enter a "stack" to await clearance for use of a runway. Queuing models can be useful in the design and management of airports and air traffic control. They can also be useful for the movement of ships arriving in harbors, trains arriving at freight yards, trucks arriving at terminals, and so forth.

Computer-Facility Queuing Systems A computer services requests for computation and data processing. If one thinks of the computer center as a queuing system, queuing models can help management determine the type of computer that is needed and the type of priorities that should be given to various jobs. By analogy, other large capital expenditures can be evaluated with insights gained from queuing models. Consider the choice of an elevator system for a new skyscraper.

Community-Services Queuing Systems In public administration, the community generates demands for such services as police protection, transportation, health care, and education. Queuing models have been useful in determining the type of service facilities that is most efficient for a specific community.

The Nature of Waiting-Line Models

Waiting-line or queuing models describe properties of queuing systems. Managers recognize that if service facilities are too slow, the queue will grow. If the queue is composed of customers, there will be ill will and waiting costs. Some, or perhaps many, customers may abandon the waiting line and take their business elsewhere. The same result can occur if the queue is filled with materials waiting to be processed into finished goods to fill orders. To reduce the time of waiting in a queue, a slow service facility might be replaced by a more-costly, faster one, or several service facilities might be added. The intention of management is to find a *queuing system* that has minimal total costs for waiting and service-facility operations.

The arrival process *concerns the number of arrivals to the system. Arrivals are usually measured in terms of number of arrivals per unit of time, and because the arrival process is typically a random process, the number of arrivals per unit of time is normally described by a probability distribution.*

To help managers evaluate queuing systems, queuing models provide the following descriptions of queuing systems.

1. The relative frequency (or probability) with which the system will be idle.
2. The probability that there will be *n* customers waiting in the queue.
3. The expected (or average) number of customers waiting in the queue.
4. The probability that it will take more than *t* units of time to service a customer.
5. The expected (or average) number of customers in the system (queue plus service facilities).
6. The average time that a customer spends in the queue waiting for service.
7. The average time required to get through the entire system.

Queue discipline *describes the manner in which items waiting for service enter and progress through queues.*

To make use of a queuing model, managers must be able to describe the arrival process, the queue discipline, the service mechanism, and the service process.

The *arrival process* concerns the number of arrivals to the system. Arrivals are usually measured in terms of number of arrivals per unit of time, and because the arrival process is typically a random process, the number of arrivals per unit of time is normally described by a probability distribution.

Balking *is the refusal of arrivals to join a queue.*

Queue discipline describes the manner in which items waiting for service enter and progress through queues. To describe queue discipline, we must state whether arrivals can join only one queue or one of several queues. For times when queues are long, we must state whether or not arrivals can *balk*. *Balking* is the refusal of arrivals to join a queue.

Reneging *is the process of abandoning a queue (and the system) after waiting for some period of time in a waiting line.*

Once in a queue, we must specify whether or not customers can *renege* or *jockey*. *Reneging* is the process of abandoning a queue (and the system) after waiting for some period of time in a waiting line. *Jockeying* is the process of jumping from one queue to another queue in the system.

Jockeying *is the process of jumping from one queue to another queue in the system.*

Finally, a description of queue discipline states how items progress through a queue. *FIFO* is a first-in-first-out ordering. *LIFO* is a last-in-first-out ordering, such as occurs in crowded elevators where the last arrival is the first to leave when the elevator reaches its destination.

