From: Kingsley E. Haynes and A. Stewart Fotheringham, Gravity and Spatial Interaction Models (Beverly Hills: Sage, 1984

GRAVITY AND SPATIAL INTERACTION MODELS

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1. GRAVITY MODEL: OVERVIEW

Spatial interaction is a broad term encompassing any movement over space that results from a human process. It includes journey-to-work, migration, information and commodity flows, student enrollments and conference attendance, the utilization of public and private facilities, and even the transmission of knowledge. Gravity models are the most widely used types of interaction models. They are mathematical formulations that are used to analyze and forecast spatial interaction patterns.

The gravity model as a concept is of fundamental importance to modern scientific geography because it makes explicit and operational the idea of relative as opposed to absolute location. All things on the face of the earth

AUTHORS' NOTE: We would like to thank John M. Hollingsworth, Department of Geography, Indiana University, for his excellent cartographic work and patience in developing the diagrams.

can be located in absolute terms by longitude and latitude coordinates, and the absolute position of things can be related to each other by reference to such coordinates. Distances can be specified in these absolute terms. It is then possible to talk about one location as being "five miles from New York City" and another as being "five miles from Bloomington, Indiana." In absolute terms, these two locations are equal in that they are both five miles from an urban area. In relative terms, however, these locations are significantly different in a multitude of ways (for example in terms of access to shopping, access to job opportunities, access to museums and theaters, access to rural life-styles, or access to wilderness opportunities). Each of these significantly differentiates absolute location from relative location. The gravity model allows us to measure explicitly such relative location concepts by integrating measures of relative distance with measures of relative scale or size.

The importance of the relative location concept and spatial interaction can be seen in the application and refinement of the gravity model over the past fifty years. Its continued use by city planners, transportation analysts, retail location firms, shopping center investors, land developers, and urban social theorists is without precedent. It is one of the earliest models to be applied in the social sciences and continues to be used and extended today. The reasons for these strong and continuing interests are easy to understand and stem from both theoretical and practical considerations.

Social scientists are interested in discovering fundamental and generalizable concepts that are basic to social relationships. One of the distinguishing features of human behavior is the ability to travel or move across the face of the earth and to exchange information and goods over distance. Such exchange processes are referred to generically as interaction, and that which occurs over a distance occurs over space. Hence, the general term "spatial interaction" has been developed to characterize this common type of geographic behavior. Shopping, migrating, commuting, distributing, collecting, vacationing, and communicating usually occur over some distance, and therefore are considered special forms of this common social behavior—spatial interaction. We seek here to describe fundamental characteristics that underlie all these forms of social behavior. We will make generalizations about those characteristics that explain or predict similar geographic behavior. Our goal is to demonstrate that spatial interaction models can be considered as the basis of important and useful social theories. The gravity model is one example of a spatial interaction model.

The gravity model, which derives its name from an analogy to the gravitational interaction between planetary bodies, appears to capture and interrelate at least two basic elements: (1) scale impacts: for example, cities with large populations tend to generate and attract more activities than cities with small populations; and (2) distance impacts: for example, the farther places, people, or activities are apart, the less they interact.

These concepts are used by urban social analysts to explain why land values are high in the central areas of cities and at other easily accessible points (Hansen 1959) and why land values are higher in larger cities than in smaller cities. They are used to explain why some public service or retail locations attract more users or customers than do others and to explain the way in which shopping centers impact the areas about them in terms of traffic and customer flows. On a larger scale, they are used to explain the movement of population in the form of migrants, visitors, business and commercial travelers, and the movement of information in the form of mail, telecommunications, and data transfers. In practical terms, these are important topics for many kinds of decision makers, both public and private; a model that purports to reduce the risk in making large capital decisions related to these topics obviously is valuable.

The applications to which the gravity model has been put are not limited to transportation, marketing, retailing, and urban analysis. In archaeology Hallam, Warren, and Renfrew (1976) have used it to help identify probable prehistoric exchange routes in the western Mediterranean, while Jochin (1976) has used it to examine the location and distribution of settlement among hunting-and-gathering peoples. Tobler and Wineberg (1971) used related methods to develop suggestions about the location of lost cities. More recently Clark (1979) has used a type of gravity model analysis to explain archeological data on the flow of goods. In a related context, Kasakoff and Adams (1977) have used location and anthropological information together with a gravity model formulation to explain marriage patterns and clan ties among Tikopians. Trudgill (1975) made a strong argument for the wider use of this method in linguistics, while Hodder (1980) has made a general plea for the wide use of this technique in trying to understand patterns and potential patterns in the spatial organization of both historic and prehistoric activities.

The Model

Figure 1.1 illustrates the basic relationships inherent in gravity models. Compare the expected level of flows between city x and city y with that between cities x and z. Without further information our intuitive expectation would be that the flows between x and y would be larger; y and z are the same distance from x—800 miles—but y has a population of 2 million while z has a population of only 1 million. If interaction were a function of pairs of individuals in any two cities then the potential sets of pairs between x and y is larger than between x and z (2,000,000 × 2,000,000 vs. 2,000,000 × 1,000,000). The potential pairs of interactions would be twice as great in the case of x and y than in the case of x and z. This is the *multiplicative* impact of scale on interaction.

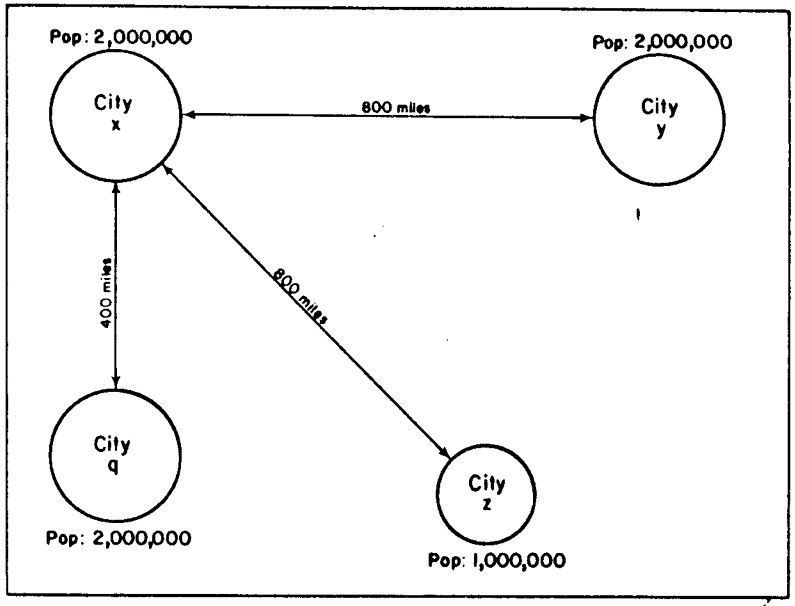
The impact of distance can be demonstrated by comparing the expected levels of flows between x and y, and x and q. The sizes of y and q are the same, so scale is constant. However, without further information we would expect more flows between x and q than between x and y because we would expect the flows between any two points to decline as distance increases. If this decline is proportional to distance, then with scale held constant we would expect half as much interaction between x and y as between x and q.

