5
Matter Waves

5-1 $\lambda=\frac{h}{p}=\frac{h}{mv}=\frac{6.63\times10^{-34}\text{ Js}}{1.67\times10^{-27}\text{ kg}}(10^{6}\text{ m/s})=3.97\times10^{-13}\text{ m}$

5-3 $\lambda=\frac{h}{p}=\frac{h}{mv}=\frac{6.63\times10^{-34}\text{ Js}}{74\text{ kg}}(5\text{ m/s})=1.79\times10^{-36}\text{ m}$

5-5 (a) $\lambda=\frac{h}{p}$ or $p=\frac{h}{\lambda}=\frac{hc}{\lambda c}=\frac{1\,240\text{ eV nm}}{(10\text{ nm})(c)}=\frac{124\text{ eV}}{c}$. As

$$K=E-mc^2=\left[p^2c^2+\left(mc^2\right)^2\right]^{1/2}-mc^2,$$

we must use the relativistic expression for *K*, when $pc\approx mc^2$. Here $pc=124\text{ eV}<<mc^2=0.511\text{ MeV}$, so we can use the classical expression for *K*, $K=\frac{p^2}{2m}$.

$$K=\frac{p^2}{2m}=\frac{p^2c^2}{2mc^2}=\frac{(124\text{ eV})^2}{2(0.511\text{ MeV})}=0.150\text{ eV}$$

(b) Electrons with $\lambda=0.10\text{ nm}$ $p=\frac{hc}{\lambda c}=\frac{12\,400\text{ eV}}{c}$ as in (a). As $pc<<mc^2=0.511\text{ MeV}$, use $K=\frac{p^2}{2m}=p^2c^2=\frac{(12\,400)^2(\text{eV})^2}{(2)\left(0.511\times10^{6}\text{ eV}\right)}=150\text{ eV}$.

(c) Electrons with $\lambda=10\text{ fm}=10\times10^{-15}\text{ m}$, $p=\frac{hc}{\lambda c}=\frac{1.24\times10^{3}\text{ MeV}}{c}$. As $pc>>mc^2=0.511\text{ MeV}$, use

$$K=\left[p^2c^2+\left(mc^2\right)^2\right]^{1/2}-mc^2=pc-mc^2=1\,240\text{ MeV}-0.511\text{ MeV}=1\,239\text{ MeV}.$$

For alphas with $mc^2 = 3\,726$ MeV:

(a) p still is $\frac{124 \text{ eV}}{c}$. As $pc << 3\,726$ MeV, we use $K = \frac{p^2}{2m}$:

$$K = \frac{p^2c^2}{2mc^2} = \frac{(124 \text{ eV})^2}{(2)(3\,726 \text{ MeV})} = 2.06 \times 10^{-6} \text{ eV}.$$

(b) For alphas with $\lambda = 0.10$ nm, $p = \frac{12\,400 \text{ eV}}{c}$. As $pc << mc^2 = 3\,726$ MeV,

$$K = \frac{p^2}{2m} = \frac{p^2c^2}{2mc^2} = \frac{(12\,400 \text{ eV})^2}{(2)(3\,726 \text{ MeV})} = 0.020\,6 \text{ eV}.$$

(c) $p = \frac{1.24 \times 10^3 \text{ MeV}}{c}$ and $pc = 1\,240 \text{ MeV} \sim mc^2 = 3\,726$ MeV. We use

$$K = \left[p^2c^2 + \left(mc^2\right)^2\right]^{1/2} - mc^2 = \left[(1\,240 \text{ MeV})^2 + (3\,726 \text{ MeV})^2\right]^{1/2} - 3\,726 \text{ MeV}$$
$$= 201 \text{ MeV}.$$

5-7 A 10 MeV proton has $K = 10 \text{ MeV} << 2mc^2 = 1\,877$ MeV so we can use the classical expression $p = (2mK)^{1/2}$. (See Problem 5-2)

$$\lambda = \frac{h}{p} = \frac{hc}{[(2)(938.3 \text{ MeV})(10 \text{ MeV})]^{1/2}} = \frac{1\,240 \text{ MeV fm}}{\left[(2)(938.3)(10)(\text{MeV})^2\right]^{1/2}} = 9.05 \text{ fm} = 9.05 \times 10^{-15} \text{ m}$$

5-9 $m = 0.20$ kg: $mgh = \frac{mv^2}{2}$: $v = (2gh)^{1/2}$

$p = mv = m(2gh)^{1/2} = (0.20)[2(9.80)(50)]^{1/2} = 6.261 \text{ kg} \cdot \text{m/s}$

$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34} \text{ J} \cdot \text{s}}{6.261 \text{ kg} \cdot \text{m/s}} = 1.06 \times 10^{-34} \text{ m}$

5-11 (a) In this problem, the electron must be treated relativistically because we must use relativity when $pc \approx mc^2$. (See problem 5-5). the momentum of the electron is

$$p = \frac{h}{\lambda} = \frac{6.626 \times 10^{-34} \text{ J} \cdot \text{s}}{10^{-14} \text{ m}} = 6.626 \times 10^{-20} \text{ kg} \cdot \text{m/s}$$

and $pc = 124 \text{ MeV} >> mc^2 = 0.511$ MeV. The energy of the electron is

$$E = \left(p^2c^2 + m^2c^4\right)^{1/2}$$
$$= \left[\left(6.626 \times 10^{-20} \text{ kg} \cdot \text{m/s}\right)^2\left(3 \times 10^8 \text{ m/s}\right)^2 + \left(0.511 \times 10^6 \text{ eV}\right)^2\left(1.602 \times 10^{-19} \text{ J/eV}\right)^2\right]^{1/2}$$
$$= 1.99 \times 10^{-11} \text{ J} = 1.24 \times 10^8 \text{ eV}$$

so that $K = E - mc^2 \approx 124$ MeV.

(b) The kinetic energy is too large to expect that the electron could be confined to a region the size of the nucleus.

5-13 A canceling wave will be produced when the path length difference between the surface reflection and the reflection from the nth plane below the surface equals some whole number of wavelengths plus $\frac{\lambda}{2}$ As the path length difference between a surface reflection and a reflection from plane n is given by $(n)(1.01\lambda)$, we find that a reflection from the 50th plane has a path difference of 50.5λ with the surface reflection, and consequently cancels the surface reflection. Essentially all waves reflected at θ will cancel as the wave reflected from the second plane will be cancelled by a reflection from the 51st plane and so on.

5-15 For a free, non-relativistic electron $E=\frac{m_e v_0^2}{2}=\frac{p^2}{2m_e}$. As the wavenumber and angular frequency of the electron's de Broglie wave are given by $p=\hbar k$ and $E=\hbar\omega$, substituting these results gives the dispersion relation $\omega=\frac{\hbar k^2}{2m_e}$. So $v_g=\frac{d\omega}{dk}=\frac{\hbar k}{m_e}=\frac{p}{m_e}=v_0$.