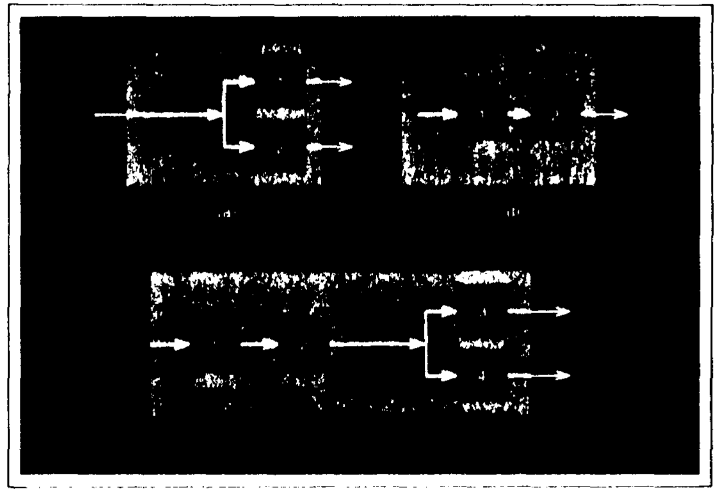

FIG. 14.2 (a) Parallel servers; (b) series servers; (c) series-parallel servers.

The service mechanism *concerns the number and configuration of the service facilities.*

Following a description of queue discipline, managers must then describe the *service mechanism*. The *service mechanism* concerns the number and configuration of the service facilities. The service mechanism can be characterized by a *single server* or by *multiple servers*. The configuration of multiple servers can be various combinations of *parallel servers*, *series servers*, or *series-parallel servers* (see Fig. 14.2).

The service process *concerns the number of customers served by a server per unit of time and/or the time that elapses between the start and completion of service (not including time waiting in a queue). The service process is usually described by a probability distribution.*

Finally managements must describe the *service process*. The *service process* concerns the number of customers served by a server per unit of time and/or the time that elapses between the start and completion of service (not including time waiting in a queue). The service process is usually described by a probability distribution.

As a result of the many possible arrival processes, queue disciplines, service mechanisms, and service processes, it might be properly suspected that no single queuing model could describe the specific nature of every possible queuing system that might be encountered in business applications. The literature of queuing models is immense and, in many instances, very complex. In this text we will confine attention to a few of the simpler models that have been found to serve as reasonable approximations in a number of applications.

In particular, the discussion will be restricted to models in which the arrival of a customer is independent of past arrivals and the service times are independent of both arrivals and past service times. This means, for example, that the probability that a customer will arrive at the system during the next hour is not influenced by the number of customers that arrived during the past hour, which also implies that arrivals are not influenced by the number of customers already waiting in the queue. Similarly, it will be assumed that the service facility does not speed up when there are many customers waiting in the queue, nor will the service facility slow down just because it has served several customers with abnormal speed and is now exhausted. Clearly, these assumptions can represent a departure from reality in many applications. If the assumptions prove to be too restrictive, the manager can turn to the exhaustive literature of queuing models for help or resort to simulation.

Probability Distributions for the Arrival Process

λ represents the average arrival rate, that is, the average number of customers that arrive at the system per period of time T.

For the case where arrivals are independent in probability—that is, where new arrivals are not influenced by the number of items already in the waiting line or the elapsed time since the last arrival—two probability distributions are available for describing the arrival process. To use the probability distributions, managers must specify a value for λ. λ represents the *average arrival rate*, that is, the average number of customers that arrive at the system per period of time T. For example,