To generalize, let the scale of each city, population, be represented by P, and the distance between cities be represented by d. Each pair of cities is designated by the subscripts i and j. Interaction between any pair of cities is specified as T_{ij}. The interaction can be expressed as a ratio of the multiplied populations over the distance between any pair of cities,

$$T_{ij} = P_i P_j / d_{ij} \qquad [1.1]$$

Modifications

Three fundamental modifications need to be made to the basic model in equation 1.1. First, the distance element is adjusted by an exponent to indicate whether the impact of distance is proportional or not. For example, the cost per mile of traveling may decrease with distance, as in air travel. Obviously the operational effect of distance would therefore not be directly proportional to airline miles and the negative aspect of distance would need to be reduced or dampened so that the model properly reflects its effects. On the other hand the effect of distance may be underestimated by mileage because the opportunity to know people in cities far away may be reduced by language, culture, and information. The impact of distance may be greater than that indicated by use of straight line mileage in the model. This "distance decay" or "friction of distance" effect will vary depending on the flows being examined—air transportation as opposed to private automobile transportation, for example. Even though distance will always have a negative influence on interaction, in some cases it may be



NOTE: This illustration demonstrates the basic principles and trade-offs between the effects of scale (population) and distance (mileage) upon expected interactions between places.

Figure 1.1 The Gravity Model Principles

more negative than in others. An exponent on the distance variable, d_{ij}^{β}, allows us to represent this variability.[1] A large theoretical and empirical literature has developed around the definition of the "correct" exponent.

Much of the literature that focused on deriving the correct exponent for the gravity model formulation was stimulated by physical science interpretations, including the Newtonian analogy where the square of distance, d_{ij}^2, is the appropriate power function. In empirical analysis, however, the exponent is generally interpreted as the responsiveness of interaction to spatial separation and is expected to vary in terms of social context. Larger exponents indicate that the friction of distance becomes increasingly important in reducing the expected level of interaction between centers. Figure 1.2, part a, indicates the impact of small and large values of the exponent β on the distance variable. Other things being equal we would expect distance

CALCULATION OF MARKET BOUNDARY BETWEEN TWO CITIES

Application of a gravity Model (Reilly's Law of Retail Gravitation) to calculating the market boundary between two cities.

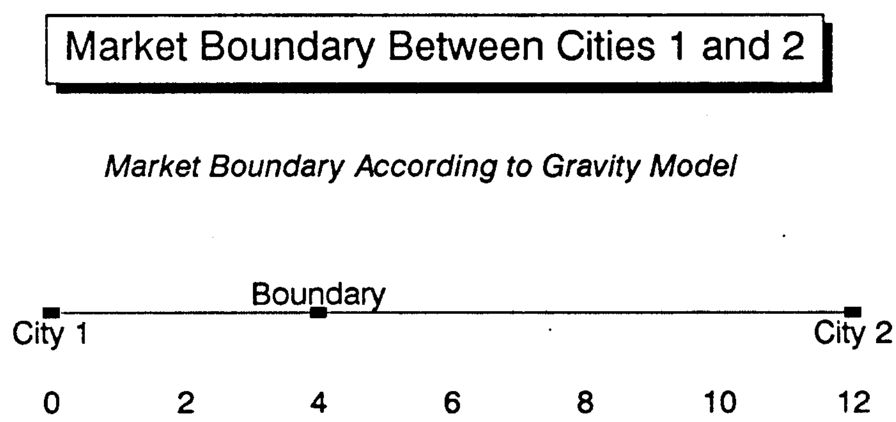
Input: (Enter information required below)

Population of City 1 =	4,000
Population of City 2 =	16,000
Distance between cities =	12

Results:

Distance from City 1 to Market Boundary =	4.00
Distance from City 2 to Market Boundary =	8.00

-- Press F10 to See Graph --



Note: Do Not Try to Do this Problem Until I have gone over it in Class & Assigned it (unless you want to)

Stipak

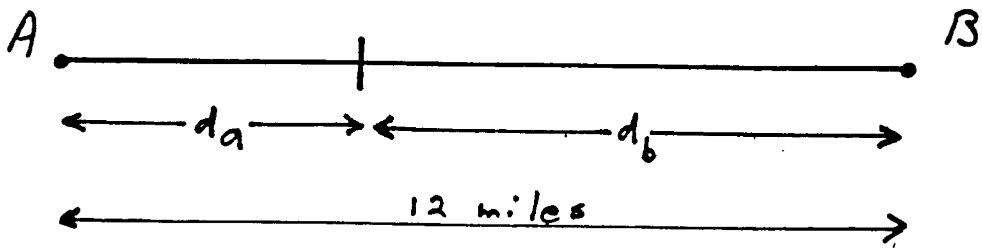
In class we discussed Reilly's law of retail gravitation:

$$F = K \frac{P}{d^2}$$

where:

F = retail attractiveness
P = size of town
d = distance to town

Using this law, determine the point on a straight line between two towns 12 miles apart which is the market boundary between the two towns. Town A has a population of 4000, Town B a population of 16,000.



