Learning about constant versus decreasing gain in a simple model of exchange rate.

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Abstract

In this paper, agents learn about forecasting models of exchange rate. Agents have correct specification of the model, their choice is about using constant or decreasing gain. I specify the choice of model using multinomial logit predictor selection dynamics (Brock and Hommes 1997, 1998) and social evolutionary learning (Arifovic and Ledyard 2004). Heterogeneity of expectations is observed in the simulations with MNL, the changes in heterogeneity depend on the value of intensity of choice. At low intensity of choice, both constant and decreasing gain updating are used equally. At high intensity of choice, agents switch between two model so that one or the other model is used almost exclusively in each period. Simulations with social learning produce heterogeneity of expectations similar to that observed in the simulations with MNL with medium values of intensity of choice.

1 Introduction.

I use a simple exchange rate model to study the question of selection of learning mechanism - decreasing gain versus constant gain learning. Agents learn about the exchange rate process by updating estimates in the regression as new data arrives. Agents use two kinds of updating - decreasing and constant gain least squares. Gain denotes weight or impact of most recent observations in the estimation. Decreasing gain means that new observations receive the same weight as past data, and as the size of the data increases with time, weight on the most recent observations decreases. Constant gain assigns the same weight to the most recent observations, thus, the weight on past observations decreases with time.

This model is self-referential: the economic outcomes depend on agents' expectations, and agents' expectations are updated based on the observations about the economic outcomes. This model is suitable for the purpose of this paper because both types of learning can be validated in this environment if used exclusively. Evans and Chakraborty (2007) show that if agents use decreasing gain and do not expect nonstationarity in the data, their expectations are confirmed, and the economy converges to a rational expectations equilibrium. If agents use constant gain and expect the economic environment to be nonstationary, their expectations are confirmed too. Adding 'learning about learning' allows agents to choose between two ways to update expectations - constant or decreasing gain. The selection between constant and decreasing gain is modeled by predictor selection based on multinomial logit approach (Manski and McFadden (1981), Brock and Hommes (1997, 1998), Branch and Evans (2006, 2007)) and social evolutionary learning (Arifovic and Ledyard 2004).

Evans and Chakraborty (2007) use this model with constant gain/perpetual learning to explain forward premium puzzle. Lewis and Markiewicz (2009) introduce model misspecification (underparameterization) and learning about misspecified models using Brock and Hommes (1997, 1998) in simple monetary exchange rate model to explain excess exchange rate volatility.

Tuinstra and Wagener (2007) use evolutionary competition between two forecasting mechanisms in a simple OLG monetary model to illustrate how learning processes can generate endogenous business cycle. One mechanism is used to forecast price level based on decreasing gain least squares updating, the other mechanism is used to forecast inflation rate based on constant gain least squares updating. Agents choose between two mechanisms based on their forecasting performance measured by the average forecast squared errors. Tuinstra and Wagener (2007) show that both estimation procedures remain in use. In the steady state, both procedures produce the same forecasts and, therefore, perform the same. Outside the steady state, mechanism using inflation rate and constant gain performs better, but once the system is in the steady state, mechanism using price level and decreasing gain is equally accurate. The system generates fluctuations as the competition between two forecasting mechanisms repeats. Tuinstra and Wagener (2007) point out that we can address the issue of infinite degrees of freedom of bounded rationality by letting the competition between the estimation procedures decide what kind of beliefs will be chosen.

LeBaron (2010) uses heterogeneous gain learning in the asset pricing model in which all agents use constant gain, but the values of constant gain are different across agents. This model is able to generate persistent price/dividend series, and replicates other stylized facts about asset prices.

Branch and Evans (2006, 2007) use MNL predictor selection between underparameterized forecasting models to explain how intrinsic heterogeneity of expectations can arise. De Grauwe and Markiewicz (2007) model agents who combine two forecasting rules - fundamentalist and chartist - to forecast future exchange rate. They specify two selection procedures - dynamic predictor selection of Brock and Hommes (1997, 1998) and statistical method (Evans and Honkapohja (2001)), and study the behavior of the exchange rate in each case.

In my paper, agents use correctly specified model. Some agents make forecast using decreasing gain, and some agents use constant gain. The proportion of agents of each type changes based on dynamic predictor selector and social learning. The focus of this paper is the selection between constant gain updating and decreasing gain updating. The main questions are the following. Will one of two learning mechanism be used exclusively? If so, which mechanism will be selected? Or will we observe heterogeneity of expectations so that both learning mechanisms are used? Under what conditions will we observe each of these outcomes?

Another aspect studied in this paper is whether model with learning about updating can match the properties of the exchange rates observed in the data. The stylized facts about the exchanges rates include unit root in the exchange rate levels, fat tails and volatility clustering in returns, the forward premium puzzle. The literature using learning to explain the properties of exchange rates includes the following papers.

As mentioned above, Evans and Chakraborty (2007) use constant gain/perpetual learning in a simple exchange rate model to explain forward premium puzzle. Chakraborty (2009) follows up Evans and Chakraborty (2007) to provide empirical evidence that supports the assumptions of the model with constant gain necessary to replicate the features of the forward and spot exchange rates and to explain forward premium puzzle.

Arifovic (1996) studies Kareken-Wallace OLG model with two currencies with genetic algorithm. Arifovic (1996) observes continuing fluctuations of the exchange rates in the simulations and in the experiments with human subjects. Unit root hypothesis cannot be rejected in experimental data and small samples of simulated data. Arifovic and Gencay (2000) study the properties of the exchange rate simulated time series in the same environment as Arifovic (1996). The properties of the simulated data show the existence of chaotic attractor which is explained by the indeterminacy of equilibrium exchange rate and arbitrage opportunities. Lux and Schornstein (2002) revisits Arifovic (1996) and uses genetic algorithm learning in the Kareken-Wallace OLG model of exchange rate. The features of the simulated data match the following stylized facts of exchange rates: unit root in the level of exchange rates, fat tails in returns and volatility clustering in returns.

Kim (2009) finds that simple model of exchange rate with regime switches and constant gain recursive least squares learning replicate such features of exchange rates as excess volatility and their persistent deviations from fundamentals. Markiewicz (2010) explains the changes in exchange rate volatility using simple exchange rate model and learning. Agents face uncertainty about exchange rate model and learn about which fundamentals to use in it. When monetary policy regime changes, the agents change the set of fundamental variables and this leads to the change in the volatility of exchange rates.

1.1 Main preliminary findings.

The analytical finding is that the proportion of agents using constant gain learning converges to 0.5. We observe that the proportion of agents using constant gain updating goes to 0.5 in real time learning for different values of the intensity of choice. At low values of intensity of choice, agents use both constant and decreasing gain updating equally. At high values of intensity of choice, agents frequently switch between two updating mechanisms, such that at each point of time almost everyone uses one of the mechanisms.

Most of the stylized facts about exchange rates are matched by the model with constant gain learning. In the simulations with learning about how to update, we observe that constant gain updating is used at low values of intensity of choice. Therefore, if we are to match the facts, it must be that economy is characterized by the low intensity of choice, i.e. agents do not fully maximize. This provides support for using constant gain learning in economic models.

In the simulations with social evolutionary learning where mutation follows imitation, proportion of agents using constant gain goes to increases with probability of mutation. This means that a substantial part of the heterogeneity is attributed to the 'diversifying' force of mutation.

The simulations with mutation before imitation show lower proportion of agents using constant gain. This means that the selection pressure provided by imitation eliminates worse performing rules with constant gain introduced by mutation.

Simulations with weighted forecast squared errors exhibit less volatility than simulations with 1-period performance measure. This could be explained as agents switching more often in the stochastic environment if they base their decision only on the last period performance.

2 Model.

The exchange rate model is based on the assumptions of purchasing power parity, covered interest rate parity, and risk-neutrality. It is described by the following equations:

$$
\begin{aligned}
F_t &= \hat{E}_t s_{t+1} && (1)\\
i_t &= i_t^* + F_t - s_t && (2)\\
m_t - p_t &= d_0 + d_1 y_t - d_2 i_t && (3)\\
p_t &= p_t^* + s_t && (4)
\end{aligned}
$$

where s_t is log of exchange rate (price of foreign currency), F_t is the forward exchange rate at t for date $t+1$, $\hat{E}_t s_{t+1}$ is the expectation of s_{t+1} at time t. i_t, i_t^* are the domestic and foreign interest rates in covered interest parity equation (2). m_t, p_t, y_t are log money supply, log price level and log real GDP in the money market equilibrium equation (3). p_t^* is log foreign price level in the purchasing power parity equation (4).

This system can be solved for exchange rate:

$$s_t = \theta \hat{E}_t s_{t+1} + v_t \tag{5}$$

where $\theta = \frac{d_2}{1+d_2}$, $0 < \theta < 0$, and $v_t = (m_t - p_t - d_0 - d_1 y_t + d_2 i_t)/(1 + d_2)$ is the fundamentals. I will follow Evans, Chakraborty (2007) to assume that v_t is exogenous stochastic process and follows AR(1) process:

$$v_t = \delta + \rho v_{t-1} + \epsilon_t$$

where $0 \leq \rho \leq 1$, $\epsilon \sim iid(0, \sigma_\epsilon^2)$, and intercept δ is normalized to 0.

