① Physics 203 T10 May 29 lecture

Relativity continued.

"Addition of velocities"

If the bus moves at speed v_1 relative to us,
The passenger moves at speed v_2 relative to bus,
What is the speed of the passenger relative to us?

In classical theory, $v = v_1 + v_2$.

But in relativity theory, when v_1 and v_2 are comparable to the speed of light c, we need to use Lorentz transformations:

<u>event 1</u>: the bus is at $x=0$,
the passenger is at $x'=0$ and $x=0$
(the bus and the passenger are at 0)

event 2: the passenger is at
earth coordinate x,
earth time t.

Relative to the bus, the same event is:

event 2: the passenger is at
bus coordinate x'
bus time t'

② Obviously, $\frac{x'}{t'} = v_2$,

$$\frac{x}{t} = v, \longrightarrow x = vt$$

$$\begin{cases} x' = \dfrac{x}{\sqrt{1-\frac{v_1^2}{c^2}}} - \dfrac{v_1 t}{\sqrt{1-\frac{v_1^2}{c^2}}} \\ t' = \dfrac{-\frac{v_1}{c^2}x}{\sqrt{1-\frac{v_1^2}{c^2}}} + \dfrac{t}{\sqrt{1-\frac{v_1^2}{c^2}}} \end{cases}$$

Divide these equations by one another:

$$v_2 = \frac{x'}{t'} = \frac{\cancel{\dfrac{x}{\sqrt{1-\frac{v_1^2}{c^2}}}} - \cancel{\dfrac{v_1 t}{\sqrt{1-\frac{v_1^2}{c^2}}}}}{\cancel{\dfrac{-\frac{v_1}{c^2}x}{\sqrt{1-\frac{v_1^2}{c^2}}}} + \cancel{\dfrac{t}{\sqrt{1-\frac{v_1^2}{c^2}}}}} =$$

$$= \frac{v\cancel{t} - v_1\cancel{t}}{-\frac{v_1 v \cancel{t}}{c^2} + \cancel{t}} = \frac{v - v_1}{1 - \frac{v v_1}{c^2}}$$

Solve $v_2 = \dfrac{v - v_1}{1 - \frac{v v_1}{c^2}}$ for v:

$$\boxed{v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}} \longleftarrow \text{Relativistic addition of velocities}$$

Examples:

Somebody on a bus turns on the headlights, emitting light at speed c. How fast is the same light moving relative to us?

$$v_{bus} = v_1$$

$$v_{light/bus} = v_2 = c$$

$$v_{light/us} = v \text{ -?}$$

$$v = \frac{v_1 + c}{1 + \frac{v_1 c}{c^2}} = \frac{v_1 + c}{1 + \frac{v_1}{c}} = \frac{(v_1 + c)c}{\left(1 + \frac{v_1}{c}\right)c} =$$

$$= \frac{\cancel{(v_1 + c)}\, c}{\cancel{c + v_1}} = c \text{ !!!}$$

$$\boxed{\text{The speed of light is the same in all inertial systems!}}$$

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Another example:

A spaceship moving at $v_1 = 0.5c$ launches a rocket at $v_2 = 0.5c$ relative to the ship.

How fast is the rocket relative to us?

Naively, we would assume $0.5c + 0.5c = c$

— is it really going at a speed of light?

No. $$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} = \frac{0.5c + 0.5c}{1 + (0.5)^2 \frac{\cancel{c^2}}{\cancel{c^2}}} =$$

$$= \frac{c}{1 + 0.25} = \frac{4}{5} c = 80\% \text{ of } c.$$

$\frac{5}{4}$ "flip"

or, $0.8c$

What about subtraction of velocities?

It's ok to use negative numbers here:

If the rocket is launched backwards,

$$v_2 = -0.5c$$

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} = \frac{0.5c - 0.5c}{1 - (0.5)^2 \frac{\cancel{c^2}}{\cancel{c^2}}} = 0$$

(5)

What about energy and momentum?

They are still conserved (good news!)

But... the formulas have changed (bad news)

$$p = \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}$$

m_0 is rest mass, mass measured at $v=0$.

$$E_{kin} = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0 c^2 \approx$$

corrections

$$\approx \frac{m_0 v^2}{2} + \frac{3 m v^4}{8 c^2} + \frac{5 m v^6}{16 c^4} + \frac{35 m v^8}{128 c^6} + \dots$$

classical kin energy

But then Einstein thought, what if...

$p = mv$ is still correct, but the object is getting heavier as it accelerates!

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$ ↑ with speed

Then $E_{kin} = mc^2 - m_0 c^2$.

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That is, at rest the object must have an energy m_0c^2, and after acceleration it is mc^2. The difference, $(m - m_0)c^2$ is simply the kinetic energy.

$\boxed{\text{Any form of energy has mass}}$

$$\boxed{E = mc^2}$$

Examples.

As we calculated in the last term, a 50-kg atomic bomb "yields" $4 \cdot 10^{15}$ J of energy, assuming all uranium took part in explosion and fissioned (not fizzled!) What percentage of mass is "converted" to energy.

$m_0 = 50$ kg $\qquad E_{rest} = 4.5 \cdot 10^{18}$ J

$E_{kin} = 4 \cdot 10^{15}$ J

$$\frac{E_{kin}}{E_{rest}} = \frac{4 \cdot 10^{15}}{4.5 \cdot 10^{18}} = 9 \cdot 10^{-4},$$

about part per 1000, or 0.1%

(7) During the Chernobyl accident, one of the dominant (and most lethal) isotopes released was ^{131}I. It has a half-life of 8 days and decays into a stable ^{131}Xe and an electron. The energy released is 0.971 MeV. Calculate the speed of electron, and the mass of such an electron.

Assume most energy goes into electron (the ^{131}Xe is ~ 250000 times heavier)

Convert units: $1\,eV = 1.6 \cdot 10^{-19}\,J$

$0.971\,MeV = 1.55 \cdot 10^{-13}\,J$

$$\frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0 c^2 = 1.55 \cdot 10^{-13}\,J$$

$8.19 \cdot 10^{-14}\,J$

$$\frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} = 1.55 \cdot 10^{-13}\,J + m_0 c^2$$

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1.55 \cdot 10^{-13} + 8.18 \cdot 10^{-14}}{8.18 \cdot 10^{-14}} = 2.9$$

$$\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{2.9} = 0.345 \qquad v = 0.939\,c$$

94% of speed of light

(8) $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = 2.9\ m_0 \rightarrow$ almost 3 times heavier!

How fast the ^{131}Xe is shooting back?

Momentum is conserved!

$$p_e + p_{^{131}Xe} = 0$$

$$|p_e| = |p_{^{131}Xe}|$$

$$p_e = 2.9 \cdot m_0 v = 2.9\ m_0 c \cdot 0.939 =$$
$$= 7.44 \cdot 10^{-22}\ kg\ \frac{m}{s}$$

Assume ^{131}Xe is moving much slower than speed of light.

$$\cancel{mv^2} \quad mv = 7.44 \cdot 10^{-22}\ kg \frac{m}{s}$$

$$v = \frac{7.44 \cdot 10^{-22}\ kg\ \frac{m}{s}}{\underbrace{131 \cdot 0.001 \cdot kg\ \frac{1}{6.02 \cdot 10^{23}}}_{2.18 \cdot 10^{-25}\ kg}} = 3420\ \frac{m}{s}$$

$$= 3.4\ \frac{km}{s}$$

$$\ll c$$

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Black holes and such...

Consider non-rotating spherical mass M (such as Earth).

The time on such a planet "flows slower" than far away from it,

$$t_{slow} = t_{far}\sqrt{1-\frac{r_0}{r}},$$

where r is the distance to the center,

$r_0 = \frac{2GM}{c^2}$ is the Schwarzschild radius of mass M.

G is the Newton's gravitational constant $= 6.67384 \times 10^{-11} \frac{N\,m^2}{kg^2}$

If the "planet" is smaller than r_0, the time on it stops, and it becomes a black hole (also, the light can't get out)

For earth, $\frac{2 \cdot 6.67 \cdot 10^{-11} (\,) \cdot 5.97\ 10^{24} kg}{(3 \cdot 10^8 \frac{m}{s})^2} =$

$= 0.00887m \approx 0.9\ cm \approx \frac{1}{3}$ inches

(10) Is flying on an airplane going to delay or accelerate an atomic clock?

$v = 500 \text{ mph} \approx 800 \frac{km}{h} \approx 225 \frac{m}{s}$

$h = 35000 \text{ ft} \approx 10 \text{ km} = 10^4 \text{ m}$

$r_{earth} = 6.4 \cdot 10^6 \text{ m}$

Dilation (slowing down) due to speed:

$$t_0 = t\sqrt{1 - \frac{v^2}{c^2}} = 0.9999999999997\, t$$

$$= \left(1 - 2.8 \cdot 10^{-13}\right) t$$

0.28 parts per trillion !!!

Dilation (slowing down) on Earth due to gravity

$$t_0 = t_\infty \sqrt{1 - \frac{r_0}{r}} = t_\infty \sqrt{1 - \frac{0.00887}{6.4 \cdot 10^6}}$$

$$= 0.9999999993 \ldots$$

$$= \left(1 - 6.9297 \cdot 10^{-10}\right)$$

(11) Dilation on an airplane:

$$t_{air} = t_\infty \sqrt{1 - \frac{r_0}{(r + 10^4 m)}} =$$

$$= \left(1 - 6.919 \cdot 10^{-10}\right) t_\infty$$

The difference is net "acceleration":

$$\frac{t_0}{t_{air}} = \frac{\left(1 - 6.9297 \cdot 10^{-10}\right)}{\left(1 - 6.919 \cdot 10^{-10}\right)} \approx 1 - \left(6.9297 \cdot 10^{-10} - 6.919 \cdot 10^{-10}\right)$$

$$= 1 - \underbrace{0.010 \cdot 10^{-10}}_{1 \cdot 10^{-12}}$$

↳ part per trillion.

So, the atomic clock on the airplane slows down by $\sim \frac{1}{4}$ parts per trillion due to speed and accelerates by ~ 1 part per trillion due to being in "less gravity" environment.

The net effect is acceleration by $\frac{3}{4}$ parts per trillion.