5-17 $E^2=p^2c^2+(m_e c^2)^2$

$E=\left[p^2c^2+(m_e c^2)^2\right]^{1/2}$. As $E=\hbar\omega$ and $p=\hbar k$

$\hbar\omega=\left[\hbar^2k^2c^2+(m_e c^2)^2\right]^{1/2}$ or

$$\omega(k)=\left[k^2c^2+\frac{(m_e c^2)^2}{\hbar^2}\right]^{1/2}$$

$$v_p=\frac{\omega}{k}=\frac{\left[k^2c^2+(m_e c^2/\hbar)^2\right]^{1/2}}{k}=\left[c^2+\left(\frac{m_e c^2}{\hbar k}\right)^2\right]^{1/2}$$

$$v_g=\left.\frac{d\omega}{dk}\right|_{k_0}=\frac{1}{2}\left[k^2c^2+\left(\frac{m_e c^2}{\hbar}\right)^2\right]^{-1/2}2kc^2=\frac{kc^2}{\left[k^2c^2+(m_e c^2/\hbar)^2\right]^{1/2}}$$

$$v_p v_g=\left\{\frac{\left[k^2c^2+(m_e c^2/\hbar)^2\right]^{1/2}}{k}\right\}\left\{\left[k^2c^2+(m_e c^2/\hbar)^2\right]^{1/2}\right\}=c^2$$

Therefore, $v_g<c$ if $v_p>c$.

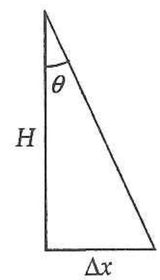
5-19 $K=\frac{mv^2}{2}=\frac{p^2}{2m}$: $(1\times10^6\ \text{eV})(1.6\times10^{-19}\ \text{J/eV})=\frac{p^2}{2(1.67\times10^{-27}\ \text{kg})}\Rightarrow p=2.312\times10^{-20}\ \text{kg}\cdot\text{m/s}$,

$\Delta p=0.05p=1.160\times10^{-21}\ \text{kg}\cdot\text{m/s}$ and $\Delta x\Delta p=\frac{h}{4\pi}$. Thus

$$\Delta x=\frac{6.63\times10^{-34}\ \text{J}\cdot\text{s}}{(1.16\times10^{-21}\ \text{kg}\cdot\text{m/s})(4\pi)}=4.56\times10^{-14}\ \text{m}.$$

Note that non-relativistic treatment has been used, which is justified because the kinetic energy is only $\frac{(1.6\times10^{-13})\times100\%}{1.50\times10^{-10}}=0.11\%$ of the rest energy.

5-21 (a) The woman tries to hold a pellet within some horizontal region Δx_i and directly above the spot on the floor. The uncertainty principle requires her to give a pellet some x velocity at least as large as $\Delta v_x=\frac{\hbar}{2m\Delta x_i}$. When the pellet hits the floor at time t, the total miss distance is $\Delta x_{total}=\Delta x_i+\Delta v_x t=\Delta x_i+\left(\frac{\hbar}{2m\Delta x_i}\right)\sqrt{\frac{2H}{g}}$. The minimum value of the function Δx_{total} occurs for $\frac{d(\Delta x_{total})}{d(\Delta x_i)}=0$ or $1-\frac{\hbar}{2m}\sqrt{\frac{2H}{g}}(\Delta x_i)^{-2}=0$.



We find $\Delta x_i=\left(\frac{\hbar}{2m}\right)^{1/2}\left(\frac{2H}{g}\right)^{1/4}$.

(b) For $H=2.0\text{ m}$, $m=0.50\text{ g}$, $\Delta x_{total}=5.2\times10^{-16}\text{ m}$.

5-23 (a) $\Delta p\Delta x=m\Delta v\Delta x\geq\frac{\hbar}{2}$

$$\Delta v\geq\frac{h}{4\pi m\Delta x}=\frac{2\pi\text{ J}\cdot\text{s}}{4\pi(2\text{ kg})(1\text{ m})}=0.25\text{ m/s}$$

(b) The duck might move by $(0.25\text{ m/s})(5\text{ s})=1.25\text{ m}$. With original position uncertainty of 1m, we can think of Δx growing to $1\text{ m}+1.25\text{ m}=2.25\text{ m}$.

5-25 To find the energy width of the γ-ray use $\Delta E\Delta t\geq\frac{\hbar}{2}$ or

$$\Delta E\geq\frac{\hbar}{2\Delta t}\geq\frac{6.58\times10^{-16}\text{ eV}\cdot\text{s}}{(2)(0.10\times10^{-9}\text{ s})}\geq3.29\times10^{-6}\text{ eV}.$$

As the intrinsic energy width of $\sim\pm3\times10^{-6}\text{ eV}$ is so much less than the experimental resolution of $\pm5\text{ eV}$, the intrinsic width can't be measured using this method.

5-27 For a single slit with width a, minima are given by $\sin\theta = \frac{n\lambda}{a}$ where $n = 1, 2, 3, \ldots$ and $\sin\theta \approx \tan\theta = \frac{x}{L}, \frac{x_1}{L} = \frac{\lambda}{a}$ and $\frac{x_2}{L} = \frac{2\lambda}{a} \Rightarrow \frac{x_2 - x_1}{L} = \frac{\lambda}{a}$ or

$$\lambda = \frac{a\Delta x}{L} = \frac{5\ \text{Å} \times 2.1\ \text{cm}}{20\ \text{cm}} = 0.525\ \text{Å}$$

$$E = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} = \frac{(hc)^2}{2mc^2\lambda^2} = \frac{\left(1.24 \times 10^4\ \text{eV}\cdot\text{Å}\right)^2}{2\left(5.11 \times 10^5\ \text{eV}\right)\left(0.525\ \text{Å}\right)^2} = 546\ \text{eV}$$

5-29 With *one* slit open $P_1 = |\Psi_1|^2$ or $P_2 = |\Psi_2|^2$. With both slits open, $P = |\Psi_1 + \Psi_2|^2$. At a maximum, the wavefunctions are in phase so

$$P_{\max} = (|\Psi_1| + |\Psi_2|)^2.$$

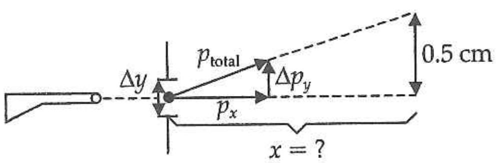
At a minimum, the wavefunctions are out of phase and

$$P_{\min} = (|\Psi_1| - |\Psi_2|)^2.$$

Now $\frac{P_1}{P_2} = \frac{|\Psi_1|^2}{|\Psi_2|^2} = 25$ or $\frac{|\Psi_1|}{|\Psi_2|} = 5$, and

$$\frac{P_{\max}}{P_{\min}} = \frac{(|\Psi_1| + |\Psi_2|)^2}{(|\Psi_1| - |\Psi_2|)^2} = \frac{(5|\Psi_2| + |\Psi_2|)^2}{(5|\Psi_1| - |\Psi_2|)^2} = \frac{6^2}{4^2} = \frac{36}{16} = 2.25.$$

5-31



$\Delta y \Delta p_y \sim \hbar \quad \Delta p_y = \frac{\hbar}{\Delta y}$. From the diagram, because the momentum triangle and space triangle are similar, $\frac{\Delta p_y}{p_x} = \frac{0.5\ \text{cm}}{x}$;

$$x = \frac{(0.5\ \text{cm})p_x}{\Delta p_y} = \frac{(0.5\ \text{cm})p_x \Delta y}{\hbar} = \frac{\left(0.5 \times 10^{-2}\ \text{m}\right)(0.001\ \text{kg})(100\ \text{m/s})\left(2 \times 10^{-3}\ \text{m}\right)}{1.05 \times 10^{-34}\ \text{J}\cdot\text{s}}$$
$$= 9.5 \times 10^{27}\ \text{m}$$

Once again we see that the uncertainty relation has no observable consequences for macroscopic systems.