TABLE 14.1

x	e^{-x}	x	e^{-x}	x	e^{-x}	x	e^{-x}
0.0	1.00000	2.5	0.08208	5.0	0.00673	7.5	0.00055
0.1	0.90483	2.6	0.07427	5.1	0.00609	7.6	0.00050
0.2	0.81873	2.7	0.06720	5.2	0.00551	7.7	0.00045
0.3	0.74081	2.8	0.06081	5.3	0.00499	7.8	0.00040
0.4	0.67032	2.9	0.05502	5.4	0.00451	7.9	0.00037
0.5	0.60653	3.0	0.04978	5.5	0.00408	8.0	0.00033
0.6	0.54881	3.1	0.04504	5.6	0.00369	8.1	0.00030
0.7	0.49658	3.2	0.04076	5.7	0.00334	8.2	0.00027
0.8	0.44932	3.3	0.03688	5.8	0.00302	8.3	0.00024
0.9	0.40656	3.4	0.03337	5.9	0.00273	8.4	0.00022
1.0	0.36787	3.5	0.03019	6.0	0.00247	8.5	0.00020
1.1	0.33287	3.6	0.02732	6.1	0.00224	8.6	0.00018
1.2	0.30119	3.7	0.02472	6.2	0.00202	8.7	0.00016
1.3	0.27253	3.8	0.02237	6.3	0.00183	8.8	0.00015
1.4	0.24659	3.9	0.02024	6.4	0.00166	8.9	0.00013
1.5	0.22313	4.0	0.01831	6.5	0.00150	9.0	0.00012
1.6	0.20189	4.1	0.01657	6.6	0.00136	9.1	0.00011
1.7	0.18268	4.2	0.01499	6.7	0.00123	9.2	0.00010
1.8	0.16529	4.3	0.01356	6.8	0.00111	9.3	0.00009
1.9	0.14956	4.4	0.01227	6.9	0.00100	9.4	0.00008
2.0	0.13533	4.5	0.01110	7.0	0.00091	9.5	0.00007
2.1	0.12245	4.6	0.01005	7.1	0.00082	9.6	0.00006
2.2	0.11080	4.7	0.00909	7.2	0.00074	9.7	0.00006
2.3	0.10025	4.8	0.00822	7.3	0.00067	9.8	0.00005
2.4	0.09071	4.9	0.00744	7.4	0.00061	9.9	0.00005
						10.0	0.00004

if, on the average, twenty customers arrive per hour, then $\lambda = 20$ customers per hour.

The probability that n items will arrive at the system during a time interval of length T is given by a ***Poisson probability distribution*** of the form

$$\text{Prob [}n\text{ arrivals during period of length }T] = \frac{e^{-\lambda T}(\lambda T)^n}{n!} \quad (14.1)$$

where $e = 2.71828$,
$n! = n(n-1)(n-2)\ldots 0!$,
$0! = 1$.

To facilitate the use of Eq. (14.1), Table 14.1 provides values for e^{-x} for various values of x.

EXAMPLE 14.1 **Production Management** On the average, six diesel engines arrive at a construction-equipment repair facility every day.

REQUIRED

(a) What is the value of λ?
(b) What is the probability that exactly two diesel engines will arrive on one day?

SOLUTION

(a) $\lambda = 6$ items per day.

(b)
$$\text{Prob [2 arrivals during 1 day]} = \frac{e^{-\lambda T}(\lambda T)^n}{n!} = \frac{e^{-[6(1)]}[6(1)]^2}{2!} = \frac{0.00247(6)^2}{2 \cdot 1 \cdot 0!} = 0.04446$$

where $e^{-6} = 0.00247$ by Table 14.1. ■

In using the Poisson relationship (Eq. 14.1), T must be expressed in the same time units as λ. For example, if λ is expressed as $\lambda = 10$ units per *hour*, and you want to know the probability of 4 units in 20 *minutes*, the 20 minutes would have to be expressed as $20/60 = 0.33$ hour.

EXAMPLE 14.2 **Computer Center Management** On the average, four large jobs arrive for computer processing at the central computer center of City Bank during an 8-hour shift.

REQUIRED

What is the probability that two large jobs will arrive within a 1-hour interval?

SOLUTION

$\lambda = 4$ arrivals per shift

$T = \frac{1}{8} = 0.125$ shift

$$\text{Prob}\,[n = 2 \text{ arrivals in } 0.125 \text{ shift}] = \frac{e^{-\lambda T}(\lambda T)^n}{n!}$$

$$= \frac{e^{-[4(0.125)]}[4(0.125)]^2}{2!}$$

$$= \frac{e^{-0.5}(0.5)^2}{2.1}$$

$$= \frac{0.60653(0.25)}{2}$$

$$= 0.0758$$ ■

It is also possible to provide probabilities for the time between arrivals. Probabilities for time between arrivals are given by the *exponential probability distribution*, which has the form,

$$\text{Prob}\begin{bmatrix}\text{no more than } T \text{ time periods}\\ \text{elapse between arrivals}\end{bmatrix} = 1 - e^{-\lambda T} \qquad (14.2)$$

where $e = 2.71828$.