The rational expectations solution is:

$$s_t = \bar{b} v_{t-1} + \bar{c} \epsilon_t$$

where $\bar{b} = \frac{\rho}{1-\rho\theta}, \bar{c} = \frac{1}{1-\rho\theta}$.

Rational agents know the value of $\bar{b}$ and form their expectations of s_{t+1} as:

$$\hat{E}_t s_{t+1} = \bar{b} v_t \tag{6}$$

I will assume that agents do not know the value of $\bar{b}$ and estimate it recursively by least squares or by constant gain least squares. The estimate at time t is computed as:

$$\begin{aligned} b_t &= b_{t-1} + \gamma R_{t-1}^{-1} v_{t-1}(s_{t-1} - b_{t-1} v_{t-1}) \\ R_t &= R_{t-1} + \gamma(v_t^2 - R_{t-1}) \end{aligned} \tag{7}$$

where γ is gain: $\gamma = 1/t$ is decreasing gain in least squares, and $\gamma > 0$ is small constant in constant gain least squares.

Now I will summarize the timing of forecasts. At the beginning of t, agents have estimate b_{t-1} based on the data up to and including $t-1$. Agents observe fundamentals v_t at time t and form their forecast as:

$$\hat{E}_t s_{t+1} = b_{t-1} v_t \tag{8}$$

The exchange rate is determined according to (5) using agents' expectations (8) and fundamentals v_t. At the end of period t, estimate b_t is updated according to (7) and will be used in the next period.

Evans, Chakraborty (2007) prove that under decreasing gain least squares learning in the limit the system converges to rational expectation equilibrium $b_t \to \bar{b}$ as $t \to \infty$ where $\bar{b} = \frac{\rho}{1-\theta\rho}$. In constant gain least squares, b_t is approximately normal $b \sim N(\bar{b}, \gamma C)$ where $C = \frac{1-\rho^2}{2(1-\rho\theta)^3}$.

3 Multinomial logit approach for predictor selection.

Following Brock and Hommes (1997) and Branch and Evans (2006, 2007), agents choose predictor using multinomial logit (MNL) model based on the predictor benefits. Agents minimize their forecast squared error:

$$Eu = -(s_t - \hat{E}_t s_{t-1})^2 \tag{9}$$

The MNL has the following mapping for the predictors $i = 1, 2$:

$$n_i = \frac{exp\{\alpha E u_i\}}{\sum_{j=1}^{2} exp\{\alpha E u_j\}} \tag{10}$$

where n_i is the fraction of agents using predictor i. Mapping (10) can be rewritten as:

$$n = \frac{1}{2}[tanh(\frac{\alpha}{2}[Eu_1 - Eu_2]) + 1] \equiv H_\alpha(Eu_1 - Eu_2)$$

where $H_\alpha : R \to [0, 1]$. Parameter α is called the intensity of choice and measures how responsive agents are to the changes in the predictor benefits. Agents fully optimize when $\alpha \to \infty$, finite α means that agents do not fully optimize. Parameter α can also be interpreted as a degree of rationality.

Agents use two predictors. The first predictor is based on decreasing gain least squares estimate, and the second predictor is based on constant gain least squares estimate. I denote n as the proportion of agents using constant gain learning, and $(1 - n)$ as the proportion of agents using decreasing gain least squares. I denote b^c_{t-1} as the estimate based on constant gain least squares, and b^d_{t-1} as the estimate based on decreasing gain least squares for data up to period $t - 1$. Using (8), the forecast of exchange rate based on constant gain model is:

$$E^c_t s_{t+1} = b^c_{t-1} v_t$$

The forecast of exchange rate based on decreasing gain model is:

$$E^d_t s_{t+1} = b^d_{t-1} v_t$$

The average forecast of exchange rate is computed as:

$$\hat{E}^{average}_t s_{t+1} = nE^c_t s_{t+1} + (1 - n)E^d_t s_{t+1} = nb^c_{t-1} v_t + (1 - n)b^d_{t-1} v_t \tag{11}$$

And so the exchange rate is determined using average forecast according to (5):

$$s_t = \theta \hat{E}^{average}_t s_{t+1} + v_t$$

The forecast squared errors are:

$$\begin{aligned} Eu_1 &= -E[\theta(nb^c_{t-1} v_t + (1 - n)b^d_{t-1} v_t) + v_t - b^c_{t-1} v_{t-1}]^2 && (12) \\ Eu_2 &= -E[\theta(nb^c_{t-1} v_t + (1 - n)b^d_{t-1} v_t) + v_t - b^d_{t-1} v_{t-1}]^2 && (13) \end{aligned}$$

Define the map $F : [0, 1] \to R$ as

$$F(n) = Eu_1 - Eu_2 \tag{14}$$

After substituting Eu_1, Eu_2 (the derivations are in the appendix), function $F(n)$ is:

$$F(n) = \frac{\sigma^2_\epsilon}{1 - \rho^2}(2\theta\bar{b}^2 - \gamma C + 2\theta n(\gamma C - 2\bar{b}^2)) \tag{15}$$

$F(n)$ is linear in n, it is continuous and well-defined for $\rho \in [0, 1)$. It is decreasing for $2\bar{b}^2 > \gamma C$.

There exists a well-defined mapping $T_\alpha : [0, 1] \to [0, 1]$ such that $T_\alpha = H_\alpha \circ F$. H_α is increasing, $F(n)$ is decreasing, and so T_α is decreasing. (I need to look into this in more detail.)

Proposition 1. The model has unique Learning Equilibrium.

This follows from $T_\alpha : [0, 1] \to [0, 1]$ is continuous and Brower's theorem. (I need to elaborate on this proof.)

Proposition 2. The unique Misspecification Equilibrium n^* has one of the following properties:

1. Case 1. If $F(0) > 0$, $F(1) < 0$, as $\alpha \to \infty$, $n^* \to \hat{n} \in (0, 1)$ where $F(\hat{n}) = 0$.

2. Case 2. If $F(0) < 0$, $F(1) < 0$, as $\alpha \to \infty$, $n^* \to 0$.

3. Case 3. If $F(0) > 0$, $F(1) > 0$, as $\alpha \to \infty$, $n^* \to 1$.

For model parameters used in Evans and Chakraborty (2007) $\gamma = 0.1, 0.05$, $\rho = 0.98, 0.99$, $\theta = 0.6, 0.9$, Case 1 is the outcome, with n^* approximately 0.5.

3.1 Simulations with MNL predictor selection.

I perform simulations with MNL dynamic predictor selection to study the real-time learning behavior in this model. I use the following values of model parameters$\theta = 0.6$, $\rho = 0.99$ as in Evans and Chakraborty (2007). Constant gain is set at $\gamma = 0.1$. The values of intensity of choice are $\alpha = 10; 1000$ as in Branch and Evans (2007) and $\alpha = 100; 100,000$ to further investigate the behavior of agents with intermediate and very high intensity of choice/degree of rationality. The length of the sample presented in the figures is 360 periods (the initial 20,000 periods are not reported in the figures).

Figure 1 illustrates a typical simulation with low intensity of choice $\alpha = 10$. We observe that the proportion of agents using constant gain is positive and varies around average value of 0.497. The first panel of Figure 1 illustrates the behavior of the forecast squared error 9 of agents using constant and decreasing gain. The forecast squared errors of agents using constant gain are similar to those of agents using decreasing gain. Sometimes the forecasting accuracy is the same for both models, sometimes one model is more accurate than the other. At the low intensity of choice, agents are not very responsive to the difference in the performance of two forecasting mechanisms, and so we observe approximately equal fractions of agents using both mechanisms.

It is interesting to know that for higher values of intensity of choiceα, average proportion of agents using constant gain learning, n^{CG}, becomes more volatile. Figure 2 shows a typical simulation for a higher value of intensity of choice $\alpha = 100$. Proportion of agents using constant gain varies around average of 0.5101. Figure 3 shows a typical simulation for a higher value of intensity of choice $\alpha = 1000$. Proportion of agents using constant gain varies around average of 0.5176. Figure 4 shows a typical simulation for a higher value of intensity of choice $\alpha = 100,000$. Proportion of agents using constant gain varies around average of 0.5249 (0.5218 for $\alpha = 10,000$).

Intensity of choice measures how responsive agents are to the changes in payoffs. Higher values of intensity of choice mean that agents respond promptly to the difference in payoffs. At high intensity of choice, agents switch between estimation frequently, and we observed that proportion of constant gain users, n, fluctuates between 0 and 1, i.e. sometimes all agents use decreasing gain updating, and sometimes all agents use constant gain updating.

We can conclude that at low intensity of choice we observe heterogeneity of forecasting models so that both updating mechanisms are used equally in each period. When agents become more responsive to the difference in the performance of two forecasting models, they frequently switch to the model with better performance every period. As a result, we observe that one of two forecasting mechanisms is used almost exclusively in each period, and both mechanisms remain in alternate use depending on their performance. We observe that the proportion of constant gain users is approximately 0.5 on average over time, not in each period.

Positive feedback from expectations to state. In this model of exchange rates, the feedback from expectations to the state is positive, $\theta > 0$. Branch and Evans (2007) study Lucas monetary model with positive feedback from expectations to the state and find that as a result of positive feedback the equilibrium forces push towards homogeneity of expectations [pp.16-17]. We observe heterogeneity of expectations for low intensity of choice. At high intensity of choice, expectations are (almost) homogeneous in each period, but heterogeneous over time.