5-33 From the uncertainty principle, $\Delta E \Delta t \sim \hbar$ $\quad \Delta mc^2 \Delta t = \hbar$. Therefore,

$$\frac{\Delta m}{m} = \frac{h}{2\pi c^2 \Delta t m} = \frac{h}{2\pi \Delta t E_{\text{rest}}} = \frac{6.63 \times 10^{-34}\ \text{J} \cdot \text{s}}{2\pi (8.7 \times 10^{-17}\ \text{s})(135 \times 10^6\ \text{eV})(1.6 \times 10^{-19}\ \text{J/eV})} = 5.62 \times 10^{-8}.$$

5-35 (a) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} a(k) e^{ikx} dk = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\alpha^2 (k-k_0)^2} e^{ikx} dk = \frac{A}{\sqrt{2\pi}} e^{-\alpha^2 k_0^2} \int_{-\infty}^{+\infty} e^{-\alpha^2 \left(k^2 - \left(2k_0 + ix/\alpha^2\right)k\right)} dk.$

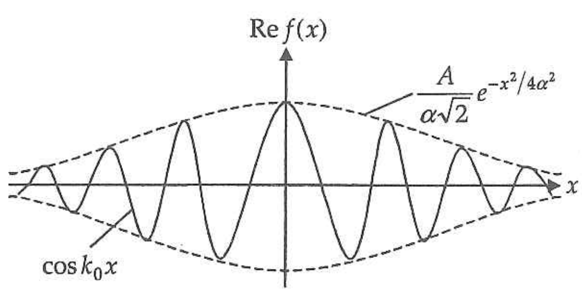
Now complete the square in order to get the integral into the standard form $\int_{-\infty}^{+\infty} e^{-az^2} dz$:

$$e^{-\alpha^2 \left(k^2 - \left(2k_0 + ix/\alpha^2\right)k\right)} = e^{+\alpha^2 \left(k_0 + ix/2\alpha^2\right)^2} e^{-\alpha^2 \left(k - \left(k_0 + ix/2\alpha^2\right)\right)^2}$$

$$f(x) = \frac{A}{\sqrt{2\pi}} e^{-\alpha^2 k_0^2} e^{\alpha^2 \left(k_0 + ix/2\alpha^2\right)^2} \int_{k=-\infty}^{+\infty} e^{-\alpha^2 \left(k - \left(k_0 + ix/2\alpha^2\right)\right)^2} dk$$

$$= \frac{A}{\sqrt{2\pi}} e^{-x^2/4\alpha^2} e^{ik_0 x} \int_{z=-\infty}^{+\infty} e^{-\alpha^2 z^2} dz$$

where $z = k - \left(k_0 + \frac{ix}{2\alpha^2}\right)$. Since $\int_{z=-\infty}^{+\infty} e^{-\alpha^2 z^2} dz = \frac{\pi^{1/2}}{\alpha}$, $f(x) = \frac{A}{\alpha\sqrt{2}} e^{-x^2/4\alpha^2} e^{ik_0 x}$. The real part of $f(x)$, Re $f(x)$ is Re $f(x) = \frac{A}{\alpha\sqrt{2}} e^{-x^2 4\alpha^2} \cos k_0 x$ and is a gaussian envelope multiplying a harmonic wave with wave number k_0. A plot of Re $f(x)$ is shown below:



Comparing $\frac{A}{\alpha\sqrt{2}} e^{-x^2 4\alpha^2}$ to $Ae^{-(x/2\Delta x)^2}$ implies $\Delta x = \alpha$.

(c) By same reasoning because $\alpha^2 = \frac{1}{4\Delta k^2}$, $\Delta k = \frac{1}{2\alpha}$. Finally $\Delta x \Delta k = \alpha\left(\frac{1}{2\alpha}\right) = \frac{1}{2}$.

5-37 We find the speed of each electron from energy conservation in the firing process:

$$0 = K_f + U_f = \frac{1}{2} mv^2 - eV$$

$$v = \sqrt{\frac{2eV}{m}} = \sqrt{\frac{2(1.6 \times 10^{-19}\ \text{C})(45\ \text{V})}{9.11 \times 10^{-31}\ \text{kg}}} = 3.98 \times 10^6\ \text{m/s}$$

The time of flight is $\Delta t = \frac{\Delta x}{v} = \frac{0.28\ \text{m}}{3.98 \times 10^{6}\ \text{m/s}} = 7.04 \times 10^{-8}\ \text{s}$. The current when electrons are 28 cm apart is $I = \frac{q}{t} = \frac{e}{\Delta t} = \frac{1.6 \times 10^{-19}\ \text{C}}{7.04 \times 10^{-8}\ \text{s}} = 2.27 \times 10^{-12}\ \text{A}$.

6
Quantum Mechanics in One Dimension

6-1 (a) Not acceptable – diverges as $x \to \infty$.

(b) Acceptable.

(c) Acceptable.

(d) Not acceptable – not a single-valued function.

(e) Not acceptable – the wave is discontinuous (as is the slope).

6-3 (a) $A\sin\left(\frac{2\pi x}{\lambda}\right) = A\sin\left(5\times10^{10}x\right)$ so $\left(\frac{2\pi}{\lambda}\right) = 5\times10^{10}\ \text{m}^{-1}$, $\lambda = \frac{2\pi}{5\times10^{10}} = 1.26\times10^{-10}\ \text{m}$.

(b) $p = \frac{h}{\lambda} = \frac{6.626\times10^{-34}\ \text{Js}}{1.26\times10^{-10}\ \text{m}} = 5.26\times10^{-24}\ \text{kg m/s}$

(c) $K = \frac{p^2}{2m}\quad m = 9.11\times10^{-31}\ \text{kg}$

$$K = \frac{\left(5.26\times10^{-24}\ \text{kg m/s}\right)^2}{\left(2\times9.11\times10^{-31}\ \text{kg}\right)} = 1.52\times10^{-17}\ \text{J}$$

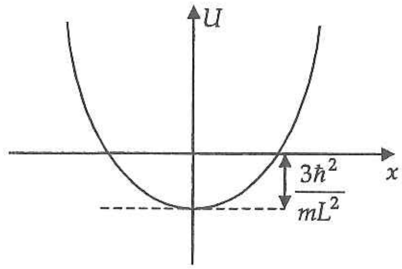
$$K = \frac{1.52\times10^{-17}\ \text{J}}{1.6\times10^{-19}\ \text{J/eV}} = 95\ \text{eV}$$

6-5 (a) Solving the Schrödinger equation for U with $E = 0$ gives

$$U = \left(\frac{\hbar^2}{2m}\right)\frac{\left(\frac{d^2\psi}{dx^2}\right)}{\psi}.$$

If $\psi = Ae^{-x^2/L^2}$ then $\frac{d^2\psi}{dx^2} = \left(4Ax^3 - 6AxL^2\right)\left(\frac{1}{L^4}\right)e^{-x^2/L^2}$, $U = \left(\frac{\hbar^2}{2mL^2}\right)\left(\frac{4x^2}{L^2} - 6\right)$.

(b) $U(x)$ is a parabola centered at $x = 0$ with $U(0) = \dfrac{-3\hbar^2}{mL^2} < 0$:



6-7 Since the particle is confined to the box, Δx can be no larger than L, the box length. With $n = 0$, the particle energy $E_n = \dfrac{n^2h^2}{8mL^2}$ is also zero. Since the energy is all kinetic, this implies $\langle p_x^2 \rangle = 0$. But $\langle p_x \rangle = 0$ is expected for a particle that spends equal time moving left as right, giving $\Delta p_x = \sqrt{\langle p_x^2 \rangle - \langle p_x \rangle^2} = 0$. Thus, for this case $\Delta p_x \Delta x = 0$, in violation of the uncertainty principle.

6-9 $E_n = \dfrac{n^2h^2}{8mL^2}$, so $\Delta E = E_2 - E_1 = \dfrac{3h^2}{8mL^2}$

$$\Delta E = (3)\frac{(1240 \text{ eV nm}/c)^2}{8(938.28 \times 10^6 \text{ eV}/c^2)(10^{-5} \text{ nm})^2} = 6.14 \text{ MeV}$$

$$\lambda = \frac{hc}{\Delta E} = \frac{1240 \text{ eV nm}}{6.14 \times 10^6 \text{ eV}} = 2.02 \times 10^{-4} \text{ nm}$$

This is the gamma ray region of the electromagnetic spectrum.

6-11 In the present case, the box is displaced from (0, L) by $\dfrac{L}{2}$. Accordingly, we may obtain the wavefunctions by replacing x with $x - \dfrac{L}{2}$ in the wavefunctions of Equation 6.18. Using

$$\sin\left[\left(\frac{n\pi}{L}\right)\left(x - \frac{L}{2}\right)\right] = \sin\left[\left(\frac{n\pi x}{L}\right) - \frac{n\pi}{2}\right] = \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi}{2}\right)$$

we get for $-\dfrac{L}{2} \le x \le \dfrac{L}{2}$

$$\psi_1(x) = \left(\frac{2}{L}\right)^{1/2} \cos\left(\frac{\pi x}{L}\right);\ P_1(x) = \left(\frac{2}{L}\right)\cos^2\left(\frac{\pi x}{L}\right)$$
$$\psi_2(x) = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{2\pi x}{L}\right);\ P_2(x) = \left(\frac{2}{L}\right)\sin^2\left(\frac{2\pi x}{L}\right)$$
$$\psi_3(x) = \left(\frac{2}{L}\right)^{1/2} \cos\left(\frac{3\pi x}{L}\right);\ P_3(x) = \left(\frac{2}{L}\right)\cos^2\left(\frac{3\pi x}{L}\right)$$

6-13 (a) Proton in a box of width $L = 0.200\text{ nm} = 2\times10^{-10}\text{ m}$

$$E_1 = \frac{h^2}{8m_pL^2} = \frac{(6.626\times10^{-34}\text{ J}\cdot\text{s})^2}{8(1.67\times10^{-27}\text{ kg})(2\times10^{-10}\text{ m})^2} = 8.22\times10^{-22}\text{ J}$$

$$= \frac{8.22\times10^{-22}\text{ J}}{1.60\times10^{-19}\text{ J/eV}} = 5.13\times10^{-3}\text{ eV}$$

(b) Electron in the same box:

$$E_1 = \frac{h^2}{8m_eL^2} = \frac{(6.626\times10^{-34}\text{ J}\cdot\text{s})^2}{8(9.11\times10^{-31}\text{ kg})(2\times10^{-10}\text{ m})^2} = 1.506\times10^{-18}\text{ J} = 9.40\text{ eV}.$$

(c) The electron has a much higher energy because it is much less massive.

6-15 (a) $$U = \left(\frac{e^2}{4\pi\varepsilon_0 d}\right)\left[-1+\frac{1}{2}-\frac{1}{3}+\left(-1+\frac{1}{2}\right)+(-1)\right] = \frac{(-7/3)e^2}{4\pi\varepsilon_0 d} = \frac{(-7/3)ke^2}{d}$$

(b) $$K = 2E_1 = \frac{2h^2}{8m\times9d^2} = \frac{h^2}{36md^2}$$

(c) $E = U + K$ and $\frac{dE}{dd} = 0$ for a minimum $\left[\frac{(+7/3)e^2k}{d^2}\right] - \frac{h^2}{18md^3} = 0$

$$d = \frac{3h^2}{(7)(18ke^2m)} \text{ or } d = \frac{h^2}{42mke^2}$$

$$d = \frac{(6.63\times10^{-34}\text{ J}\cdot\text{s})^2}{(42)(9.11\times10^{-31}\text{ kg})(9\times10^9\text{ N}\cdot\text{m}^2\cdot\text{C}^{-2})(1.6\times10^{-19}\text{ C})^2} = 0.5\times10^{-10}\text{ m} = 0.050\text{ nm}$$

(d) Since the lithium spacing is a, where $Na^3 = V$ and the density is $\frac{Nm}{V}$ where m is the mass of one atom, we get

$$a = \left(\frac{Vm}{Nm}\right)^{1/3} = \left(\frac{m}{\text{density}}\right)^{1/3} = \left(1.66\times10^{-27}\text{ kg}\times\frac{7}{530\text{ kg/m}^3}\right)^{1/3}\text{ m} = 2.8\times10^{-10}\text{ m}$$

$$= 0.28\text{ nm}$$

(2.8 times larger than $2d$)

6-17 (a) The wavefunctions and probability densities are the same as those shown in the two lower curves in Figure 6.16 of the text.

(b) $$P_1 = \int_{1.5\text{ Å}}^{3.5\text{ Å}} |\psi|^2 dx = \frac{2}{10\text{ Å}}\int_{1.5\text{ Å}}^{3.5\text{ Å}} \sin^2\left(\frac{\pi x}{10}\right)dx$$

$$\frac{1}{5}\left[\frac{x}{2} - \frac{10}{4\pi}\sin\left(\frac{\pi x}{5}\right)\right]_{1.5}^{3.5}$$

In the above result we used $\int \sin^2 ax dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax)$. Therefore,

$$P_1 = \frac{1}{10}\left[x - \frac{5}{\pi}\sin\left(\frac{\pi x}{5}\right)\right]_{1.5}^{3.5} = \frac{1}{10}\left\{3.5 - \frac{5}{\pi}\sin\left[\frac{\pi(3.5)}{5}\right] - 1.5 + \frac{5}{\pi}\sin\left[\frac{\pi(1.5)}{5}\right]\right\}$$
$$= \frac{1}{10}\left[2.0 + \frac{5}{\pi}(\sin 0.3\pi - \sin 0.7\pi)\right] = \frac{1}{10}[2.00 + 0.0] = 0.200$$

(c) $$P_2 = \frac{1}{5}\int_{1.5}^{3.5}\sin^2\left(\frac{\pi x}{5}\right)dx = \frac{1}{5}\left[\frac{x}{2} - \frac{5}{4\pi}\sin(0.4\pi x)\right]_{1.5}^{3.5} = \frac{1}{10}\left[x - \frac{5}{2\pi}\sin(0.4\pi x)\right]_{1.5}^{3.5}$$
$$= \frac{1}{10}\{2.0 + (0.798)\{\sin[0.4\pi(1.5)] - \sin[0.4\pi(3.5)]\}\} = 0.351$$

(d) Using $E = \frac{n^2h^2}{8mL^2}$ we find $E_1 = 0.377$ eV and $E_2 = 1.51$ eV.

6-19 The allowed energies for this system are given by Equation 6.17, or $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} = \frac{n^2h^2}{8mL^2}$. Using $E_n = 10^{-3}$ J, $m = 10^{-3}$ kg, $L = 10^{-2}$ m and solving for n gives

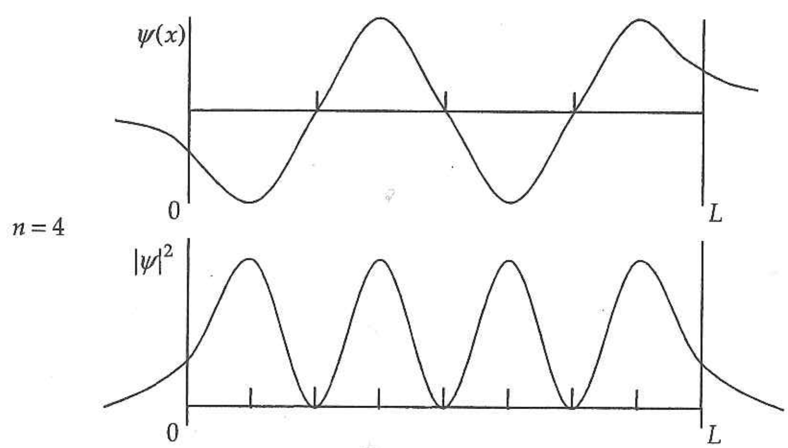
$$n = \frac{\left\{8\left(10^{-3}\text{ kg}\right)\left(10^{-2}\text{ m}\right)^2\left(10^{-3}\text{ J}\right)\right\}^{1/2}}{6.63\times10^{-34}\text{ J}\cdot\text{s}} = 4.27\times10^{28}.$$

The excitation energy is $\Delta E = E_{n+1} - E_n$, or

$$\Delta E = \frac{h^2}{8mL^2}\left\{(n+1)^2 - n^2\right\} = \left(\frac{h^2}{8mL^2}\right)\{2n+1\} = E_n\left(\frac{2n+1}{n^2}\right) \approx \frac{2}{n}E_n \text{ for } n >> 1.$$

Thus, $\Delta E \approx \frac{(2)\left(10^{-3}\text{ J}\right)}{4.27\times10^{28}} = 4.69\times10^{-32}$ J.

6-21 $n = 4$



Note that the $n = 4$ wavefunction has three nodes and is antisymmetric about the midpoint of the well.

6-23 Inside the well, the particle is free and the Schrödinger waveform is trigonometric with wavenumber $k=\left(\frac{2mE}{\hbar^2}\right)^{1/2}$:

$$\psi(x)=A\sin kx+B\cos kx \quad 0\le x\le L.$$

The infinite wall at $x=0$ requires $\psi(0)=B=0$. Beyond $x=L$, $U(x)=U$ and the Schrödinger equation $\frac{d^2\psi}{dx^2}=\left(\frac{2m}{\hbar^2}\right)\{U-E\}\psi(x)$, which has exponential solutions for $E<U$

$$\psi(x)=Ce^{-\alpha x}+De^{+\alpha x}, \qquad x>L$$

where $\alpha=\left[\frac{2m(U-E)}{\hbar^2}\right]^{1/2}$. To keep ψ bounded at $x=\infty$ we must take $D=0$. At $x=L$, continuity of ψ and $\frac{d\psi}{dx}$ demands

$$A\sin kL=Ce^{-\alpha L}$$
$$kA\cos kL=-\alpha Ce^{-\alpha L}$$

Dividing one by the other gives an equation for the allowed particle energies: $k\cot kL=-\alpha$. The dependence on *E* (or *k*) is made more explicit by noting that $k^2+\alpha^2=\frac{2mU}{\hbar^2}$, which allows the energy condition to be written $k\cot kL=-\left[\left(\frac{2mU}{\hbar^2}\right)-k^2\right]^{1/2}$. Multiplying by *L*, squaring the result, and using $\cot^2\theta+1=\csc^2\theta$ gives $(kL)^2\csc^2(kL)=\frac{2mUL^2}{\hbar^2}$ from which we obtain $\frac{kL}{\sin kL}=\left(\frac{2mUL^2}{\hbar^2}\right)^{1/2}$. Since $\frac{\theta}{\sin\theta}$ is never smaller than unity for *any* value of θ, there can be no bound state energies if $\frac{2mUL^2}{\hbar^2}<1$.

6-25 At its limits of vibration $x=\pm A$ the classical oscillator has all its energy in potential form: $E=\frac{1}{2}m\omega^2A^2$ or $A=\left(\frac{2E}{m\omega^2}\right)^{1/2}$. If the energy is quantized as $E_n=\left(n+\frac{1}{2}\right)\hbar\omega$, then the corresponding amplitudes are $A_n=\left[\frac{(2n+1)\hbar}{m\omega}\right]^{1/2}$.

6-29 (a) Normalization requires $1=\int_{-\infty}^{\infty}|\psi|^2dx=C^2\int_0^{\infty}e^{-2x}\left(1-e^{-x}\right)^2dx=C^2\int_0^{\infty}\left(e^{-2x}-2e^{-3x}+e^{-4x}\right)dx$. The integrals are elementary and give $1=C^2\left\{\frac{1}{2}-2\left(\frac{1}{3}\right)+\frac{1}{4}\right\}=\frac{C^2}{12}$. The proper units for *C* are those of $(\text{length})^{-1/2}$ thus, normalization requires $C=(12)^{1/2}\ \text{nm}^{-1/2}$.

(b) The most likely place for the electron is where the probability $|\psi|^2$ is largest. This is also where ψ itself is largest, and is found by setting the derivative $\frac{d\psi}{dx}$ equal zero:

$$0=\frac{d\psi}{dx}=C\{-e^{-x}+2e^{-2x}\}=Ce^{-x}\{2e^{-x}-1\}.$$

The RHS vanishes when $x=\infty$ (a minimum), and when $2e^{-x}=1$, or $x=\ln 2$ nm. Thus, the most likely position is at $x_p=\ln 2\text{ nm}=0.693\text{ nm}$.

(c) The average position is calculated from

$$\langle x\rangle=\int_{-\infty}^{\infty}x|\psi|^2dx=C^2\int_0^{\infty}xe^{-2x}(1-e^{-x})^2dx=C^2\int_0^{\infty}x(e^{-2x}-2e^{-3x}+e^{-4x})dx.$$

The integrals are readily evaluated with the help of the formula $\int_0^{\infty}xe^{-ax}dx=\frac{1}{a^2}$ to get $\langle x\rangle=C^2\left\{\frac{1}{4}-2\left(\frac{1}{9}\right)+\frac{1}{16}\right\}=C^2\left\{\frac{13}{144}\right\}$. Substituting $C^2=12\text{ nm}^{-1}$ gives

$$\langle x\rangle=\frac{13}{12}\text{ nm}=1.083\text{ nm}.$$

We see that $\langle x\rangle$ is somewhat greater than the most probable position, since the probability density is skewed in such a way that values of x larger than x_p are weighted more heavily in the calculation of the average.

6-31 The symmetry of $|\psi(x)|^2$ about $x=0$ can be exploited effectively in the calculation of average values. To find $\langle x\rangle$

$$\langle x\rangle=\int_{-\infty}^{\infty}x|\psi(x)|^2dx$$

We notice that the integrand is antisymmetric about $x=0$ due to the extra factor of x (an odd function). Thus, the contribution from the two half-axes $x>0$ and $x<0$ cancel exactly, leaving $\langle x\rangle=0$. For the calculation of $\langle x^2\rangle$, however, the integrand is symmetric and the half-axes contribute equally to the value of the integral, giving

$$\langle x\rangle=\int_0^{\infty}x^2|\psi|^2dx=2C^2\int_0^{\infty}x^2e^{-2x/x_0}dx.$$

Two integrations by parts show the value of the integral to be $2\left(\frac{x_0}{2}\right)^3$. Upon substituting for C^2, we get $\langle x^2\rangle = 2\left(\frac{1}{x_0}\right)(2)\left(\frac{x_0}{2}\right)^3 = \frac{x_0^2}{2}$ and $\Delta x = \left(\langle x^2\rangle - \langle x\rangle^2\right)^{1/2} = \left(\frac{x_0^2}{2}\right)^{1/2} = \frac{x_0}{\sqrt{2}}$. In calculating the probability for the interval $-\Delta x$ to $+\Delta x$ we appeal to symmetry once again to write

$$P = \int_{-\Delta x}^{+\Delta x} |\psi|^2 dx = 2C^2 \int_0^{\Delta x} e^{-2x/x_0} dx = -2C^2\left(\frac{x_0}{2}\right)e^{-2x/x_0}\bigg|_0^{\Delta x} = 1 - e^{-\sqrt{2}} = 0.757$$

or about 75.7% independent of x_0.

6-33 (a) Since there is no preference for motion in the leftward sense vs. the rightward sense, a particle would spend equal time moving left as moving right, suggesting $\langle p_x\rangle = 0$.

(b) To find $\langle p_x^2\rangle$ we express the average energy as the sum of its kinetic and potential energy contributions: $\langle E\rangle = \left\langle \frac{p_x^2}{2m}\right\rangle + \langle U\rangle = \frac{\langle p_x^2\rangle}{2m} + \langle U\rangle$. But energy is sharp in the oscillator ground state, so that $\langle E\rangle = E_0 = \frac{1}{2}\hbar\omega$. Furthermore, remembering that $U(x) = \frac{1}{2}m\omega^2 x^2$ for the quantum oscillator, and using $\langle x^2\rangle = \frac{\hbar}{2m\omega}$ from Problem 6-32, gives $\langle U\rangle = \frac{1}{2}m\omega^2\langle x^2\rangle = \frac{1}{4}\hbar\omega$. Then $\langle p_x^2\rangle = 2m(E_0 - \langle U\rangle) = 2m\left(\frac{\hbar\omega}{4}\right) = \frac{m\hbar\omega}{2}$.

(c) $\Delta p_x = \left(\langle p_x^2\rangle - \langle p_x\rangle^2\right)^{1/2} = \left(\frac{m\hbar\omega}{2}\right)^{1/2}$

6-35 Applying the momentum operator $[p_x] = \left(\frac{\hbar}{i}\right)\frac{d}{dx}$ to each of the candidate functions yields

(a) $[p_x]\{A\sin(kx)\} = \left(\frac{\hbar}{i}\right)k\{A\cos(kx)\}$

(b) $[p_x]\{A\sin(kx) - A\cos(kx)\} = \left(\frac{\hbar}{i}\right)k\{A\cos(kx) + A\sin(kx)\}$

(c) $[p_x]\{A\cos(kx) + iA\sin(kx)\} = \left(\frac{\hbar}{i}\right)k\{-A\sin(kx) + iA\cos(kx)\}$

(d) $[p_x]\{e^{ik(x-a)}\} = \left(\frac{\hbar}{i}\right)ik\{e^{ik(x-a)}\}$

In case (c), the result is a multiple of the original function, since

$$-A\sin(kx) + iA\cos(kx) = i\{A\cos(kx) + iA\sin(kx)\}.$$

The multiple is $\left(\frac{\hbar}{i}\right)(ik) = \hbar k$ and is the eigenvalue. Likewise for (d), the operation $[p_x]$ returns the original function with the multiplier $\hbar k$. Thus, (c) and (d) are eigenfunctions of $[p_x]$ with eigenvalue $\hbar k$, whereas (a) and (b) are not eigenfunctions of this operator.

6-37 (a) Normalization requires

$$1 = \int_{-\infty}^{\infty} |\Psi|^2 dx = C^2 \int_{-\infty}^{\infty} \{\psi_1^* + \psi_2^*\}\{\psi_1 + \psi_2\} dx$$
$$= C^2 \left\{ \int |\psi_1|^2 dx + \int |\psi_2|^2 dx + \int \psi_2^* \psi_1 dx + \int \psi_1^* \psi_2 dx \right\}$$

The first two integrals on the right are unity, while the last two are, in fact, the same integral since ψ_1 and ψ_2 are both real. Using the waveforms for the infinite square well, we find

$$\int \psi_2 \psi_1 dx = \frac{2}{L} \int_0^L \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{2\pi x}{L}\right) dx = \frac{1}{L} \int_0^L \left\{ \cos\left(\frac{\pi x}{L}\right) - \cos\left(\frac{3\pi x}{L}\right) \right\} dx$$

where, in writing the last line, we have used the trigonometric exponential identities of sine and cosine. Both of the integrals remaining are readily evaluated, and are zero. Thus, $1 = C^2\{1+0+0+0\} = 2C^2$, or $C = \frac{1}{\sqrt{2}}$. Since $\psi_{1,2}$ are stationary states, they develop in time according to their respective energies $E_{1,2}$ as $e^{-iEt/\hbar}$. Then $\Psi(x, t) = C\left\{\psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_2 t/\hbar}\right\}$.

(c) $\Psi(x, t)$ is a stationary state only if it is an eigenfunction of the energy operator $[E] = i\hbar \frac{\partial}{\partial t}$. Applying $[E]$ to Ψ gives

$$[E]\Psi = C\left\{ i\hbar\left(\frac{-iE_1}{\hbar}\right)\psi_1 e^{-iE_1 t/\hbar} + i\hbar\left(\frac{-iE_2}{\hbar}\right)\psi_2 e^{-iE_2 t/\hbar} \right\} = C\left\{E_1 \psi_1 e^{-iE_1 t/\hbar} + E_2 \psi_2 e^{-iE_2 t/\hbar}\right\}.$$

Since $E_1 \neq E_2$, the operations $[E]$ does *not* return a multiple of the wavefunction, and so Ψ is not a stationary state. Nonetheless, we may calculate the average energy for this state as

$$\langle E \rangle = \int \Psi^* [E] \Psi dx = C^2 \int \left\{\psi_1^* e^{+iE_1 t/\hbar} + \psi_2^* e^{+iE_2 t/\hbar}\right\}\left\{E_1 \psi_1 e^{-iE_1 t/\hbar} + E_2 \psi_2 e^{-iE_2 t/\hbar}\right\} dx$$
$$= C^2 \left\{ E_1 \int |\psi_1|^2 dx + E_2 \int |\psi_2|^2 dx \right\}$$

with the cross terms vanishing as in part (a). Since $\psi_{1,2}$ are normalized and $C^2 = \frac{1}{2}$ we get finally $\langle E \rangle = \frac{E_1 + E_2}{2}$.