Note: $$d_a = \frac{d\sqrt{P_a}}{\sqrt{P_b} + \sqrt{P_a}}$$

BASIC INVENTORY MODEL

variable or parameter	description
Q units	constant order quantity
d units/time	constant, continuous demand rate
c \$/order	fixed order cost
h \$/unit/time	holding or storage cost

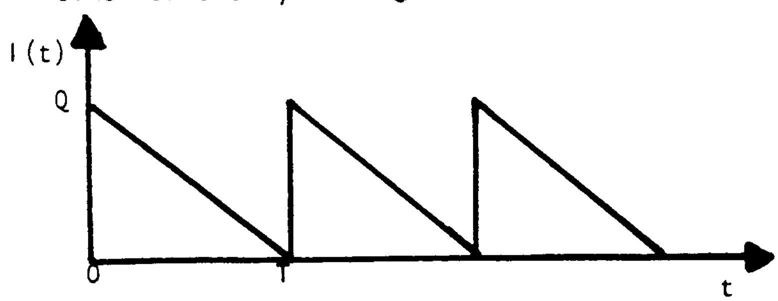
No lead time is necessary in placing orders. Inventory orders are filled immediately.

Negative inventory is not allowed--i.e. inventory is not allowed to remain out-of-stock. An order of size Q is immediately placed and filled when the inventory is depleted.

Let:

I(t) = size of inventory at time t

T = time for one cycle = Q/d



Objective: Minimize the total inventory cost (order and holding costs) per unit of time.

Let:

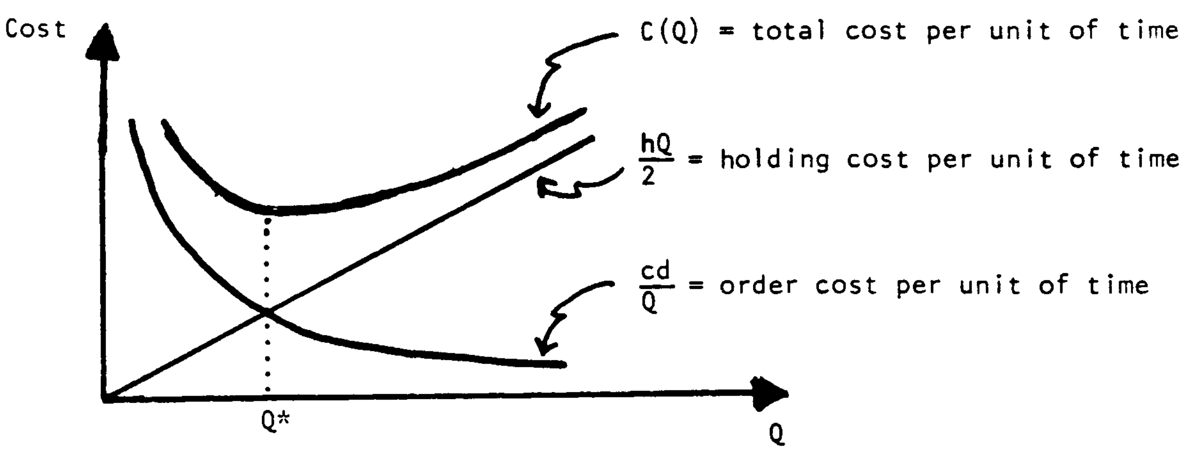
C(Q) = total inventory cost per unit of time, averaged over one cycle

Then:

$$C(Q) = \frac{\text{order costs/cycle}}{\text{time/cycle}} + (\text{holding costs/unit/time})(\text{average number units})$$

$$= \frac{c}{T} + h(Q/2) = \frac{c}{Q/d} + \frac{hQ}{2} = \frac{cd}{Q} + \frac{hQ}{2}$$

Procedure: Choose Q to minimize $C(Q) = \frac{cd}{Q} + \frac{hQ}{2}$



OVER

If only a small number of values of Q are in the feasible set, we can compute C(Q) and choose the value of Q, Q*, that minimizes C(Q).
Using calculus, the value of Q that minimizes C(Q) is easily shown to be:

$$Q^* = \sqrt{\frac{2cd}{h}}$$

Alternative, equivalent, procedure: Mimimize the average total inventory cost per unit.
Let:

$$U(Q) = \frac{\text{order costs/order}}{\text{units/order}} + (\text{holding costs/unit/time})(\text{ave. time in inventory})$$

$$= \frac{c}{Q} + h(T/2) = \frac{c}{Q} + \frac{hQ}{2d}$$

We now choose Q to minimize $U(Q) = \frac{c}{Q} + \frac{hQ}{2d}$

Using calculus, the value of Q that minimizes U(Q) is found to be the same Q* that mimimizes C(Q):

$$Q^* = \sqrt{\frac{2cd}{h}}$$

Homework Problem

You are the manager of a federally-funded local job training program. The program places trainees in local job openings which occur at a dependable, constant rate of 12 openings/month. The training session lasts 3 weeks. The costs of the training program consist of 3 parts:

1) a fixed cost of $400 for each training session regardless of the number of people trained
2) an additional cost of $700 per trainee which includes a stipend paid the trainee
3) waiting costs, as explained below

Training is performed to just keep ahead of demand, so that no openings that arise are ever vacant. Therefore, trainees usually have to wait after finishing the session before they are placed in an opening. The cost to the program for a trainee who is waiting is $800/month.

Because of the facilities available, no two training sessions can overlap in time, and the largest possible class size is 14.

As manager of the program, you have to decide how many trainees to admit to each session. Since the program is on a tight budget, you want to set the class size to minimize total program costs.

a) Explain how you would structure the problem by identifying the different problem elements: decision variable, parameters, constraints, objective function

b) Determine the optimum class size.

c) Do a sensitivity analysis by calculating the average increase in costs per trainee for each of the sub-optimum alternatives, compared to the optimum alternative.

d) Use the formula for Q* to calculate the optimum unconstrained value of the decision variable. Is the calculated Q* in the feasible set? If not, what is the unconstrained optimum class size? If the unconstrained optimum you calculated is not within the feasible range, what implications does this have for the possibility of lowering costs by overlapping training sessions, or by increasing class size?