To further investigate the behavior of this system, I have collected summary statistics about proportion of agents using constant gain learning based on 1000 simulations. For each of these simulations, I have computed average proportion of agents using constant gain for different sample sizes after simulation has run for 20,000 periods. In other words, the initial 20,000 periods of data were discarded that was done to avoid dependance on the starting values and to study long-term behavior. Using average values from each simulation, I have computed median proportion of constant gain users over 1000 simulation.

Tables 1 and 2 present the median proportion of agents using constant gain updating in sample size $T = 120$. Tables 3 and 4 present the same data in sample size $T = 360$.[1] These tables show that the proportion of agents using constant gain goes approximately to 0.5 for most parameter combinations. For $\theta = 0.9$, increase in intensity of choice leads to increase in the proportion of constant gain users.

For higher values of constant gain, γ, the proportion of agents using constant gain updating is lower and falls non-monotonically. This negative relationship is more pronounced for higher values of intensity of choice, α. For higher value of feedback parameter θ (feedback from expectations to the state), the proportion of agent using constant gain is higher. This is consistent with the conclusion of Branch and Evans (2000) that in presence of positive feedback from expectations to state the system moves toward homogeneity of expectations. Here I find that when the value of positive feedback from expectations to state is larger, the system moves stronger towards more homogeneous use of constant gain updating.

The proportions of agents using constant gain stay the same for sample sizes $T = 120$ (Tables 1 and 2) and $T = 360$ (Tables 3 and 4).[2] This means that this outcome is stable over time.

The relationship between AR(1) coefficient in AR(1) process for fundamentals, ρ, and proportion of agents using constant gain is not monotonic. For most of the parameter combinations, when ρ increases, the proportion of agents using constant gain goes up. When ρ is equal 1, the proportion of constant gain users is the highest. This means when fundamentals are random walk, agents switch more to use constant gain updating. The explanation is that the fundamentals become very persistent, constant gain updating would perform very well.

[1] I have also collected data for sample sizes $T = 2000$ and $T = 20000$. The results are qualitatively the same as in sample sizes $T = 120$ and $T = 360$.

[2] In sample sizes $T = 2000$ and $T = 20000$, this conclusion remains valid: proportion of agents using constant gain stays the same.

4 Social evolutionary learning as predictor selection.

This section describes the use of social evolutionary learning as a predictor selection. Both social learning and dynamic MNL predictor selection are based on the idea that mechanism with better performance is more likely to be selected and, thus, will prevail in the population.

Evolutionary learning is based on genetic algorithm. Genetic algorithm is a numerical optimization technique first introduced by Holland (1975) and described in Goldberg (1987), Michalewicz (1996), Back et. al. (2000). Among the advantages of using genetic algorithm for optimization are that it starts with a set of random solutions and so does not rely on the starting point, and that it is applicable for discontinuous, nondifferentiable, noisy, multimodal and other unconventional surfaces (Schwefel 2000). In other words, when the nature of landscape is not known in advance, the application of a genetic algorithm is appropriate. For example, Bullard and Duffy (2004) use simulated method of moments with a genetic algorithm to estimate growth model with structural breaks. Evolutionary learning is convenient to study economies with heterogeneous agents, it does not impose high information and computational requirements on agents and is able to explain actual and experimental data better than models with rational expectations. Evolutionary learning has been used in different economic environments. Arifovic (1994, 1995, 1996), LeBaron (2000), Arifovic, Bullard and Duffy (1997, 1998 a,b,c), Dawid (2006), Lux and Marchesi (2000), Lux and Schornstein (2002) are some examples of research using evolutionary learning agents.

The social learning is implemented in the following way. The initial population of N agents is generated randomly such that each agent has equal probability to start with either constant or decreasing gain (i.e. probability to start with either is 0.5).

I make assumption that all agents have access to all past data since the beginning of the simulation, and the only choice they need to make is whether in current period to forecast based on the model with decreasing gain or constant gain, i.e. whether to use all of the data or discard past data and use more recent data. This means that if agent switches from constant gain to decreasing gain at time t, he uses gain $\gamma = 1/t$ in period t, he does not treat period t as the first period in his updating and does not use $\gamma = 1$ at t and then decreases it starting from time t. And if agent switches from decreasing gain to constant gain at time t, his forecast at t is computed from the model with constant gain that has been recursively estimated starting from period 1. In other words, each agent simultaneously updates two models - with constant gain and with decreasing gain - and in each period he makes choice to forecast based on one of the models. This assumption about recursive updating is the same as in the simulations with MNL predictor selection.

In each period, the estimates of coefficients b are updated according to (7) using constant and decreasing gain. There are only 2 different estimates of the coefficient in the population at each point of time. And average forecast is computed using the proportion of agents of different type from previous period according to (11). At the end of period, the forecast squared errors are computed. These are used as performance measures (fitness) of agents with different forecasting models.

Before next period starts, the population of agents learns about their forecasting models. Learning is implemented by applying genetic algorithm operators - imitation followed by experimentation.

Imitation aims to model the behavior where economic agents learn from each other. Here it is implemented by tournament selection. Agents revise their choice of forecasting models by observing forecasting models of other agents and their performance. An agent is matched randomly (with replacement) with another agent, it can compare forecasting performance of its own model and other agent's model, and adopts the model of other agent if its performance is better (higher fitness).

This process is repeated for each of N agents. The resulting updated population represents changes in ideas held by the agents. Tournament selection provides all the selection pressure in the social evolutionary learning, as model with lower fitness is discarded, while model with higher fitness remains in the population over time.

Experimentation represents the behavior where people arrive at new ideas by chance. The new ideas can be good innovation or accidental mistakes. Experimentation is implemented as follows. Each agent can change its decision about the forecasting model to the other one with some probability, *mprob*. Generally, the value of probability of experimentation represents the frequency of the arrival of new ideas. In the setup of my model, there are two competing models - constant and decreasing gain. Thus, the value of the probability of experimentation represents how likely it is that agents can switch to a different forecasting model. The benefit of experimentation in genetic algorithm is that algorithm can explore the entire range of values and, thus, more likely to find the global optimum. However, experimentation also introduces additional volatility and can disrupt the convergence.

The values of model parameters are the same as in the simulations with MNL predictor selector. The genetic algorithm parameters are as follows. The number of agents is $N = 100$. I have varied the probability of experimentation for sensitivity analysis. The data is collected for different sample sizes after the initial 20,000 periods of data are discarded for the same reasons as stated above.

Figures 5, 6 and 7 illustrate typical simulations for different values of the probability of experimentation of 0.01, 0.05 and 0.1 respectively. For $mprob = 0.01$, the proportion of agents using constant gain learning varies around average value of 0.4132. For $mprob = 0.05$, the average proportion is 0.4237. For $mprob = 0.1$, the average proportion of constant gain users is 0.4445. Based on these observations, agents both updating mechanisms almost equally for different experimentation rates. When probability of experimentation is high, the simulations show more heterogeneity. This heterogeneity of expectations, however, is due to diversifying or disrupting force of higher experimentation rate. The first panels of Figures 5, 6 and 7 illustrate the forecasting errors of constant and decreasing gain updating. The forecast errors of constant gain are similar to those of decreasing gain. Most of the time they are the same, and sometimes one is higher than the other. Therefore, both updating mechanisms remain in use, and we observe heterogeneity of expectations in these simulations. Increase of experimentation rates does not show noticeable impact on the behavior of the proportion of constant gain users as illustrated on the figures, but it leads to slight increase of the average proportion of constant gain users. This could be attributed to the 'diversifying' impact of experimentation.

To investigate this possibility, I run simulations where imitation follows mutation. Imitation provides selection pressure, and so some of the 'bad ideas' coming from experimentation are eliminated by the imitation. Figure 8 illustrates typical behavior in the simulation with $mprob = 0.1$ where mutation is followed by imitation. The average proportion of constant gain users is 0.4246 and is lower than 0.4445 in the simulation where experimentation follows imitation. Thus, imitation 'changes mind' of agents experimenting with using constant gain updating. Therefore, constant gain updating might perform slightly worse than decreasing gain updating because the proportion of agents using it decreases in the simulation with imitation after mutation.

To further investigate the behavior of this system and the impact of different parameter values, I have collected summary statistics about average proportion of agents using constant gain learning. I have run 1000 simulations; for each of these simulations, I have computed average proportion of agents using constant gain during the last 100 periods of the simulation. Based on this data, I have computed median proportion of constant gain users over 1000 simulation. This data is presented in Table **??** for different parameter sets.

From this Table, we can observe that the only parameter that affects the proportion of agents using constant gain is the probability of mutation. The rest of the parameters and sample size do not have any impact. When probability of mutation is higher, the proportion of agents using constant gain updating is higher too. The role of mutation is to introduce new ideas, but it also introduces noise. When probability of mutation is low, almost all agents use decreasing gain which means that decreasing gain updating brings higher payoff. Higher probability of mutation means that more agents use constant gain through experimentation. Therefore, higher proportion of agents using constant gain reflects that noise introduced by higher rate of mutation.