EXAMPLE 14.3 **Warehouse Management** On the average, twenty-five trucks per hour arrive at the loading dock of a large fruit warehouse.

REQUIRED

What is the probability that no more than 0.02 hours (1.2 minutes) will elapse between arrivals?

SOLUTION

$$\text{Prob}\begin{bmatrix}\text{no more than } T = 0.02 \text{ hours}\\ \text{between arrivals}\end{bmatrix} = 1 - e^{-\lambda T}$$

$$= 1 - e^{-[25(0.02)]}$$

$$= 1 - e^{-0.5}$$

$$= 1 - 0.606531 \text{ (from Table 14.1)}$$

$$= 0.393469$$ ■

Probability Distributions for the Service Process

The same probability distributions that were used to describe the arrival process can be used to describe the service process when service is independent in probability, that is, when the number of customers served or the time required to serve a customer is not influenced by past service rates or the length of the waiting line. In place of λ (the average arrival rate used in describing the *arrival* process), the symbol μ is used for the service process. μ represents the *average service rate* of a a service facility, that is, the average number of customers served per time period T. With the substitution of μ for λ in Eqs. (14.1) and (14.2), we obtain

μ represents the average service rate of a service facility, that is, the average number of customers served per time period T

$$\text{Prob}\begin{bmatrix}n \text{ customers served in}\\ \text{period of length } T\end{bmatrix} = \frac{e^{-\mu T}(\mu T)^n}{n!} \qquad (14.3)$$

and

$$\text{Prob}\begin{bmatrix}\text{no more than } T \text{ time periods}\\ \text{required to serve a customer}\end{bmatrix} = 1 - e^{-\mu T} \qquad (14.4)$$

Equation 14.3 assumes that there are always customers waiting to be served, so that the service facility is never idle.

EXAMPLE 14.4 **Bank Management** The manager of a bank has observed that, on the average, a busy teller serves fifteen customers per hour.

REQUIRED

(a) What is the probability that exactly five customers will be served in $T = 0.5$ hours?

(b) What is the probability that a customer will be served in 2 minutes or less?

SOLUTION

(a) For $\mu = 15$ customers per hour, and $T = 0.5$ hour,

$$\text{Prob}\begin{bmatrix}n = 5 \text{ customers served}\\ \text{in } T = 0.5 \text{ hour}\end{bmatrix} = \frac{e^{-\mu T}(\mu T)^n}{n!}$$

$$= \frac{e^{-[15(0.5)]}[15(0.5)]^5}{5!}$$

$$= \frac{e^{-7.5}[7.5]^5}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= \frac{0.00055[23730.5]}{120}$$

$$= 0.1087$$

(b) For $\mu = 15$ customers per hour, and $T = 2/60 = 1/30 = 0.0333$ hour,

$$\text{Prob}\begin{bmatrix}\text{no more than 0.0333 hour}\\ \text{required to serve a customer}\end{bmatrix} = 1 - e^{-\mu T}$$

$$= 1 - e^{-[15(1/30)]}$$

$$= 1 - e^{-0.5}$$

$$= 1 - 0.606531$$

$$= 0.393469$$

■

Characteristics of a Single-Queue, Single-Server System

In this section some useful characteristics of a single-queue, single-server waiting-line system are presented. A description of the system is presented in Table 14.2 and Figure 14.3.

TABLE 14.2

Description of the Single-Queue, Single-Server System

The arrival process:	**Arrivals are independent, with λ denoting the average number of arrivals per period of time T.**
Queue discipline:	**Arrivals join one queue. There is no balking, reneging, or jockeying. Customers progress through the waiting line on a FIFO basis.**
Service mechanism:	**There is only one server.**
Service process:	**Service times are independent, with μ ($\mu > \lambda$) denoting the average number served per period of time T.**

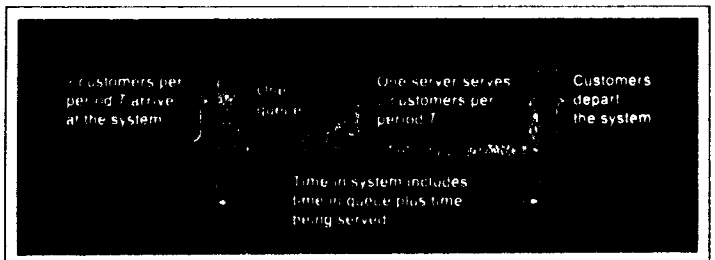

FIG. 14.3 The single-queue, single-server system.

The system utilization factor, denoted by ρ, is the probability that the system is busy, that is, the probability that there are one or more customers in the queue or service facility.

The *system utilization factor*, denoted by ρ, is the probability that the system is busy, that is, the probability that there are one or more customers in the queue or service facility.

The system utilization factor is computed by Eq. (14.5).

$$\rho = \text{Prob}\begin{bmatrix}\text{system}\\ \text{is busy}\end{bmatrix} = \frac{\lambda}{\mu} \tag{14.5}$$

When $\rho \geq 1$, the number of customers in the queue grows without bound (approaches infinity). To prevent this condition, the description of the service process in Table 14.2 specifies that $\mu > \lambda$ so that ρ will be less than 1.

When ρ has a value that is less than 1, the following characteristics of the system can be computed.

$$P_o = \text{Prob}\begin{bmatrix}\text{system is}\\ \text{empty (idle)}\end{bmatrix} = 1 - \frac{\lambda}{\mu} \tag{14.6}$$

$$L_q = \begin{matrix}\text{average number}\\ \text{in the queue}\end{matrix} = \frac{\lambda^2}{\mu(\mu - \lambda)} \tag{14.7}$$

$$L = \begin{matrix}\text{average number}\\ \text{in the system}\end{matrix} = \frac{\lambda}{\mu - \lambda} \tag{14.8}$$

$$W_q = \begin{matrix}\text{average time}\\ \text{in the queue}\end{matrix} = \frac{\lambda}{\mu(\mu - \lambda)} \tag{14.9}$$

$$W = \begin{matrix}\text{average time}\\ \text{in the system}\end{matrix} = \frac{1}{\mu - \lambda} \tag{14.10}$$

EXAMPLE 14.5

Educational Administration To complete registration at Dart University, students must report to the cashier's office to pay their fees. On the average, 100 students arrive at the cashier's office each hour; on the average, the cashier's office can process 120 students per hour. The registration procedure can be modeled as a single-queue, single-server waiting-line system.

REQUIRED

(a) What is the probability that a student will find the system idle when he or she appears to pay the fees?

(b) On the average, how many students will be waiting to be served?

(c) On the average, how long will a student wait in line before arriving at the cashier?

(d) On the average, how much time will a student spend waiting *and* conducting transactions with the cashier?

SOLUTION

$\lambda = 100$, and $\mu = 120$.

(a) $P_0 = 1 - \dfrac{\lambda}{\mu} = 1 - \dfrac{100}{120} = 1 - 0.833 = 0.167$

(b) $L_q = \dfrac{\lambda^2}{\mu(\mu - \lambda)} = \dfrac{100^2}{120(120 - 100)} = 4.16$ students

(c) $W_q = \dfrac{\lambda}{\mu(\mu - \lambda)} = \dfrac{100}{120(120 - 100)} = \dfrac{1}{24}$ hour = 2.5 minutes

(d) $W = \dfrac{1}{\mu - \lambda} = \dfrac{1}{120 - 100} = \dfrac{1}{20}$ hour = 3 minutes ■

KEY CONCEPT

Even when the average service capacity exceeds the average arrival rate, potentially long queues can form. For this reason, managers must use queuing models, instead of merely comparing average arrival and service rates, if they want to develop efficient waiting-line systems.

Managers are often astonished by the facts that are revealed by queuing models. In Example 14.5, the registration procedure can accommodate more students per hour, *on the average*, than the average number of arrivals. Regardless of this seeming excess of service capacity, the model shows that, on the average, there will be 4.16 students waiting for service! The reason for this seeming disparity is that there will be idle moments when no students are waiting, and the service capability will be wasted. Eventually there will come a time when many students arrive almost concurrently. The service capacity that was wasted when the queue was empty cannot be reclaimed, so the queue will grow. Over time, the number of students waiting for service will fluctuate around an average of 4.16 students in the queue.

EXAMPLE 14.6

Equipment Selection A retailer is evaluating two alternative computerized cash register systems. The firm expects that about 2500 customers per hour will require service and estimates a cost of $2 per hour per customer in ill will caused by waiting to complete a transaction. Each of the two systems can be considered to be a single-queue, single-server system.

System 1	*System 2*
$\mu = 2800$ per hour	$\mu = 3500$ per hour
Operating cost = $100 per hour	Operating cost = $125 per hour

REQUIRED

Which system would you recommend?

SOLUTION

For System 1

$\lambda = 2500$ and $\mu = 2800$ per hour

$$W = \frac{1}{\mu - \lambda} = \frac{1}{2800 - 2500} = \frac{1}{300} = 0.0033333 \text{ hour}$$

On the average, 2500 customers will each wait 0.0033333 hour for completion of service at a cost of $2 per customer per hour.

The average hourly cost of ill will = 2500(0.0033333)(2)	$ 16.67
Operating cost	100.00
Total system 1 cost per hour	$116.67

For System 2

Without even computing the cost of ill will on the faster system, it is seen that hourly operating costs will exceed the total cost of system 1.

Conclusion: Recommend system 1. ■

It is instructive to observe that, in Example 14.6, the average hourly cost of ill will could have been computed by an alternative approach.

$$\text{Average hourly cost of ill will} = \$2\begin{pmatrix}\text{average number}\\ \text{of customers}\\ \text{in the system}\end{pmatrix}$$

$$= \$2(L)$$

$$= \$2\left(\frac{\lambda}{\mu - \lambda}\right)$$

$$= \$2\left(\frac{2500}{2800 - 2500}\right)$$

$$= \$16.67$$

Characteristics of a Single-Queue, Double-Server System

The waiting-line system that is described in this section is similar to the system described in the previous section, except that now there are two service facilities that operate in parallel. A description of the system is presented in Table 14.3 and Figure 14.4.

TABLE 14.3 Description of the Single-Queue, Double-Server System

The arrival process	Arrivals are independent, with λ denoting the average number of arrivals per period of time T.
Queue discipline	Arrivals join one queue. There is no balking, reneging, or jockeying. Customers progress through the queue on a FIFO basis.
Service mechanism	There are two servers. The servers are identical and operate in parallel. Whenever a server completes service to a customer, it begins serving the next customer at the head of the waiting line.
The service process	Service times are independent, with μ denoting the average number served by *one service facility* per period of time T. Both service facilities have the same service rate, μ. $2\mu > \lambda$.

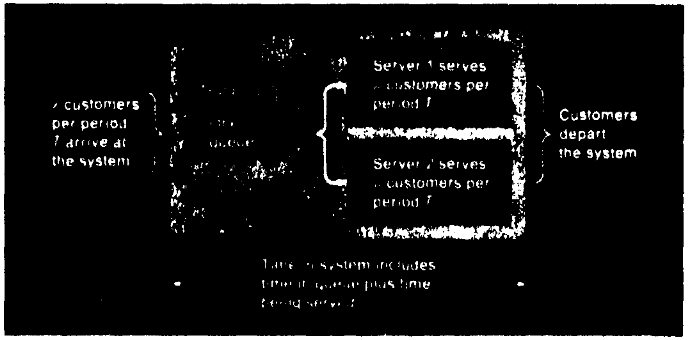

FIG. 14.4 The single-queue, double-server system.