Example of Spreadsheet Application of Basic Inventory Model

Parameters:

c =	$1,500	fixed cost/session
d =	10	trainees/month
h =	$800	holding cost/trainee/month

Optimal Analysis:

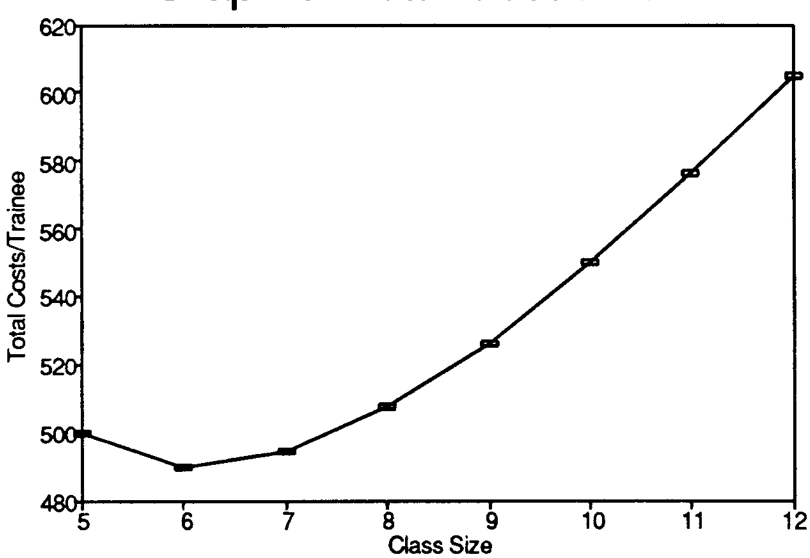
Q* = 6.12

Class Size	Fixed Costs	Waiting Costs	Total Costs	$ Over Optimum
5	$300	$200	$500	$10
6	$250	$240	$490	$0
7	$214	$280	$494	$4
8	$188	$320	$508	$18
9	$167	$360	$527	$37
10	$150	$400	$550	$60
11	$136	$440	$576	$86
12	$125	$480	$605	$115

To see a graph of how total costs vary by class size press the F10 key.

Note: All costs in the table are given as costs per trainee.

Graph of Total Costs/Trainee



Spreadsheet Application of Basic Inventory Model, Modified to Allow for a Variable Purchase Price Using a Lookup Table

Parameters:

c =	$1,500	fixed cost/session
d =	10	trainees/month
h =	$800	holding cost/trainee/month

Optimal Analysis, NOT including purchase cost:
Q* = 6.12

Class Size	Fixed Costs/ Session	Fixed Costs/ Trainee	Waiting Costs	Total Costs	$ Over Optimum
5	$300	$100	$200	$600	$23
6	$250	$100	$240	$590	$13
7	$214	$100	$280	$594	$17
8	$188	$70	$320	$578	$0
9	$167	$70	$360	$597	$19
10	$150	$70	$400	$620	$43
11	$136	$70	$440	$646	$69
12	$125	$50	$480	$655	$78

To see the lookup table for fc/train. press PgDn.

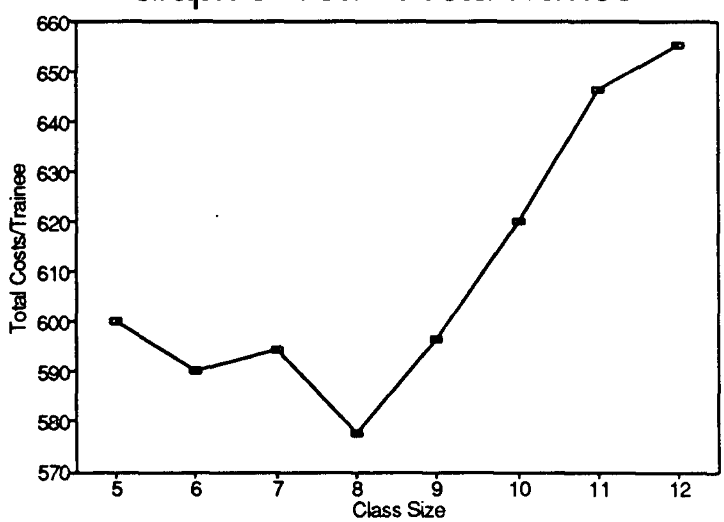
To see a graph of how total costs vary by class size press the F10 key.

Note: All costs in the table are given as costs per trainee.

Lookup Table for Fixed Costs per Trainee
(Analogous to purchase cost)

Class Size	Fixed Costs/ Trainee
1	$300
3	$200
5	$100
8	$70
12	$50
20	$40

Graph of Total Costs/Trainee



From Michael J. White et al., Managing Public Systems

Duxbury, North Scituate, 1980

Systems-analytic techniques

TWELVE

Operations Research in Action: Executive Replacement Planning

In recent years, the federal government has sought to bring more rationality, greater sophistication, and increased efficiency to the management area of *executive development.* No longer viewed simply as "shipping a manager off to school," executive development is now regarded as a comprehensive process involving identification, counseling, individual planning, training and development, and utilization. In a broad sense, executive development activities are geared to (1) increasing individuals' effectiveness in their current jobs, and (2) preparing people for higher level responsibilities.

Executive development efforts have traditionally emphasized the first objective. However, current efforts are increasingly aimed at the second. In the mid-1970s the U.S. Civil Service Commission (now the Office of Personnel Management, OPM) required federal agencies to establish high-potential identification programs and to identify all managerial positions and the skills, knowledge, and abilities they require.[1] In

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this way, training and developmental activities were to be closely tailored to the future responsibilities of the individual.

With strong and continuing support of the Office of Management and Budget and the Civil Service Commission, executive development became a high-priority effort.

With minor exception, current executive development strategies focus on training and development processes. But there are now several other areas that deserve serious attention—including executive manpower planning and program evaluation. As expenditures for training and development increase, resources for executive development will become issues that involve critical choices. Concurrently, as executive development becomes more institutionalized within organizations, it will develop vested interests, constituencies, and dependencies. Thus, executive development must increasingly be capable of making a case for itself. Yet, planning and evaluation techniques in this regard are fairly primitive. To date, central guidance has essentially involved exhortation—encouragement of agencies to do planning and evaluation—but has offered little assistance on how planning and evaluation should be done.

If the state of the art is to be advanced, it is time to look at executive development from new vantage points. One such perspective is offered by operations research (OR). Assuming an OR lens, executive development specialists can assess executive development in a rigorous, quantitative manner. Planning, training option, and resource allocation decisions could become more rational, and, it is hoped, more effective.