Based on these observations I can conclude that decreasing gain learning is selected in the simulations with social learning. This outcome is similar to the that in the simulations with MNL predictor selection. In case of social learning, heterogeneity of expectations can be observed when agents experiment more. In case of MNL predictor selection, heterogeneity of expectations is observed at low intensity of choice when agents do not fully maximize and respond slowly to the difference in payoffs.

5 Extensions and robustness.

The results presented above are for the simulations where agents evaluate the performance of their models based on forecast squared error in the most recent period. This can make agent's choice too sensitive to stochastic shocks. Therefore, agents may want to evaluate their models over more than 1 previous period. This can be done using weighted past forecast errors computed as:

$$u_t = u_{t-1} + \lambda_t(-(s_t - \hat{E}_t s_{t+1})^2 - u_{t-1}) \tag{16}$$

When $\lambda_t = 1$, one period forecast squared error is recovered. When $\lambda_t = t^{-1}$, (16) computes average of past forecast squared errors with equal weights, $\lambda_t = \lambda$ weighs recent forecast errors more. This weighted performance measure makes more sense if agents want good forecasting model that performs well over time horizons longer than 1 period. This is especially important in the stochastic environment.

I use value of $\lambda = 0.35$ (as in Branch, Evans 2007). The rest of the parameters are the same as before. Typical simulations with MNL predictor selection are illustrated in Figure 9 for $\alpha = 10$, Figure 10 for $\alpha = 100$ and Figure 11 for $\alpha = 1000$. These simulations present less volatility than those based on one-period performance measure. This means that when agents make decisions based on weighted performance in the stochastic environment, they switch less often than in the simulations based on one-period performance measure because they do not overreact to one-period shocks. In the simulations with one-period performance measure, the proportion of constant gain users becomes more volatile when intensity of choice increases that is the same as in the simulations with one-period performance.

Typical simulations with social learning are illustrated in Figure 12 for $mprob = 0.01$ and Figure 13 for $mprob = 0.1$. These simulations do not show any difference in the behavior of the proportion of constant gain users.

6 Conclusion.

This paper shows that if agents have the choice between forecasting based on estimation with constant gain or decreasing gain, we can observe heterogeneity of forecasting methods. Agents use both models if selection pressure is not too high in MNL predictor selection. The proportion of agents using constant gain model is close to 0.5 for $\alpha = 100$ that indicates that two models perform equally well. For high selection pressure with $\alpha = 100,000$, agents switch between the models more frequently, but the average proportion of constant gain users is 0.5.

In case of social evolutionary learning, the proportion of agents using constant gain converges to different values depending on the rate of experimentation. At high probability of experimentation (0.10), the proportion of agents using constant gain is higher than at low probability of experimentation. This means that some heterogeneity is attributed to experimentation. The simulations with experimentation before imitation show lower proportion of agents using constant gain. This means that the selection pressure provided by imitation eliminates worse performing rules with constant gain introduced by experimentation.

When agents base their decisions on weighted forecast error, we observe less volatility in the decision about which model to use. One-period forecast error as a measure leads to more switching of the forecasting model in the stochastic environment.

7 Appendices.

Derivation of function $F(n)$.

Function $F(n)$ is defined as:

$$F(n) = Eu_1 - Eu_2$$

where

$$\begin{aligned} Eu_1 &= -E[s_t - E_t^c s_{t+1}]^2 = -E[\theta(nb^c_{t-1}v_t + (1-n)b^d_{t-1}v_t) + v_t - b^c_{t-1}v_t]^2 \\ Eu_2 &= -E[s_t - E_t^d s_{t+1}]^2 = -E[\theta(nb^c_{t-1}v_t + (1-n)b^d_{t-1}v_t) + v_t - b^d_{t-1}v_t]^2 \end{aligned}$$

where $E_t^c s_{t+1}$ and $E_t^d s_{t+1}$ are expectations of s_{t+1} at time t formed using constant gain and decreasing gain models respectively. I will first manipulate each of Eu_1, Eu_2 separately and then substitute them into $F(n)$.

$$\begin{aligned} Eu_1 &= -E[\theta(nb^c_{t-1}v_t + (1-n)b^d_{t-1}v_t) + v_t - b^c_{t-1}v_t]^2 = \\ &= -E[\{(\theta n - 1)b^c_{t-1} + \theta(1-n)b^d_{t-1} + 1\}v_t]^2 = \\ &= -E[\{(\theta n - 1)^2 b^{c2}_{t-1} + \theta^2(1-n)^2 b^{d2}_{t-1} + 2\theta(1-n)(\theta n - 1)b^c_{t-1}b^d_{t-1} + 1 + \\ &\quad +2((\theta n - 1)b^c_{t-1} + \theta(1-n)b^d_{t-1})\}v_t^2] = \\ &= -E[(\theta n - 1)^2 b^{c2}_{t-1}v_t^2 + \theta^2(1-n)^2 b^{d2}_{t-1}v_t^2 + 2\theta(1-n)(\theta n - 1)b^c_{t-1}b^d_{t-1}v_t^2 + \\ &\quad +v_t^2 + 2(\theta n - 1)b^c_{t-1}v_t^2 + 2\theta(1-n)b^d_{t-1}v_t^2] \end{aligned}$$

$$\begin{aligned} Eu_2 &= -E[\theta(nb^c_{t-1}v_t + (1-n)b^d_{t-1}v_t) + v_t - b^c_{t-1}v_t]^2 = \\ &= -E[\{\theta n b^c_{t-1} + (\theta(1-n) - 1)b^d_{t-1} + 1\}v_t]^2 = \\ &= -E[\{\theta^2 n^2 b^{c2}_{t-1} + (\theta(1-n) - 1)^2 b^{d2}_{t-1} + 2\theta n(\theta(1-n) - 1)b^c_{t-1}b^d_{t-1} + 1 + \\ &\quad +2(\theta n b^c_{t-1} + (\theta(1-n) - 1)b^d_{t-1})\}v_t^2] = \\ &= -E[\theta^2 n^2 b^{c2}_{t-1}v_t^2 + (\theta(1-n) - 1)^2 b^{d2}_{t-1}v_t^2 + \\ &+ 2\theta n(\theta(1-n) - 1)b^c_{t-1}b^d_{t-1}v_t^2 + v_t^2 + 2\theta n b^c_{t-1}v_t^2 + 2(\theta(1-n) - 1)b^d_{t-1}v_t^2] \end{aligned}$$

After subtracting Eu_2 from Eu_1 and collecting terms, i get:

$$\begin{aligned} Eu_1 - Eu_2 &= E[(\theta^2 n^2 - (\theta n - 1)^2) b_{t-1}^{c2} v_t^2 + ((\theta(1-n)-1)^2 - \theta^2(1-n)^2) b_{t-1}^{d2} v_t^2 + \\ &\quad +(2\theta n(\theta(1-n)-1) - 2\theta(1-n)(\theta n - 1)) b_{t-1}^c b_{t-1}^d v_t^2 + \\ &\quad +(2\theta n - 2(\theta n - 1)) b_{t-1}^c v_t^2 + (2(\theta(1-n)-1) - 2\theta(1-n)) b_{t-1}^d v_t^2] = \\ &= E[(2\theta n - 1) b_{t-1}^{c2} v_t^2 + (1 - 2\theta(1-n)) b_{t-1}^{d2} v_t^2 + (-4\theta n + 2\theta) b_{t-1}^c b_{t-1}^d v_t^2 + \\ &\quad +2 b_{t-1}^c v_t^2 - 2 b_{t-1}^d v_t^2] \end{aligned}$$

Using that $E(b_{t-1}^{c2}) = \gamma C$, $E(b_{t-1}^{d2}) = 0$, $E(v_t^2) = \frac{\sigma_\epsilon^2}{1-\rho^2}$, $E(b_{t-1}^c) = \bar{b}$, $E(b_{t-1}^d) = \bar{b}$, I receive that:

$$Eu_1 - Eu_2 = [(2\theta n - 1)\gamma C \frac{\sigma_\epsilon^2}{1-\rho^2} + 0 + (-4\theta n + 2\theta)\bar{b}^2 \frac{\sigma_\epsilon^2}{1-\rho^2} + 2\bar{b}\frac{\sigma_\epsilon^2}{1-\rho^2} - 2\bar{b}\frac{\sigma_\epsilon^2}{1-\rho^2}] \quad (17)$$

And finally after collecting terms, function $F(n)$ is:

$$F(n) = \frac{\sigma_\epsilon^2}{1-\rho^2}(2\theta\bar{b}^2 - \gamma C + 2\theta n(\gamma C - 2\bar{b}^2))$$

References

[1] Alfarano, S., Lux, T. Extreme value theory as a theoretical background for power law behavior. Prepared for *Power laws in the social science: discoving complexity and non-equilibrium dynamics in the social universe* edited by Claudio Cioffi-Revilla.

[2] Arifovic, J.1996. The Behavior of the Exchange Rate in the Genetic Algorithm and Experimental Economics. *Journal of Political Economy*, 104: 510-541.

[3] Arifovic, J. 2000. Evolutionary algorithms in macroeconomic models. *Macroeconomic Dynamics* 4(3): 373-414.