As noted in the description of the service process in Table 14.3, 2μ must be greater than λ; otherwise, the number of customers waiting in the queue will approach infinity.

For the single-queue, double-server system, the following characteristics are easily computed:

$$P_o = \text{Prob}\begin{bmatrix}\text{system is}\\ \text{empty}\end{bmatrix} = \frac{1}{1 + \frac{\lambda}{\mu} + \left[\frac{\lambda}{\mu}\right]^2\left[\frac{\mu}{2\mu - \lambda}\right]} \tag{14.11}$$

$$P_b = \text{Prob}\begin{bmatrix}\text{both servers}\\ \text{are busy}\end{bmatrix} = \text{Prob}\begin{bmatrix}\text{new arrivals must}\\ \text{wait in queue}\end{bmatrix}$$

$$= \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\left[\frac{2\mu}{2\mu - \lambda}\right]P_o \tag{14.12}$$

$$L_q = \begin{matrix}\text{average number}\\ \text{in queue}\end{matrix} = \frac{\lambda\mu\left[\frac{\lambda}{\mu}\right]^2 P_o}{(2\mu - \lambda)^2} \tag{14.13}$$

$$L = \begin{matrix}\text{average number}\\ \text{in system}\end{matrix} = L_q + \frac{\lambda}{\mu} \tag{14.14}$$

$$W_q = \begin{matrix}\text{average time}\\ \text{in queue}\end{matrix} = \frac{\mu\left[\frac{\lambda}{\mu}\right]^2 P_o}{(2\mu - \lambda)^2} \tag{14.15}$$

$$W = \begin{matrix}\text{average time}\\ \text{in system}\end{matrix} = W_q + \frac{1}{\mu} \tag{14.16}$$

It is important to observe that for the double-server system,

$$\text{Prob}\begin{bmatrix}\text{both servers}\\ \text{are busy}\end{bmatrix} \neq 1 - \text{Prob}\begin{bmatrix}\text{both servers}\\ \text{are idle}\end{bmatrix}$$

That is,

$$P_b \neq 1 - P_o$$

The reason for the above condition is that there can be occasions when only one of the servers is busy (the other server is idle). The probability that one server is idle is given by

$$\text{Prob}\begin{bmatrix}\text{one server}\\ \text{is idle}\end{bmatrix} = 1 - \text{Prob}\begin{bmatrix}\text{both servers}\\ \text{are idle}\end{bmatrix} - \text{Prob}\begin{bmatrix}\text{both servers}\\ \text{are busy}\end{bmatrix}$$

$$= 1 - P_o - P_b$$

EXAMPLE 14.7

Production Management The final step in the production of electronic calculators at Lyndon Industry's Grove City plant is the quality-control station. The arrival of units at the quality-control station and the automatic quality-control testing of units by two identical testing machines satisfies the characteristics of the single-queue, double-server system described in Table 14.3. Figure 14.5 illustrates the quality-control station (system).

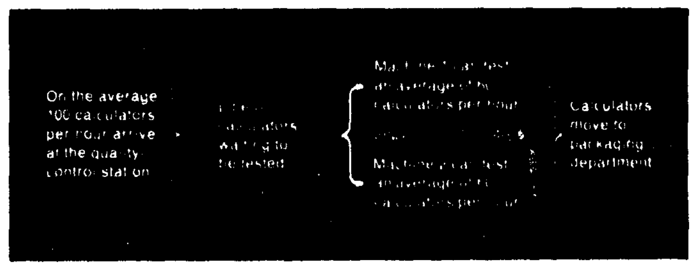

FIG. 14.5

REQUIRED

(a) What is the probability that the system is empty?
(b) What is the probability that both testing machines are busy?
(c) What is the average number of calculators waiting to be tested?
(d) What is the average number of calculators in the system?
(e) What is the average waiting time in the queue?
(f) What is the average waiting time in the system?