Summation Notation Worksheet

1 Introduction

Sigma notation is used as a convenient shorthand notation for the summation of terms. For example:

1. We write $\sum_{n=1}^{5} n = 1 + 2 + 3 + 4 + 5$.

   Here the symbol $\sum$ (sigma) indicates a sum. The variable $n$ is called the *index* and increments by 1 with each iteration. The numbers at the top and bottom of the sigma are called upper and lower bounds, respectively. These tell us the starting and ending values of the index. What comes after the sum is an algebraic expression representing the terms in the sum.

2. $\sum_{n=1}^{5} n^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3$

3. $\sum_{n=3}^{5} n^3 = 3^3 + 4^3 + 5^3$

4. $\sum_{i=1}^{4} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

   Note that we used $i$ instead of $n$ in the last example. The name of the indexing variable is not important, though it is customary to use $i, m, n, \text{ or } k$ as index variables.
It is often the case that we have a sum of a list of terms that we wish to express using $\sum$ notation.

5. Write the expression $3 + 6 + 9 + 12 + \cdots + 60$ using $\sum$ notation.
   - Notice that we are adding multiples of 3;
   - then this sum can be written as $\sum_{n=1}^{20} 3n$. (Alternatively, you may index with $i$ or $k$, etc.)

6. Write the expression $1 + \frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \cdots + \frac{1}{3n+1}$ in $\sum$ notation.
   - Notice that we are adding fractions with a numerator of 1 and denominators which start at 1 and increase by 3 each time.
   - Thus the denominator may be represented by $3k + 1$ for $k = 0, 1, \ldots, n$;
   - then this sum can be written as $\sum_{k=0}^{n} \frac{1}{3k + 1}$

Here we see a case where one of the bounds is itself a variable. Be careful to distinguish between this variable and the index for the sum!

We can also use $\sum$ notation when we have variables in our terms.

7. Write the expression $3x + 6x^2 + 9x^3 + 12x^4 + \cdots + 60x^{20}$ in $\sum$ notation.
   - The coefficients are successive multiples of 3, while the exponents on the $x$-term go up by 1 each time;
   - then this sum can be written as $\sum_{i=1}^{20} (3i)x^i$
Exercises

Write out each of the following sums long-hand:

1. \[ \sum_{n=1}^{6} n^4 \]
2. \[ \sum_{k=3}^{7} \frac{k + 1}{k} \]
3. \[ \sum_{i=2}^{n} (2i - 1) \]
4. \[ \sum_{k=0}^{n} 2^{k+1}x^k \]
5. \[ \sum_{k=0}^{n} \frac{(-1)^k x^k}{2k + 1} \]

Write each of the following sums using \( \sum \) notation.

6. \[ 1 + 4 + 9 + 16 + 25 + 36 \]
7. \[ 3 - 5 + 7 - 9 + 11 - 13 + 15 \]
8. \[ \frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} + \frac{1}{17} \]
9. \[ \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots + \frac{n+1}{n+2} \]
10. \[ 2 - 2^2 + 2^3 - 2^4 + \cdots + 2^{2n+1} \]
11. \[ 2x^3 + 4x^5 + 6x^7 + \cdots + 30x^{31} \]
12. \[ 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} \]