[4] Arifovic, J., Gencay, R. 2000. Statistical properties of genetic learning in a model of exchange rate. *Journal of Economic Dynamics and Control*, 24:981-1005

[5] Arifovic, J., Ledyard, J. 2004. Computer Testbeds: The Dynamics of Groves-Ledyard Mechanisms. Manuscript, April.

[6] Back,T., Fogel, D.B., and Z. Michalewicz. 2000. *Evolutionary Computation 1 Bacis Algorithms and Operators.* Institute of Physics Publishing, Bristol and Philadelphia

[7] Branch, W., and G.Evans. 2006. Intrinsic heterogeneity in expectation formation. *Journal of Economic Theory*, 127: 264-295.

[8] Branch, W., and G. Evans. 2007. Model uncertainty and endogenous volatility. *Review of Economic Dynamics*, 19: 207-237.

[9] Brock, W.A., and C.H.Hommes. 1997. A rational route to randomness. *Econometrica*, 65: 1059-1160

[10] Brock, W.A., and C.H.Hommes. 1998. Heterogeneous beliefs and routed to chaos in a simple asset pricing model. *Journal of Economic Dynamics and Control*, 22: 1235-1274

[11] Chakraborty, A. 2009. Learning, the forward premium puzzle, and market efficiency. *Macroeconomic Dynamics*, 13(1): 31-57

[12] Chakraborty, A.. and G. Evans. 2007. Can perpetual learning explain the forward premium puzzle? Working paper available at http://www.uoregon.edu/~gevans/

[13] De Grauwe, P., and A. Markiewicz. 2007. Learning to forecast the exchange rate: two competing approaches. Working paper available at https://www.econ.kuleuven.be/ew/academic/intecon/Markiewicz/Default.htm

[14] Goldberg, D.E. 1989. *Genetic Algorithms in Search, Optimizations, and Machine Learning.* Reading, Mass.: Addison-Wesley Pub.Co.

[15] Holland, J.H. 1975. *Adaptation in Natural and Artificial Systems.* University of Michigan Press, Ann Arbor

[16] Kim, Y.S. 2009. Exchange rates and fundamentals under adaptive learning. *Journal of Economic Dynamics and Control*, 33: 843-863

[17] LeBaron, B. 2000. Agent-based Computational Finance: Suggested Readings and Early Research. *Journal of Economic Dynamics and Control*, 24: 679-702.

[18] LeBaron, B. 2010. Heterogeneous gain learning and long swings in asset prices. Manuscript.

[19] Lewis, V., and A. Markiewicz. 2009. Model Misspecification, Learning and the Exchange Rate Disconnect Puzzle. *B.E. Journal of Macroeconomics*, 9(1):

[20] Lux, Th. and S. Schornstein. 2002. Genetic learning as an explanation of stylized fact of foreign exchange markets. Discussion paper 29/02 Economic Research Center of the Deutsche Bundesbank

[21] Markiewicz,A. 2010. Monetary policy, model uncertainty and exchange rate volatility. CESifo Working Paper Series, CESifo Group Munich Available at

[22] Lewis, V., Markiewicz, A. 2009. Model misspecification, learning and the exchange rate disconnect puzzle. Research series 200907-01, National Bank of Belgium. Available at http://www.nbb.be/doc/oc/repec/reswpp/wp168En.pdf

[23] Michalewicz, Z. 1996. *Genetic Algorithms + Data Structures = Evolution Programs.* 3rd ed. Springer-Verlag

[24] Schwefel, H.P. 2000. Advantages (and disadvantages) of evolutionary computation over other approaches. in *Evolutionary Computation 1 Bacis Algorithms and Operators.* edited by Back,T., Fogel, D.B., and Z. Michalewicz. Institute of Physics Publishing, Bristol and Philadelphia

[25] Tuinstra, J., and Florian A.A. Wagener. 2007. On learning equilibria. *Economic Theory*, 30: 493-513

		$\rho = 0.98$							$\rho = 0.99$						
					α							α			
θ	γ	1	10	100	10^3	10^4	10^5	10^6	1	10	100	10^3	10^4	10^5	10^6
0.6	0.01	0.500	0.499	0.496	0.496	0.498	0.500	0.500	0.500	0.499	0.496	0.496	0.499	0.500	0.500
	0.02	0.500	0.499	0.493	0.492	0.494	0.495	0.495	0.500	0.499	0.494	0.495	0.497	0.498	0.498
	0.03	0.500	0.498	0.491	0.490	0.492	0.492	0.492	0.500	0.498	0.492	0.493	0.493	0.493	0.492
	0.04	0.500	0.498	0.491	0.490	0.491	0.492	0.492	0.500	0.498	0.491	0.492	0.492	0.492	0.492
	0.05	0.500	0.497	0.490	0.489	0.491	0.491	0.492	0.500	0.497	0.490	0.490	0.490	0.492	0.492
	0.10	0.499	0.495	0.488	0.488	0.488	0.489	0.492	0.499	0.495	0.488	0.487	0.487	0.487	0.485
	0.20	0.499	0.491	0.483	0.483	0.483	0.483	0.483	0.499	0.492	0.484	0.484	0.483	0.483	0.483
0.9	0.01	0.504	0.511	0.523	0.525	0.525	0.525	0.525	0.506	0.515	0.528	0.529	0.532	0.533	0.533
	0.02	0.502	0.505	0.512	0.514	0.516	0.517	0.517	0.502	0.507	0.515	0.518	0.518	0.517	0.517
	0.03	0.501	0.502	0.506	0.506	0.508	0.508	0.508	0.501	0.505	0.510	0.513	0.514	0.516	0.517
	0.04	0.500	0.501	0.504	0.505	0.506	0.508	0.508	0.501	0.502	0.507	0.508	0.508	0.508	0.508
	0.05	0.500	0.499	0.503	0.504	0.506	0.508	0.508	0.500	0.502	0.504	0.505	0.507	0.508	0.508
	0.10	0.499	0.498	0.500	0.501	0.500	0.500	0.500	0.500	0.499	0.503	0.505	0.504	0.501	0.500
	0.20	0.499	0.506	0.507	0.510	0.510	0.508	0.508	0.500	0.509	0.514	0.515	0.517	0.517	0.517

Table 1: Results from 1000 simulations for $\rho = 0.98, 0.99$ with sample size T=120 after discarding the first 20,000 data points. Table gives median values of proportion of agents using constant gain updating in the simulations with MNL predictor selection.

		$\rho = 0.995$							$\rho = 1$						
		α							α						
θ	γ	1	10	100	10^3	10^4	10^5	10^6	1	10	100	10^3	10^4	10^5	10^6
0.6	0.01	0.500	0.500	0.497	0.496	0.499	0.500	0.500	0.500	0.499	0.494	0.493	0.494	0.495	0.492
	0.02	0.500	0.499	0.495	0.496	0.497	0.499	0.500	0.500	0.499	0.492	0.493	0.495	0.496	0.495
	0.03	0.500	0.498	0.492	0.493	0.494	0.493	0.492	0.500	0.498	0.491	0.492	0.492	0.492	0.492
	0.04	0.500	0.498	0.491	0.492	0.492	0.492	0.492	0.500	0.498	0.490	0.490	0.491	0.492	0.492
	0.05	0.500	0.497	0.490	0.490	0.491	0.492	0.492	0.500	0.497	0.489	0.490	0.491	0.492	0.492
	0.10	0.499	0.495	0.488	0.488	0.488	0.485	0.483	0.499	0.495	0.487	0.487	0.486	0.485	0.483
	0.20	0.499	0.492	0.485	0.484	0.483	0.483	0.483	0.499	0.492	0.486	0.488	0.489	0.490	0.492
0.9	0.01	0.507	0.521	0.533	0.535	0.533	0.533	0.533	0.576	0.624	0.627	0.625	0.625	0.625	0.625
	0.02	0.503	0.508	0.520	0.525	0.525	0.525	0.525	0.554	0.598	0.600	0.600	0.600	0.600	0.600
	0.03	0.501	0.505	0.510	0.514	0.515	0.516	0.517	0.547	0.585	0.589	0.586	0.583	0.583	0.583
	0.04	0.501	0.504	0.511	0.511	0.512	0.511	0.510	0.540	0.574	0.578	0.578	0.581	0.579	0.579
	0.05	0.500	0.503	0.509	0.510	0.509	0.508	0.508	0.535	0.563	0.567	0.569	0.574	0.575	0.575
	0.10	0.499	0.501	0.504	0.505	0.506	0.508	0.508	0.518	0.525	0.529	0.528	0.532	0.533	0.529
	0.20	0.501	0.513	0.516	0.517	0.517	0.517	0.517	0.508	0.524	0.525	0.526	0.525	0.525	0.525

Table 2: Results from 1000 simulations for $\rho = 0.995, 1$ with sample size T=120 after discarding the first 20,000 data points. Table gives median values of proportion of agents using constant gain updating in the simulations with MNL predictor selection.