7

Tunneling Phenomena

7-1 (a) The reflection coefficient is the ratio of the reflected intensity to the incident wave intensity, or $R=\frac{|(1/2)(1-i)|^2}{|(1/2)(1+i)|^2}$. But $|1-i|^2=(1-i)(1-i)^*=(1-i)(1+i)=|1+i|^2=2$, so that $R=1$ in this case.

(b) To the left of the step the particle is free. The solutions to Schrödinger's equation are $e^{\pm ikx}$ with wavenumber $k=\left(\frac{2mE}{\hbar^2}\right)^{1/2}$. To the right of the step $U(x)=U$ and the equation is $\frac{d^2\psi}{dx^2}=\frac{2m}{\hbar^2}(U-E)\psi(x)$. With $\psi(x)=e^{-kx}$, we find $\frac{d^2\psi}{dx^2}=k^2\psi(x)$, so that $k=\left[\frac{2m(U-E)}{\hbar^2}\right]^{1/2}$. Substituting $k=\left(\frac{2mE}{\hbar^2}\right)^{1/2}$ shows that $\left[\frac{E}{(U-E)}\right]^{1/2}=1$ or $\frac{E}{U}=\frac{1}{2}$.

(c) For 10 MeV protons, $E=10$ MeV and $m=\frac{938.28\text{ MeV}}{c^2}$. Using $\hbar=197.3\text{ MeV fm}/c\left(1\text{ fm}=10^{-15}\text{ m}\right)$, we find
$$\delta=\frac{1}{k}=\frac{\hbar}{(2mE)^{1/2}}=\frac{197.3\text{ MeV fm}/c}{\left[(2)\left(938.28\text{ MeV}/c^2\right)(10\text{ MeV})\right]^{1/2}}=1.44\text{ fm}.$$

7-3 With $E=25$ MeV and $U=20$ MeV, the ratio of wavenumber is $\frac{k_1}{k_2}=\left(\frac{E}{E-U}\right)^{1/2}=\left(\frac{25}{25-20}\right)^{1/2}=\sqrt{5}=2.236$. Then from Problem 7-2 $R=\frac{\left(\sqrt{5}-1\right)^2}{\left(\sqrt{5}+1\right)^2}=0.146$ and $T=1-R=0.854$. Thus, 14.6% of the incoming particles would be reflected and 85.4% would be transmitted. For electrons with the same energy, the transparency and reflectivity of the step are unchanged.

7-5 (a) The transmission probability according to Equation 7.9 is $\frac{1}{T(E)}=1+\left[\frac{U^2}{4E(U-E)}\right]\sinh^2\alpha L$ with $\alpha=\frac{[2m(U-E)]^{1/2}}{\hbar}$. For $E<<U$, we find $(\alpha L)^2\approx\frac{2mUL^2}{\hbar^2}>>1$ by hypothesis. Thus, we may write $\sinh\alpha L\approx\frac{1}{2}e^{\alpha L}$. Also $U-E\approx U$, giving $\frac{1}{T(E)}\approx1+\left(\frac{U}{16E}\right)e^{2\alpha L}\approx\left(\frac{U}{16E}\right)e^{2\alpha L}$ and a probability for transmission $P=T(E)=\left(\frac{16E}{U}\right)e^{-2\alpha L}$.

(b) Numerical Estimates: $(\hbar = 1.055\times10^{-34}\ \text{Js})$

1) For $m = 9.11\times10^{-31}$ kg, $U - E = 1.60\times10^{-21}$ J, $L = 10^{-10}$ m;
$\alpha = \frac{[2m(U-E)]^{1/2}}{\hbar} = 5.12\times10^{8}\ \text{m}^{-1}$ and $e^{-2\alpha L} = 0.90$

2) For $m = 9.11\times10^{-31}$ kg, $U - E = 1.60\times10^{-19}$ J, $L = 10^{-10}$ m; $\alpha = 5.12\times10^{9}\ \text{m}^{-1}$ and $e^{-2\alpha L} = 0.36$

3) For $m = 6.7\times10^{-27}$ kg, $U - E = 1.60\times10^{-13}$ J, $L = 10^{-15}$ m; $\alpha = 4.4\times10^{14}\ \text{m}^{-1}$ and $e^{-2\alpha L} = 0.41$

4) For $m = 8$ kg, $U - E = 1$ J, $L = 0.02$ m; $\alpha = 3.8\times10^{34}\ \text{m}^{-1}$ and $e^{-2\alpha L} = e^{-1.5\times10^{33}} \approx 0$

7-7 The continuity requirements from Equation 7.8 are

$$A + B = C + D \qquad [\text{continuity of } \Psi \text{ at } x = 0]$$
$$ikA - ikB = \alpha D - \alpha C \qquad \left[\text{continuity of } \frac{\partial\Psi}{\partial x} \text{ at } x = 0\right]$$
$$Ce^{-\alpha L} + De^{+\alpha L} = Fe^{ikL} \qquad [\text{continuity of } \Psi \text{ at } x = L]$$
$$\alpha De^{+\alpha L} - \alpha Ce^{-\alpha L} = ikFe^{ikL} \qquad \left[\text{continuity of } \frac{\partial\Psi}{\partial x} \text{ at } x = L\right]$$

To isolate the transmission amplitude $\frac{F}{A}$, we must eliminate from these relations the unwanted coefficients *B*, *C*, and *D*. Dividing the second line by *ik* and adding to the first eliminates *B*, leaving *A* in terms of *C* and *D*. In the same way, dividing the fourth line by α and adding the result to the third line gives *D* (in terms of *F*), while subtracting the result from the third line gives C (in terms of *F*). Combining these results finally yields *A*:
$A = \frac{1}{4}Fe^{ikL}\left\{\left[2-\left(\frac{\alpha}{ik}+\frac{ik}{\alpha}\right)\right]e^{+\alpha L} + \left[2+\left(\frac{\alpha}{ik}+\frac{ik}{\alpha}\right)\right]e^{-\alpha L}\right\}$. The transmission probability is $T = \left|\frac{F}{A}\right|^2$.
Making use of the identities $e^{\pm\alpha L} = \cosh\alpha L \pm \sinh\alpha L$ and $\cosh^2\alpha L = 1 + \sinh^2\alpha L$, we obtain

$$\frac{1}{T} = \left|\frac{A}{F}\right|^2 = \frac{1}{4}\left|2\cosh\alpha L + i\left(\frac{\alpha}{k}-\frac{k}{\alpha}\right)\sinh\alpha L\right|^2 = \cosh^2\alpha L + \frac{1}{4}\left(\frac{\alpha}{k}-\frac{k}{\alpha}\right)^2 \sinh\alpha L$$
$$= 1 + \frac{1}{4}\left[\frac{U-E}{E} + \frac{E}{U-E} + 2\right]\sinh^2\alpha L = 1 + \frac{1}{4}\left[\frac{U^2}{E(U-E)}\right]\sinh^2\alpha L$$

7-11 (a) The matter wave reflected from the trailing edge of the well $(x = L)$ must travel the extra distance 2*L* before combining with the wave reflected from the leading edge $(x = 0)$. For $\lambda_2 = 2L$, these two waves interfere destructively since the latter suffers a phase shift of 180° upon reflection, as discussed in Example 7.3.

(b) The wave functions in all three regions are free particle plane waves. In regions 1 and 3 where $U(x) = U$ we have

$$\Psi(x,\ t) = Ae^{i(k'x-\omega t)} + Be^{i(-k'x-\omega t)} \qquad x < 0$$
$$\Psi(x,\ t) = Fe^{i(k'x-\omega t)} + Ge^{i(-k'x-\omega t)} \qquad x < 0$$

with $k' = \frac{[2m(E-U)]^{1/2}}{\hbar}$. In this case $G = 0$ since the particle is incident from the left. In region 2 where $U(x) = 0$ we have

$$\Psi(x, t) = Ce^{i(-kx-\omega t)} + De^{i(kx-\omega t)} \qquad 0 < x < L$$

with $k = \frac{(2mE)^{1/2}}{\hbar} = \frac{2\pi}{\lambda_2} = \frac{\pi}{L}$ for the case of interest. The wave function and its slope are continuous everywhere, and in particular at the well edges $x = 0$ and $x = L$. Thus, we must require

$$A + B = C + D \qquad [\text{continuity of } \Psi \text{ at } x = 0]$$
$$k'A - k'B = kD - kC \qquad \left[\text{continuity of } \frac{\partial\Psi}{\partial x} \text{ at } x = 0\right]$$
$$Ce^{-ikL} + De^{ikL} = Fe^{ik'L} \qquad [\text{continuity of } \Psi \text{ at } x = L]$$
$$kDe^{ikL} - kCe^{-ikL} = k'Fe^{ik'L} \qquad \left[\text{continuity of } \frac{\partial\Psi}{\partial x} \text{ at } x = L\right]$$

For $kL = \pi$, $e^{\pm ikL} = -1$ and the last two requirements can be combined to give $kD - kC = k'C + k'D$. Substituting this into the second requirement implies $A - B = C + D$, which is consistent with the first requirement only if $B = 0$, i.e., no reflected wave in region 1.

7-13 As in Problem 7-12, waveform continuity and the slope condition at the site of the delta well demand $A + B = F$ and $ik(A - B) - ikF = -\left(\frac{2mS}{\hbar^2}\right)F$. Dividing the second of these equations by ik and subtracting from the first gives $2B + F = F + \frac{(2mS/\hbar^2)F}{ik}$, or $B = -i\left(\frac{mS}{\hbar^2 k}\right)F = -iF\left(\frac{-E_0}{E}\right)^{1/2}$. Thus, the reflection coefficient R is $R(E) = \left|\frac{B}{A}\right|^2 = \left|\frac{B}{F}\right|^2 \left|\frac{F}{A}\right|^2 = \left(\frac{-E_0}{E}\right)\left[1 + \left(\frac{-E_0}{E}\right)\right]^{-1}$. Then, with $T(E)$ from Problem 7-12, $T(E) = \left[1 + \left(\frac{-E_0}{E}\right)\right]^{-1}$, we find $R(E) + T(E) = \left(1 - \frac{E_0}{E}\right)\left[1 + \left(\frac{-E_0}{E}\right)\right]^{-1} = 1$.

7-15 Divide the barrier region into N subintervals of length $\Delta x = x_{i+1} - x_i$. For the barrier in the i^{th} subinterval, denote by A_i and F_i the incident and transmitted wave amplitudes, respectively. The transmission coefficient for this interval is then $T_i = \left|\frac{F_i}{A_i}\right|^2$, and that for the entire barrier is $T(E) = \left|\frac{F_N}{A_1}\right|^2$. Now consider the product $\Pi T_i = T_1 T_2 T_3 \ldots T_N = \left(\frac{|F_1|^2}{|A_1|^2}\right)\left(\frac{|F_2|^2}{|A_2|^2}\right)\left(\frac{|F_3|^2}{|A_3|^2}\right)\ldots\left(\frac{|F_N|^2}{|A_N|^2}\right)$. Assuming the transmitted wave intensity for one barrier becomes the incident wave intensity for the next, we have $|F_1|^2 = |A_2|^2$, $|F_2|^2 = |A_3|^2$ etc., so that $T(E) = \left|\frac{F_N}{A_1}\right|^2 = T_1 T_2 T_3 \ldots T_N$. Next, we assume that Δx is sufficiently small and that $U(x)$ is sensibly constant over each interval (so that the square barrier result can be used for T_i), yet large enough to approximate $\sinh \alpha_i \Delta x$ with $\frac{1}{2}e^{\alpha_i \Delta x}$, where α_i, is the value taken by α in the i^{th} subinterval: $\alpha_i = \frac{[2m(U_i - E)]^{1/2}}{\hbar}$.

Then, $\frac{1}{T_i}=1+\left[\frac{U_i^2}{4E(U_i-E)}\right]\sinh^2(\alpha_i\Delta x)\approx\left[\frac{U_i^2}{16E(U_i-E)}\right]e^{2\alpha_i\Delta x}$ and the transmission coefficient for the entire barrier becomes $T(E)\approx\Pi\left\{\left[\frac{16E(U_i-E)}{U_i^2}\right]e^{-2\alpha_i\Delta x}\right\}\approx\left[\frac{\Pi 16E(U_i-E)}{U_i^2}\right]e^{-\Sigma 2\alpha_i\Delta x}$. To recover Equation 7.10, we approximate the sum in the exponential by an integral, and note that the product in square brackets is a term of order 1: $T(E)\sim e^{\Sigma 2\alpha_i\Delta x}\approx e^{-\int 2\alpha(x)dx}$ where now $\alpha(x)=\frac{2m[U(x)-E]^{1/2}}{\hbar}$.

7-17 The collision frequency f is the reciprocal of the transit time for the alpha particle crossing the nucleus, or $f=\frac{v}{2R}$, where v is the speed of the alpha. Now v is found from the kinetic energy which, inside the nucleus, is not the total energy E but the difference $E-U$ between the total energy and the potential energy representing the bottom of the nuclear well. At the nuclear radius $R=9$ fm, the Coulomb energy is

$$\frac{k(Ze)(2e)}{R}=2Z\left(\frac{ke^2}{a_0}\right)\left(\frac{a_0}{R}\right)=2(88)(27.2\text{ eV})\left(\frac{5.29\times10^4\text{ fm}}{9\text{ fm}}\right)=28.14\text{ MeV}.$$

From this we conclude that $U=-1.86$ MeV to give a nuclear barrier of 30 MeV overall. Thus an alpha with $E=4.05$ MeV has kinetic energy $4.05+1.86=5.91$ MeV inside the nucleus. Since the alpha particle has the combined mass of 2 protons and 2 neutrons, or about 3 755.8 MeV/c^2 this kinetic energy represents a speed

$$v=\left(\frac{2E_k}{m}\right)^{1/2}=\left[\frac{2(5.91)}{3\,755.8\text{ MeV}/c^2}\right]^{1/2}=0.056c.$$

Thus, we find for the collision frequency $f=\frac{v}{2R}=\frac{0.056c}{2(9\text{ fm})}=9.35\times10^{20}$ Hz.