This chapter explores the use of various operations research techniques in analyzing executive replacement planning. As such, executive development will be viewed as a means of identifying and preparing managers to assume executive jobs.

The Executive Replacement Planning Problem

As personnel costs soar, government is increasingly concerning itself with manpower management—the effective utilization of human resources. Executive manpower management is especially critical. Turnover in the executive ranks of the federal government is high, due, among other things, to the executive salary situation and the attractiveness of being a retired annuitant. In fact, in the late 1970s, in several large agencies a majority of the executives became eligible for retirement. At the same time, the job of the career executive becomes increasingly complex. Key executive vacancies should be filled expeditiously by people who can "hit the ground running."

Without compromising the merit promotion principles, it is possible to regard executive development as an integral part of executive replacement planning. The scenario would go something like this: Planners would identify key executive jobs in an organization and articulate re-

quired knowledge, skills, and abilities. Those managers with executive potential would be identified and assessed in terms of the qualifications required for executive positions. High-potential managers would receive training and developmental experiences necessary so that they could step into executive vacancies and perform effectively. These managers would constitute an executive replacement pool.

Unfortunately, executive replacement planning in the majority of federal agencies is, at best, informal and ad hoc. One frequently hears executive development specialists talk about executive manpower planning as follows: "Since our organization has thirty executive positions, thirty people ought to be standing in the wings so that when vacancies occur, someone can take over. However, maybe the ratio shouldn't be 1:1 since there is the possibility that some of the people we have trained for executive positions might leave the organization. So, let's set the ratio at 3:1 (three trained managers for each executive position)." This line of reasoning is not rigorous. It results in many costs to the organization and ignores the dynamics of executive replacement. If you think this is strictly a government failing, you might check the news media during recent recessions to learn of the substantial "surplusing" of executives by many leading corporations during those difficult periods.

A common complaint about high-potential identification is that too many people are identified and trained for too few real opportunities. Such people are trained for jobs they might not assume. Thus, they are operating in a position that does not take full advantage of their training. The managerial dilemma is to insure that key executive vacancies are filled quickly with qualified people and that the replacement pool does not have too much overdeveloped and underutilized talent.

The question is, "How many managers should be in the pool?" The answer to this question would have a significant bearing on selection procedures, training strategies, and resource requirements.

The approach of this chapter is primarily conceptual. The data used in the examples, while hypothetical, are realistic and are based on familiarity with this subject. Such data are available in federal organizations, so that the techniques outlined here should work in actuality.

Executive Replacement Planning Viewed as a Queueing Situation

Executive replacement planning involves the identification of executive positions, the projection of executive vacancies, and a conscious strategy to train and develop potential executives. Thus, when positions become vacant, they can be filled expeditiously with qualified people.

This replacement process involves two key elements of a queueing situation: service stations (executive vacancies) and a waiting line (a pool of people ready and qualified to fill the executive vacancies). This queueing model is portrayed in detail in Figure 12-1. It is apparent from the dia-

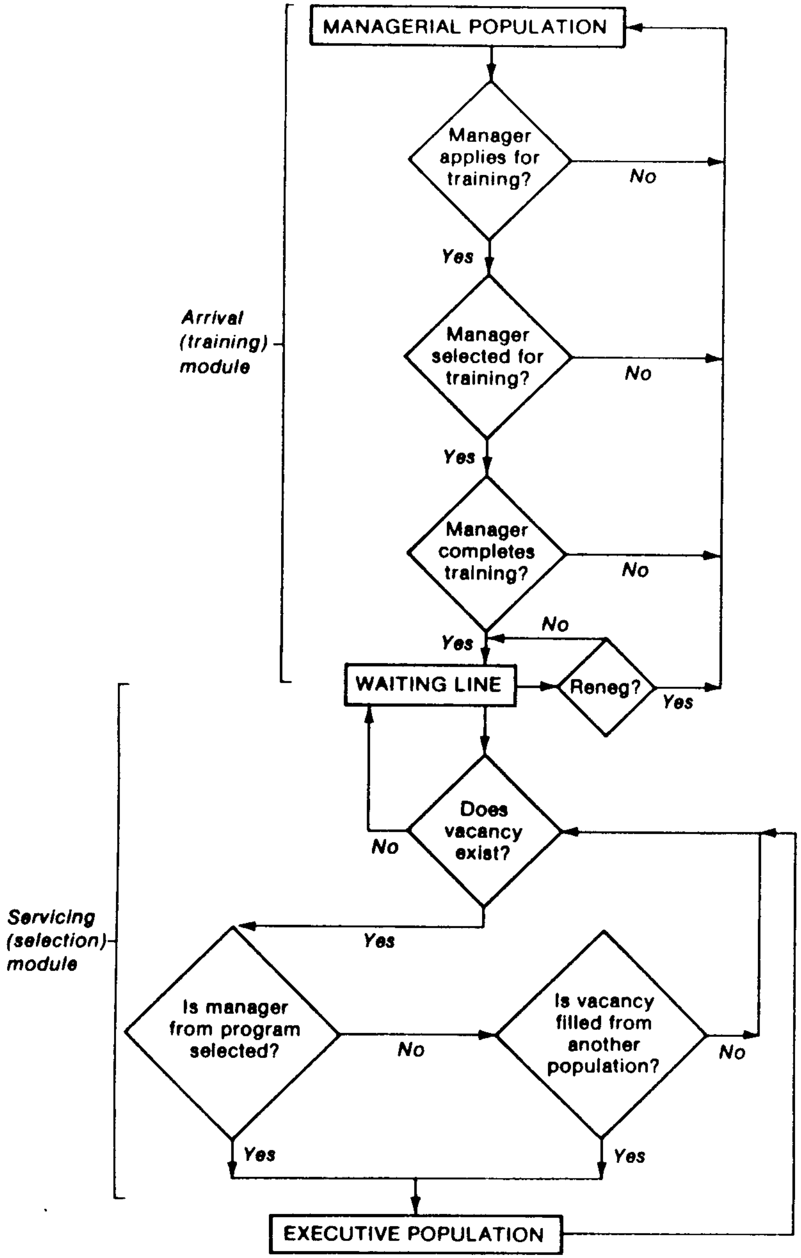


Figure 12-1. *Flow Diagram of Training and Queueing for Executive Replacement*

gram that the customers have little control over queue processes. In fact, there are only two decision points: whether or not to apply for the program and whether or not to continue to wait in the queue. The executive queue discipline is not "first come, first served." Rather, service is determined by the organization's selection procedures. Management may decide to fill the executive position with anyone who has completed the training program, with someone in the organization who has not participated in the training, or with someone from outside of the organization.

Both the arrival rate and the service rate are independent of the customer. Management ascertains how many people are to be selected for the training program. And the service rate is a function of executive positions becoming vacant.

A queueing model highlights the fact of waiting and organizes a picture of the executive replacement situation. Since executives are trained as a batch (class) and we wish to schedule these classes for time and size based on the utilization rate, the situation no longer fits the queueing model. Neither service time (training period) nor arrival rates (enrollments) are stochastic variables. Instead, they are policy decisions, and we have an inventory situation to model.

Executive Replacement Planning Viewed as an Inventory Model

An inventory model for this executive replacement problem considers relevant factors and provides the basis for determining optimum replacement pool size. Qualified people who are ready to assume executive vacancies are produced in batches (classes) via an executive training program. These people then constitute an executive replacement pool. The number of people who should be in each batch depends upon many factors, including the projected number of vacancies, the costs of the training program, and the costs associated with not fully using the talents of the people who have been trained.

An economic lot size model is a very useful way of determining the optimum batch (class) size. First, some assumptions need to be articulated:

- When an executive position becomes vacant, it will be filled;
- The organization encourages promoting from within and wishes to strengthen the credibility of its executive training program; therefore, all executive vacancies will be filled with persons who have completed the training program;
- Because of the previous assumption, the relationship between training and selection is high. Therefore, there are strong incentives for those selected for training to complete the training program.

These assumptions are not unrealistic and they often hold in a real-life situation. Nonetheless, they are not universally true. Organizations, for one reason or another, may decide to keep an executive position vacant for some time. And, quite frequently, executive vacancies are not filled with persons completing special training programs but rather by other persons within or outside the organization. Finally, persons selected for training programs may decide not to complete the developmental experience but to pursue other opportunities.

For the purposes of this analysis, let's proceed with the assumptions stated. You should recognize that many other situations could prevail and that they could be accommodated in an inventory model.

The objective, then, is to find the size of the executive training class that, given a certain annual level of demand, yields the minimum combination of two divergent costs: the cost of carrying the inventory (keeping a manager in the pool) and the costs of originating and producing each order (training and processing for filling executive vacancies).

Total Costs and the Economic Lot Size Formula

The economic lot size formula from inventory theory will yield a solution to this problem. But first, before using it, you should understand where this formula comes from.

The two costs generally associated with an inventory can be combined into total costs:

$$TC = C_O + C_H,$$

where

TC = total costs,
C_O = ordering costs,
C_H = holding costs.

Ordering costs are the fixed costs of assembling and training a class, regardless of class size. Holding costs are the costs associated with having an "item in inventory"; in this case it is the cost of having a person trained but not using the training. For now, we shall make some further assumptions. On the average, let us assume that the need (demand) for these specially trained managers is relatively constant, so many per week with very little deviation. Given this assumption, if the agency starts a new class at regular intervals, the inventory will take on the sawtooth pattern illustrated in Figure 12-2. According to this portrayal, during each period t, approximately Q managers are selected from those who have taken the training program; these are promoted to the executive classification. Further, the rate of appointment is a constant, so many per week. Finally, Figure 12-2 suggests that the new class graduates just at the time that the supply of previously graduated managers runs out. The agency might want

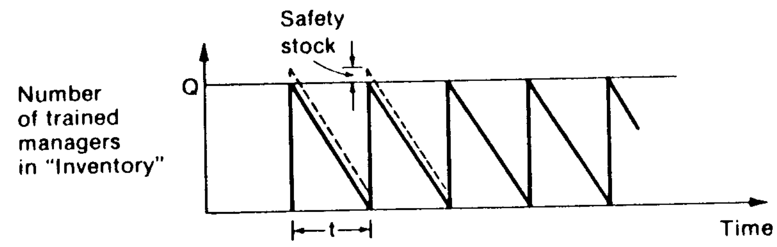


Figure 12-2. *Sawtooth Inventory Pattern for the Appointment of the Specially Trained Executives*

to keep a *safety stock* just in case demand for the trained managers is not as predictable as Figure 12-2 portrays. In that case, the pattern would follow the dotted line in Figure 12-2. Safety stock will be discussed in the following section on reorder point. Note here that the sawtooth model presumes that demand is deterministic rather than stochastic.

What are the ordering costs? Since our agency seeks to minimize total costs, it is most convenient to treat costs on an annual basis. Order costs for the year, C_O, will be cost per order, O, times the number of orders per year. The number of orders per year is the annual number of units ordered, N, divided by the size of the order, Q. We seek to determine Q:

$$C_O = \frac{N}{Q}(O).$$

Holding costs are the costs of carrying inventory. Normally, with a physical inventory holding costs, C_H, would include the costs of space; insurance; security; interest on the money tied up; loss through deterioration, pilferage, or outdating; and similar factors. Sometimes these costs are represented as a percentage of the purchase costs or of the costs of the average inventory (purchase costs times average inventory level). Here it is more convenient to consider the annual cost of holding an item, H, and multiply this by the average number of items in inventory. From Figure 12-2 you can see that if there is no safety stock, the average number of items in inventory equals one-half the size of the regular order, Q. (Verify this for yourself using the graph.) The holding costs are then

$$C_H = H\frac{Q}{2}.$$

Total costs, then, are

$$TC = C_O + C_H = O\frac{N}{Q} + H\frac{Q}{2}. \quad (12.1)$$

Mathematically, equation (12.1) can be solved for that value of Q that minimizes total costs. Most introductory operations research texts and any introductory calculus text will explain the derivation. The formula for the *economic order quantity* is

$$Q = \sqrt{\frac{2 \times N \times O}{H}}. \qquad (12.2)$$

To apply this formula, some numbers are necessary. And since this is a hypothetical example, we will supply some hypothetical numbers.

Assigning Costs and Applying the Formula

You might think at first that the ordering costs of a class are the costs of selecting, assembling, and housing the trainees—the "purchase costs" would then be the costs of the training itself. But, it is really the reverse here. Within broad limits, the training costs will not vary with the size of the class. The training curriculum will be the same whether there are thirty or thirty-five trainees. The instructors can handle a few more or a few less. But selection, transportation, housing, and the costs of removal from the job all vary directly with the number of trainees! So the fixed costs per class are the training costs. These are the order costs for this example. The variable costs are the unit price and are composed of the costs of assembling the trainees and removing them from work.

Assume the training runs for four weeks (twenty class days) and that trainees return home on Friday but spend Sunday through Thursday evenings at a hotel conference center at $50 per day for room and board. This figure also rents a conference room. Trainees pick up their own bar tabs. The training sessions are conducted largely by hired professionals at an average cost of $300 per day. In addition, two training division staff members coordinate the sessions daily, at a budgeted cost of $100 per day each. Finally, there are various overhead costs associated with retaining the professional trainers, revising the curriculum, and so forth, budgeted at $2000. The total fixed costs are given in Table 12-1.

Table 12-1. *"Ordering Costs" for a Class of Trainees: The Fixed Costs of the Training Course*

Item	Fixed Cost
Professional trainers, 20 days, @ $300/day	$6000
Training division staff, 2/day, 20 days, @ $100/day	4000
Budgeted overhead	2000
Ordering costs per class	$12,000

Among the other figures needed are the number of new executives needed per year and the holding costs of an executive for a year. Assume the agency needs 100 new executives each year. For holding costs, no easy figure is available. One alternative might be to assume that the value of a trained manager is accurately represented by the salary increment attendant upon promotion. In a perfect market—which a civil service system most likely neither faces nor contributes to—the incremental wage would be related to the incremental productivity of the trainee. For sake of argument, let us assume a perfect market here. Upon promotion, the trainees have their salary and benefits boosted by $4000. If they are trained but not promoted, then the agency is missing an opportunity to contribute incrementally at that rate to the community. Therefore, we assign a value of $4000 per unit as the annual holding cost. Now all the letters in equation (12.2) have numbers attached and you can solve for Q:

$$Q = \sqrt{\frac{2 \times N \times O}{H}} = \sqrt{\frac{2 \times 100 \times 12{,}000}{4{,}000}} = 24.5.$$

According to this analysis, anyway, each class should contain twenty-four and one-half managers! Well, let's make it twenty-five.

This solution could also have been approximated graphically. Using the given values for H, N, and O, various figures for Q have been inserted in equation (12.1) and the resulting terms calculated, displayed, and graphed in Figure 12-3. The point at which total costs (TC) are a minimum can be read off the graph, and the optimal value of Q (24.5) corresponding to it determined.

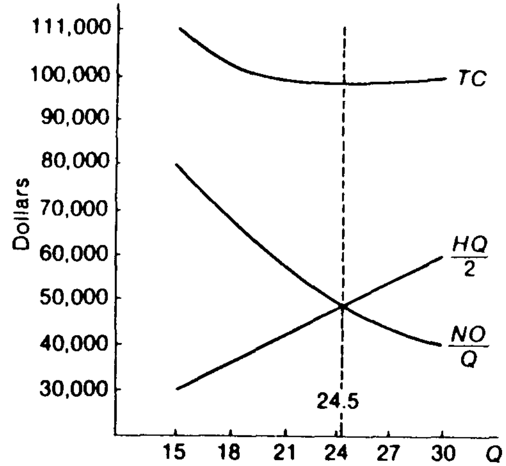
Reorder Point

Since it takes a whole month to prepare a new class, the agency also wants to know when to begin a class. Since people must be selected and they must also arrange for their leave, the agency adds another two weeks to the four weeks of seminar, for a lead time of six weeks.

If the agency assumes that its rate of appointment of trainees really looks like the sawtooth inventory of Figure 12-2, with no stochastic fluctuation, then it should begin assembling a new class whenever it has exactly six weeks worth of trained managers in inventory—which, mathematically at least, turns out to be 11.54 trainees:

$$\frac{\text{100 managers needed/year}}{\text{52 weeks}} \times 6 \text{ weeks} = 11.54.$$

But the agency will not face such a regular demand. Demand may show a seasonal pattern. Myrtle discusses seasonal variation in Chapter 17, so



Q	HQ/2	NO/Q	TC
15	30,000	80,000	110,000
18	36,000	66,667	102,667
21	42,000	57,143	99,143
24	48,000	50,000	98,000
27	54,000	44,444	98,444
30	60,000	40,000	100,000

Figure 12-3. *Solution of the Economic Order Quantity Formula by Table and Graph*

this topic will not be emphasized here. The demand will also vary randomly from week to week. Table 12-2 represents some research by the personnel department. Their analysts noted that in the past the agency had overstocked trained managers and had not "run short." They documented the number of managers needed by week, and then tabulated these weekly figures into six-week moving sums. This means that they summed the demand for weeks one through six, for weeks two through seven, and so on, through weeks fifty-two to fifty-seven. Seven was the lowest number of executives needed and fifteen was the highest number.

If this is a representative distribution, we can use it to help the agency determine how much safety stock to hold. When the costs of being short are difficult to determine, an organization might adopt a service level policy. Such a policy states what percent of the time the organization is willing to run short. For this training situation, the agency might be willing to run short 10 percent of the time. In that case, a new class would be started whenever the supply of trained executives dropped to thirteen. The

Table 12-2. *Number of Executives Needed for Fifty-Two Rolling Six-Week Periods: Frequency by Number Needed*

Number Needed in Six-Week Period (n)	Frequency (F)	Number × Frequency (n × F)	Percentage of Periods This Number Is Needed	Probability This Number Is Needed [P(n)]
7	1	7	1.9	.019
8	1	8	1.9	.019
9	3	27	5.8	.058
10	7	70	13.5	.135
11	15	165	28.8	.288
12	14	168	26.9	.269
13	6	78	11.5	.115
14	3	42	5.8	.058
15	3	30	3.8	.038
	52	595	99.9%[a]	.999[a]

[a]Total less than 100 due to rounding.

chance of fourteen or more executives being demanded in any six-week period is

$$P(14) + P(15) = .058 + .038 = .096 \quad \text{or} \quad 9.6\%.$$

What is the average number of managers the agency will be short during each six-week order period if it reorders when the inventory equals thirteen? Well, if they reorder at thirteen and the demand is thirteen or less, there is no shortage. If the demand is fourteen, they will be short one manager. The chance of that, from Table 12-2, is only $P(14) = .058$. The *expected value* is the product of the outcome and the probability: One person short times .058 equals .058 persons. If demand equals fifteen, then the agency runs two persons short. The chance that the demand will be fifteen is $P(15) = .038$, and the expected value is $2 \times .038 = .076$ persons. The total expected shortage is .058 plus .076, or .134 persons short each six-week period if the agency reorders at an inventory level of thirteen.

With a similar analysis, the agency could assess the costs of needing fewer than thirteen executives during the order period, as shown in Table 12-3. That surplus is held for the entire six weeks. Taking the opportunity cost figure used above for holding costs—$4000 per person per year—then the "long" costs associated with a reorder point of thirteen are

$$1.691 \times \$4000 = \$6764.$$

Perhaps this agency can also assign a cost to being short when an executive is needed. On the average, the agency will be short an executive for about four days each order period. This is a length of time that likely falls short of executive days lost to illness absence. After all, any agency that replaces 100 executives a year must have a pretty large number of executives! Therefore, you might decide that a reorder point of thirteen is conservative enough for your taste. If you care to be more risk taking or

Table 12-3. *Cost Associated with a Demand of Less Than Thirteen Executives*

Executives Available	*Executives Demanded*	*Probability of Demand [P(S)]*	*Surplus (S)*	*P(S) × S*
13	12	.269	1	.269
	11	.288	2	.576
	10	.135	3	.405
	9	.058	4	.232
	8	.019	5	.095
	7	.019	6	.114
		Expected surplus		1.691

less, you can replicate the preceding analysis using a different reorder point. You might also prepare your own demand distribution, expanding and then contracting the variance of the demand to see what effect that has on the desirable reorder point. You will find that the greater the variance in demand, the more expensive it is to pursue a given service-level policy.

Conclusion

While queueing theory offers a good way to obtain a picture of an executive replacement planning situation, the inventory model appropriately highlights the dynamics of executive replacement planning, considers both demand and costs, and is effective in answering the question related to the optimum size of the executive replacement pool. The inventory model also provides a means for determining when a new training program should be started.

To end, though, by applauding the inventory model for giving us these answers would be a disservice. For, its potential benefits are more far reaching. The inventory model can encourage rigor, comprehensiveness, and cost effectiveness in executive development and manpower planning. It also provides an opportunity for an agency to scrutinize its forecasting techniques, training strategies, and utilization policies.

To reinforce these points, let us go back to the economic lot size equation (12.2). If projected demand should have been 150 positions instead of 100, thirty people would be required for each class. A forecasting mistake could have significant negative consequences. Similarly, the model can encourage management to explore various training strategies. If a less expensive training program were used, the pool would need fewer people. If the training were done in-house at a savings of, say $4000, economic lot size drops to twenty. Finally, the organization may want to reduce the opportunity costs of those waiting in the pool. For example, a policy that encouraged top officials to appoint pool members to executive positions on an acting basis would lessen the underutilization costs.

Needless to say, one technique can solve only a few problems. But, you have seen the potential of the inventory model in addressing one human resources question. Other OR techniques can be similarly effective in other decision situations by identifying critical variables; by reducing guesswork without sacrificing creativity; and by assisting in the exploration of efficient, defensible solutions.

Exercises

1. Enumerate all the costs that might be incurred in ordering and holding each of the following things: explosives, to be used by a road building crew; vaccines, to be used for innoculations of people thought to have been exposed to a dangerous virus; personalized executive stationery in an agency with a high executive turnover rate; ball-point pens of a popular variety. Pay special attention to the costs of spoilage, the costs of being caught short, and other contingent and/or hidden costs.

2. The Department of Parks and Recreation uses 100,000 bags of fertilizer per year. Each fertilizer order costs $50 to place. In California, in contrast to the eastern United States, fertilizing is a year-round job. The director is concerned about where to store fertilizer and how much to store at each location. An investigation of the present activity shows that it costs $1 per year per bag to maintain the present storage facility. As a start to the solution of the problem, the director wants to know the economic order quantity and how long that will last assuming constant usage.

3. Efficient City uses 20,000 printed computer forms per year, which cost $100 per 1000 forms. Ordering these forms costs $15 per order. Storage cost is estimated to be 5 percent of the average inventory. What is the optimum number of forms to be ordered each time?

4. The fleet manager for Progressive City is concerned about fuel supplies, from the standpoint of both ordering and storing, in this period of rapidly changing fuel prices. Presently, the city is using 100,000 gallons a year. The city purchasing office tells the manager that ordering costs are $10 per order. The manager knows that storage costs are 5 percent of his average fuel stock when he pays $1 per gallon for fuel. What change would he have to make in his storage costs if the cost increases to $1.25 per gallon and he maintains the same order size? Does this appear to be a good solution?

5. A federal agency has a nationwide investigative staff. Currently, workload is neither increasing nor decreasing. Budget and workload are in reasonable balance. The director of training is consulting with the personnel director in trying to arrive at a recruitment and training schedule. The agency needs 800 new investigators each year. Training classes of twenty individuals each are conducted in the larger offices of the agency. No new facilities are required and the instructor time is usually not considered a cost since present investigational staff is used and the instruction is considered part of refresher training for them. However, each time a training cycle begins (several classes) central office staff is sent out, materials are revised, and considerable staff work is done. These costs come to $125,000 each time. The agency considers the salary of new investigators waiting for assignment as a cost; salary is $14,000 per year. The director of training decides she needs to know:

The most economical size of group to be recruited and trained at one time;
How many training cycles to start during the year; and
How much the waiting time of unassigned investigators is costing.

NOTES

1. U.S. Civil Service Commission, "Executives and Management Development," Federal Personnel Manual System Letter 412-2, Washington, D.C., January 29, 1974.

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