		$\rho = 0.98$							$\rho = 0.99$						
		α							α						
θ	γ	1	10	100	10^3	10^4	10^5	10^6	1	10	100	10^3	10^4	10^5	10^6
0.6	0.01	0.500	0.499	0.497	0.495	0.496	0.496	0.496	0.500	0.499	0.496	0.495	0.497	0.496	0.497
	0.02	0.500	0.499	0.494	0.492	0.493	0.493	0.492	0.500	0.499	0.494	0.493	0.494	0.494	0.494
	0.03	0.500	0.498	0.492	0.491	0.492	0.492	0.492	0.500	0.498	0.493	0.491	0.491	0.491	0.491
	0.04	0.500	0.498	0.491	0.489	0.490	0.490	0.491	0.500	0.498	0.492	0.490	0.490	0.491	0.491
	0.05	0.500	0.497	0.491	0.488	0.489	0.489	0.489	0.500	0.497	0.490	0.488	0.489	0.489	0.489
	0.10	0.499	0.495	0.487	0.487	0.487	0.487	0.486	0.499	0.495	0.488	0.486	0.486	0.486	0.486
	0.20	0.499	0.491	0.482	0.483	0.483	0.483	0.483	0.499	0.492	0.483	0.482	0.482	0.481	0.482
0.9	0.01	0.508	0.518	0.524	0.525	0.526	0.525	0.525	0.512	0.525	0.532	0.533	0.533	0.533	0.533
	0.02	0.505	0.508	0.512	0.513	0.513	0.513	0.514	0.507	0.513	0.517	0.519	0.519	0.520	0.519
	0.03	0.503	0.505	0.506	0.508	0.508	0.508	0.508	0.505	0.508	0.511	0.513	0.513	0.513	0.514
	0.04	0.503	0.503	0.504	0.505	0.504	0.504	0.503	0.504	0.505	0.508	0.509	0.509	0.508	0.508
	0.05	0.502	0.501	0.501	0.501	0.502	0.503	0.503	0.503	0.504	0.506	0.506	0.507	0.506	0.506
	0.10	0.500	0.498	0.500	0.501	0.501	0.500	0.500	0.502	0.501	0.503	0.503	0.503	0.503	0.503
	0.20	0.500	0.506	0.510	0.511	0.511	0.511	0.511	0.502	0.510	0.515	0.515	0.514	0.514	0.514

Table 3: Results from 1000 simulations for $\rho = 0.98, 0.99$ with sample size T=360 after discarding the first 20,000 data points. Table gives median values of proportion of agents using constant gain updating in the simulations with MNL predictor selection.

		$\rho = 0.995$							$\rho = 1$						
		α							α						
θ	γ	1	10	100	10^3	10^4	10^5	10^6	1	10	100	10^3	10^4	10^5	10^6
0.6	0.01	0.500	0.499	0.497	0.497	0.497	0.497	0.497	0.500	0.500	0.496	0.495	0.497	0.497	0.497
	0.02	0.500	0.499	0.494	0.494	0.494	0.493	0.493	0.500	0.499	0.495	0.493	0.494	0.494	0.494
	0.03	0.500	0.498	0.493	0.492	0.492	0.491	0.492	0.500	0.499	0.494	0.492	0.493	0.493	0.492
	0.04	0.500	0.498	0.492	0.491	0.491	0.491	0.492	0.500	0.498	0.493	0.491	0.492	0.492	0.492
	0.05	0.500	0.497	0.491	0.489	0.490	0.490	0.489	0.500	0.498	0.492	0.490	0.491	0.491	0.492
	0.10	0.499	0.496	0.488	0.487	0.486	0.487	0.486	0.500	0.496	0.490	0.489	0.488	0.489	0.489
	0.20	0.499	0.492	0.484	0.483	0.483	0.483	0.483	0.499	0.493	0.487	0.486	0.488	0.488	0.486
0.9	0.01	0.514	0.533	0.540	0.540	0.540	0.539	0.539	0.576	0.627	0.629	0.630	0.631	0.631	0.631
	0.02	0.509	0.518	0.522	0.524	0.524	0.524	0.525	0.556	0.603	0.606	0.606	0.606	0.606	0.606
	0.03	0.506	0.512	0.516	0.517	0.517	0.517	0.517	0.550	0.589	0.593	0.594	0.594	0.593	0.593
	0.04	0.505	0.509	0.512	0.513	0.513	0.514	0.514	0.544	0.576	0.582	0.582	0.583	0.583	0.583
	0.05	0.504	0.507	0.510	0.510	0.510	0.511	0.511	0.538	0.565	0.569	0.570	0.570	0.569	0.569
	0.10	0.502	0.504	0.505	0.506	0.505	0.506	0.506	0.523	0.528	0.529	0.529	0.528	0.528	0.528
	0.20	0.504	0.514	0.517	0.517	0.517	0.517	0.517	0.511	0.525	0.529	0.530	0.529	0.529	0.528

Table 4: Results from 1000 simulations for $\rho = 0.995, 1$ with sample size T=360 after discarding the first 20,000 data points. Table gives median values of proportion of agents using constant gain updating in the simulations with MNL predictor selection.

	$\rho = 0.98$										$\rho = 0.99$									
					$mprob$										$mprob$					
γ	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
$\theta = 0.6$																				
0.01	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.02	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.03	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.04	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.05	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.10	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.20	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
$\theta = 0.9$																				
0.01	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.02	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.03	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.04	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.05	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.10	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.20	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149

Table 5: Results from 1000 simulations for $\rho = 0.98, 0.99$ with sample size T=120 after discarding the first 20,000 data points. Table gives median values of proportion of agents using constant gain updating in the simulations with social learning.

	$\rho = 0.995$										$\rho = 1$									
					$mprob$										$mprob$					
γ	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
$\theta = 0.6$																				
0.01	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.01	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.03	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.04	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.05	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.10	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.20	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
$\theta = 0.9$																				
0.01	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.02	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.03	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.04	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.05	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.10	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149
0.02	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.133	0.141	0.149

Table 6: Results from 1000 simulations for $\rho = 0.995, 1$ with sample size T=120 after discarding the first 20,000 data points. Table gives median values of proportion of agents using constant gain updating in the simulations with social learning.

	$\rho = 0.98$										$\rho = 0.99$									
	$mprob$										$mprob$									
γ	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
$\theta = 0.6$																				
0.01	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.02	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.03	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.04	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.05	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.10	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.20	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
$\theta = 0.9$																				
0.01	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.02	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.03	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.04	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.05	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.10	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.20	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149

Table 7: Results from 1000 simulations for $\rho = 0.98, 0.99$ with sample size T=360 after discarding the first 20,000 data points. Table gives median values of proportion of agents using constant gain updating in the simulations with social learning.

	$\rho = 0.995$										$\rho = 1$									
	$mprob$										$mprob$									
γ	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
$\theta = 0.6$																				
0.01	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.02	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.03	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.04	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.05	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.06	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.07	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
$\theta = 0.9$																				
0.01	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.02	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.03	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.04	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.05	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.10	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149
0.20	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149	0.046	0.067	0.082	0.094	0.106	0.116	0.125	0.134	0.142	0.149

Table 8: Results from 1000 simulations for $\rho = 0.995, 1$ with sample size T=360 after discarding the first 20,000 data points. Table gives median values of proportion of agents using constant gain updating in the simulations with social learning.

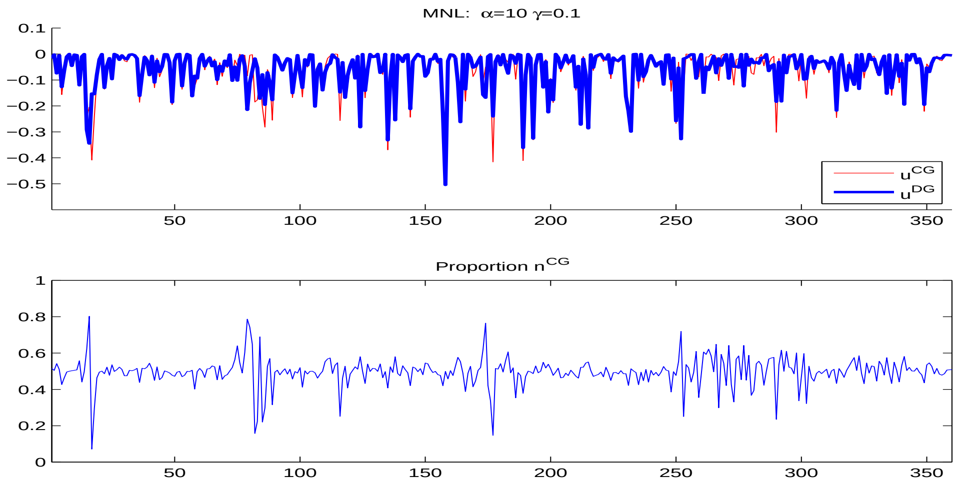

Figure 1:

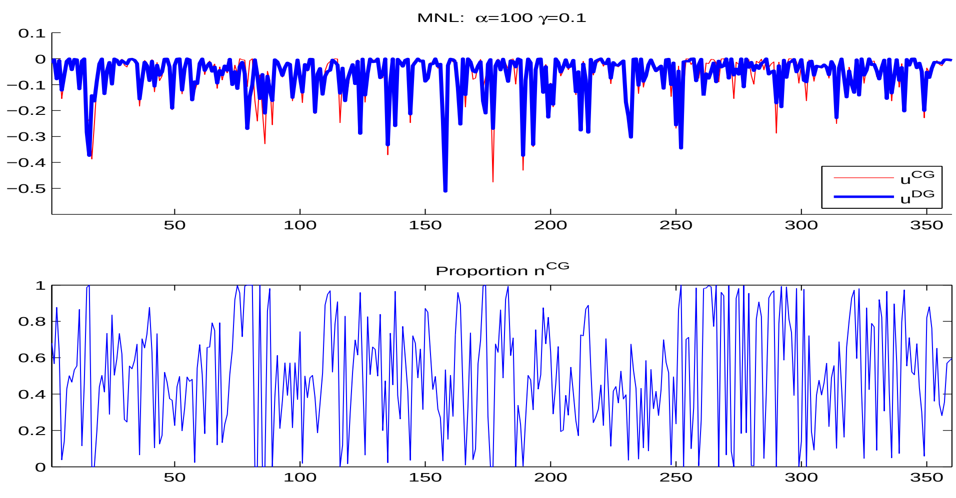

Figure 2:

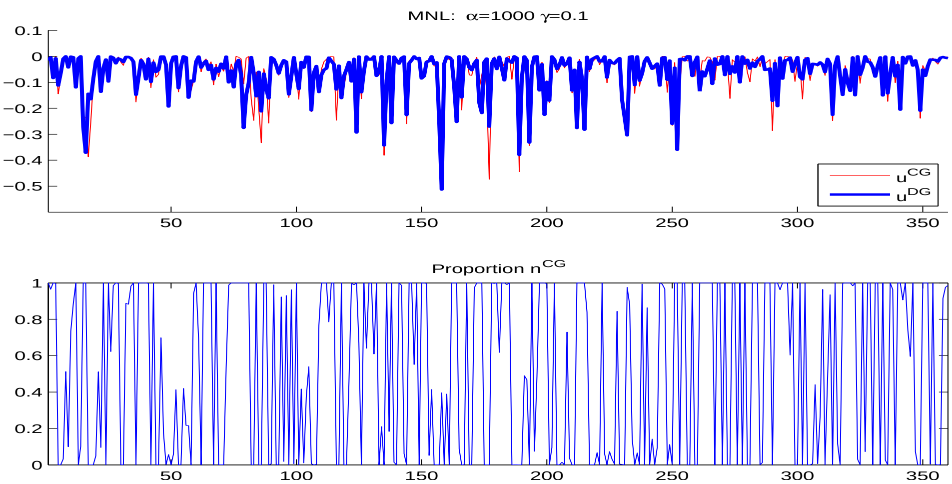

Figure 3:

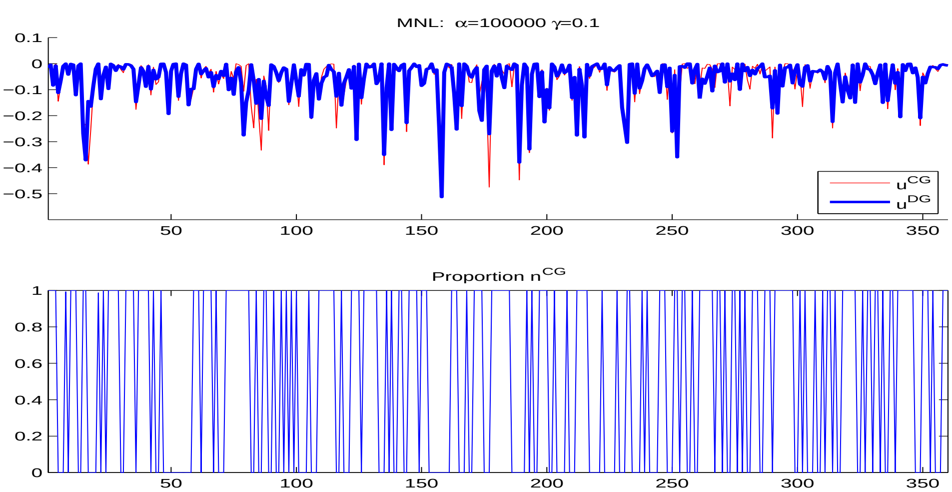

Figure 4:

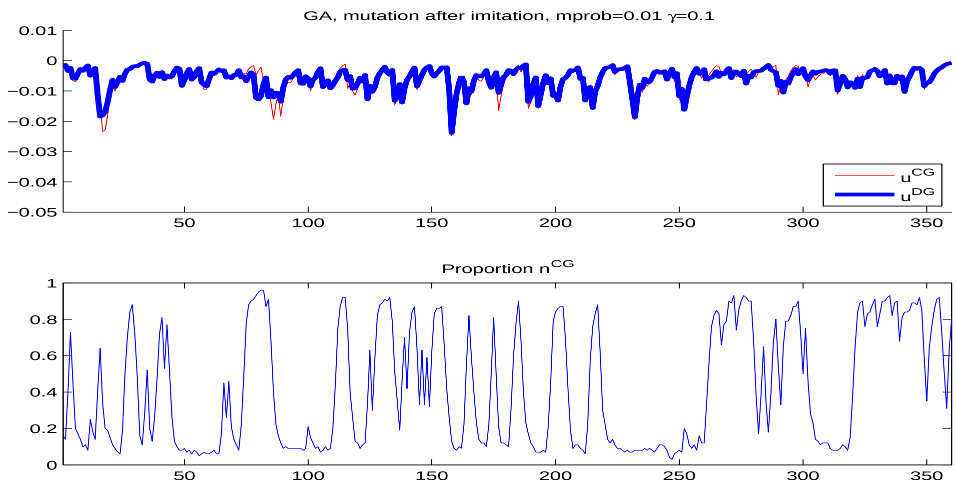

Figure 5:

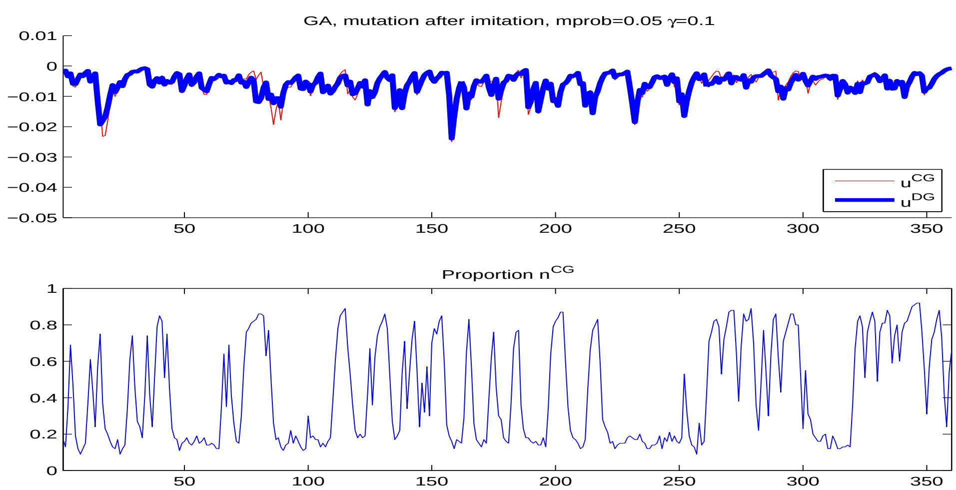

Figure 6:

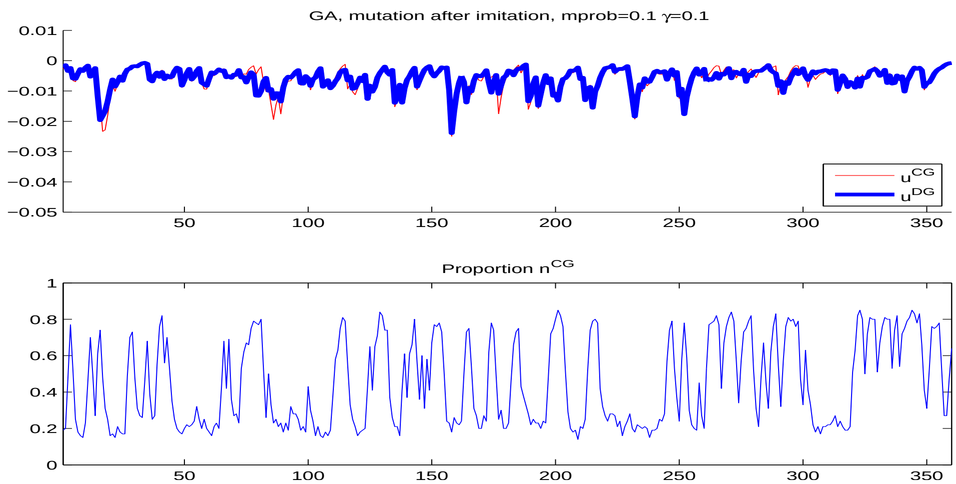

Figure 7:

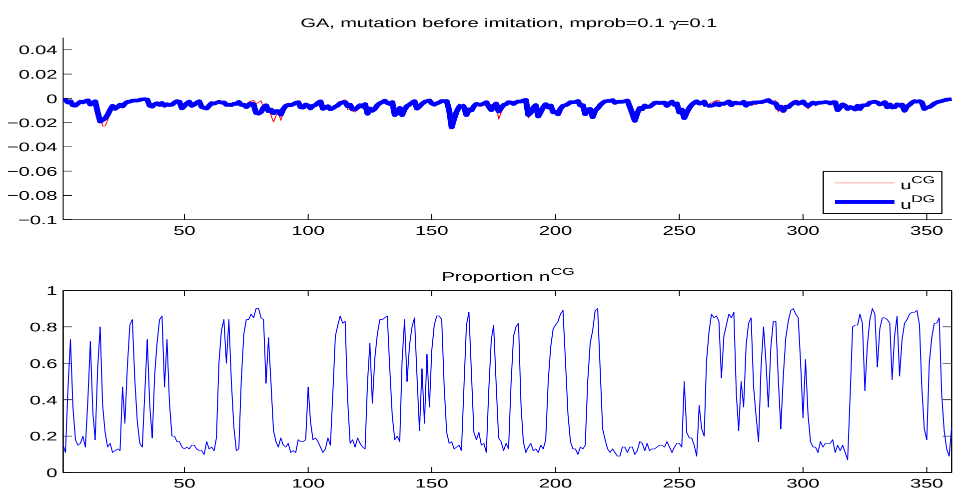

Figure 8:

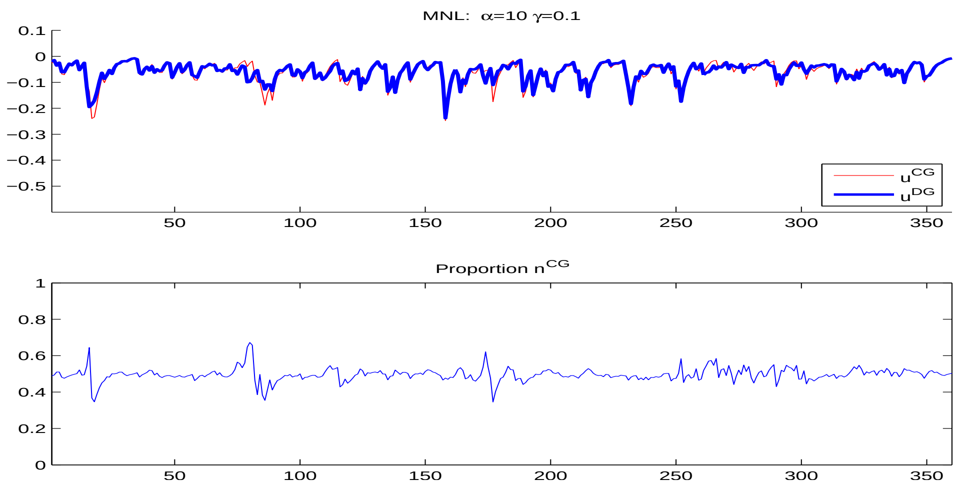

Figure 9:

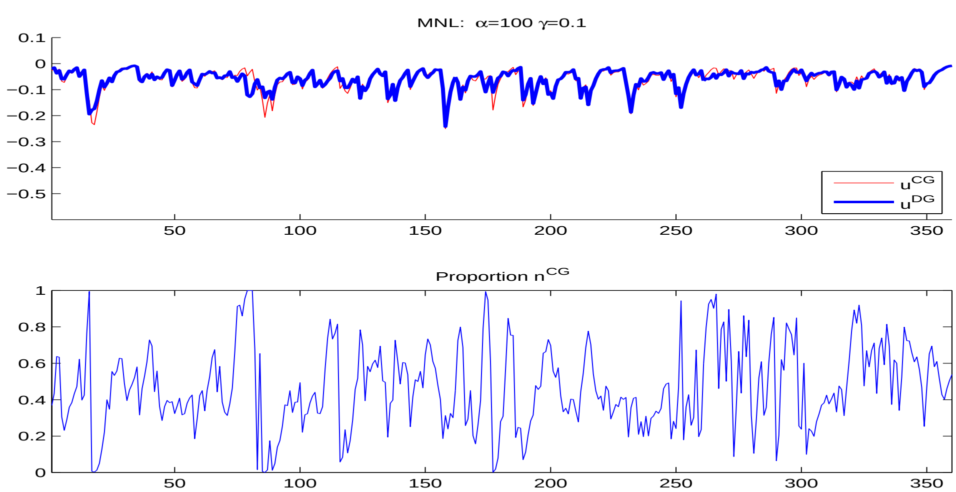

Figure 10:

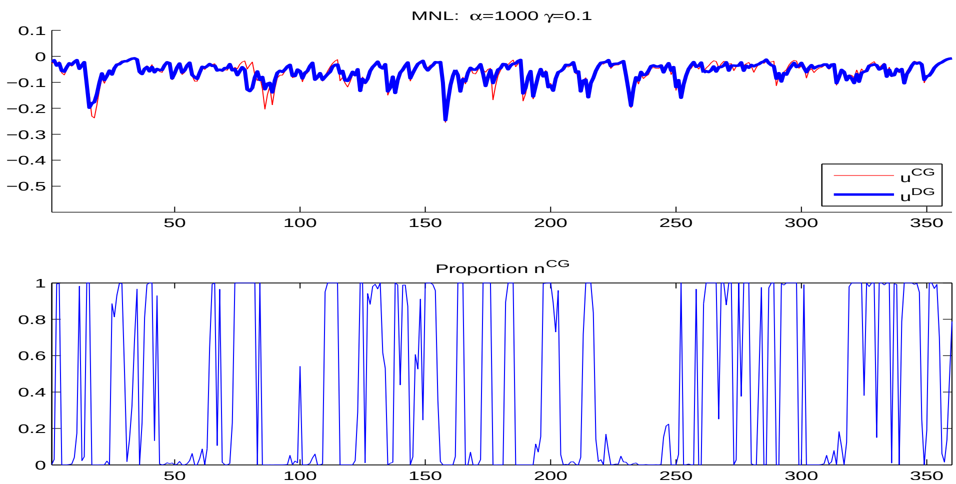

Figure 11:

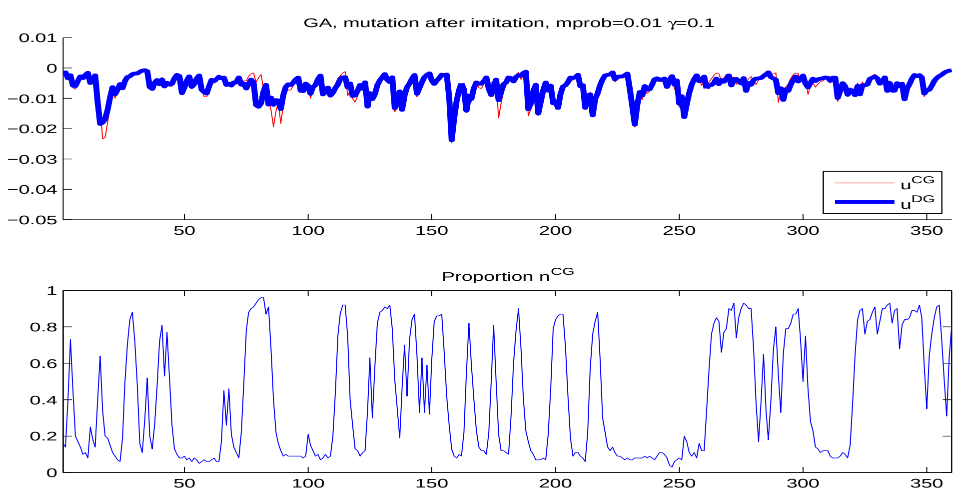

Figure 12:

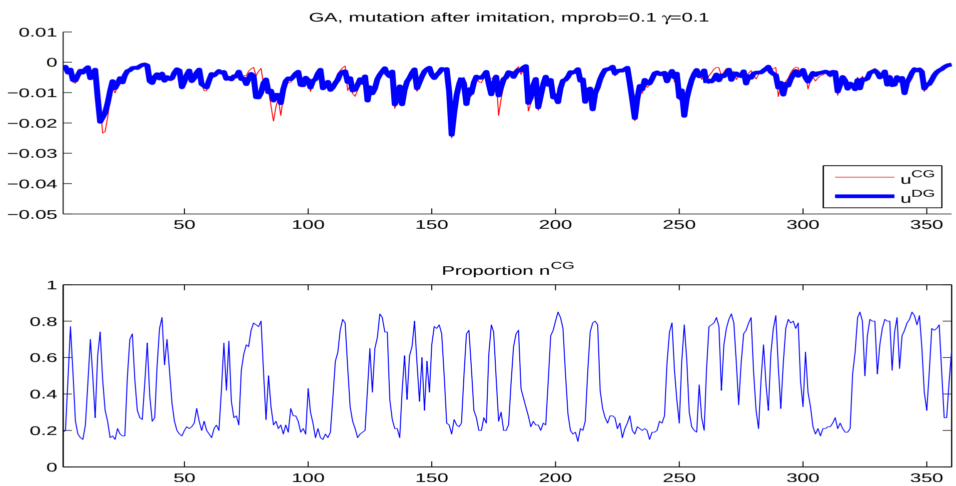
GA, mutation after imitation, mprob=0.1 γ=0.1
0.01
0
−0.01
−0.02
−0.03
−0.04
−0.05
50
100
150
200
250
300
350
u^CG
u^DG
Proportion n^CG
1
0.8
0.6
0.4
0.2
0
50
100
150
200
250
300
350

Figure 13: