**Random Effects Likelihood Ratio Test Examples**

The result of maximum likelihood estimation is a -2 log likelihood value, which is a summary of the fit of the observed to the expected values. These values can be used for comparing different models that are nested (see the "Significance Testing in Multilevel Regression" handout). The difference in likelihood values can be evaluated against the chi-square distribution for significance—the likelihood ratio test. If the models differ only in the random effects, REML estimation is fine. If the fixed effects differ at all, then full ML should be used.

**SPSS**

To illustrate the likelihood ratio test approach, I use the HSB data to compare the model with SES as a level-1 predictor (uncentered) with varying slopes.¹

**Test of Slope Variance using the Wald Test**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Std. Error</th>
<th>Wald Z</th>
<th>Sig.</th>
<th>95% Confidence Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Lower Bound</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Upper Bound</td>
</tr>
<tr>
<td>Residual</td>
<td>36.630165</td>
<td>.629312</td>
<td>58.524</td>
<td>.000</td>
<td>35.617160</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>38.084480</td>
</tr>
<tr>
<td>Intercept + ses [subject = schooid]</td>
<td>4.828637</td>
<td>.672123</td>
<td>7.184</td>
<td>.000</td>
<td>3.675711</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>6.343190</td>
</tr>
<tr>
<td></td>
<td>-1.54275</td>
<td>.672123</td>
<td>-5.161</td>
<td>.000</td>
<td>-.739984</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>.431434</td>
</tr>
<tr>
<td></td>
<td>.412929</td>
<td>.250141</td>
<td>1.757</td>
<td>.079</td>
<td>-.135340</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.259873</td>
</tr>
</tbody>
</table>

¹ Dependent Variable: mathach.

The UN() notation refers to the rows and columns of the variance-covariance matrix. The UN(1,1) row refers to the intercept variance, because \( \beta_{j0} \) is the first parameter and an element with the same row and column number refers to the variance, called \( \tau^2_0 \) in the text. The UN(2,2) row refers to the second row and second column of the variance-covariance matrix, so if \( \beta_{j0} \) is the first row and first column and \( \beta_{j1} \) is the second row and second column, then UN(2,2) refers to the variance of the \( \beta_{j1} \) slope, known as \( \tau^2_1 \). UN(2,1) then refers to the covariance between the intercept \( \beta_{j0} \) and the slope \( \beta_{j1} \), known as \( \pi_{01} \). So, the table indicates that the intercept variance is significant, even after halving the p-value (one-tailed p-value = .000/2 = <.01 at least), the variance of the slope is significant with a one-tailed test (.079/2 = .0395). The p-value for the covariance between the intercept and slope \( p = .606 \) (two-tailed test that is not halved) is not significant.

**Likelihood Ratio Test of the Just the Covariance**

It may be of interest, particular with lower power, to examine the likelihood ratio test of the covariance between intercept and slope even though a Wald test is automatically provided in the "Estimates of Covariance Parameters" box when COVTYPE(UN) is used on the RANDOM subcommand. A model without the covariance is tested by specifying COVTYPE(VC), for variance components, on the RANDOM subcommand.

Here is the original model with the random effect for slope and the intercept-slope covariance estimated because COVTYPE(UN) was used.

```
MIXED mathach WITH ses
/METHOD = REML
/PRINT = SOLUTION TESTCOV
/FIXED = ses | SSTYPE(3)
/RANDOM = INTERCEPT ses | SUBJECT(schooid) COVTYPE(UN).
```

¹ I do not generally recommend uncentered predictors in most circumstances, but SES in the HSB data set was pre-standardized so it has a mean of zero.
The model is retested using a different specification on the **RANDOM** subcommand **COVTYPE (VC)** which requests a diagonal matrix assuming covariance between intercept and slope is equal to 0 (not a reasonable assumption usually).

MIXED mathach WITH ses  
/CRITERIA=MXITER(1000) SCORING(1)  
/METHOD = REML  
/PRINT = SOLUTION TESTCOV HISTORY  
/FIXED = ses | SSTYPE(3)  
/RANDOM = INTERCEPT ses | SUBJECT(schoolid) COVTYPE(VC).

In the results, notice that there is no covariance estimate—the UN(2,1) value is absent.

The likelihood ratio test subtracts the -2 log likelihood value for the previous model with the covariance estimated (same as D1 below), from this more restricted model 46640.398 with the covariance not estimated (set to 0), 46640.663. The resulting chi-square test can be compared to a standard chi-square table. The difference in this case is not significant, \( \chi^2(1) = .265, \text{ ns.} \)

Likelihood Ratio Test of the Slope Variance and Intercept-Slope Covariance Together

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Std. Error</th>
<th>Wald Z</th>
<th>Sig.</th>
<th>95% Confidence Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>Residual</td>
<td>36.822862</td>
<td>.628871</td>
<td>58.553</td>
<td>.000</td>
<td>35.610122 - 38.075712</td>
</tr>
<tr>
<td>Intercept</td>
<td>4.852841</td>
<td>.673828</td>
<td>7.202</td>
<td>.000</td>
<td>3.696621 - 6.370702</td>
</tr>
</tbody>
</table>

\( ^a \) Dependent Variable: mathach.

D\(_a\) model in which slope is non-varying

MIXED mathach WITH ses  
/CRITERIA=MXITER(1000) SCORING(1)  
/METHOD = REML  
/PRINT = SOLUTION TESTCOV HISTORY  
/FIXED = ses | SSTYPE(3)  
/RANDOM = INTERCEPT | SUBJECT(schoolid) COVTYPE(UN).

\( ^a \) Dependent Variable: mathach.

D\(_1\) model in which slope is allowed to vary

MIXED mathach WITH ses  
/CRITERIA=MXITER(1000) SCORING(1)  
/METHOD = REML  
/PRINT = SOLUTION TESTCOV HISTORY  
/FIXED = ses | SSTYPE(3)  
/RANDOM = INTERCEPT ses | SUBJECT(schoolid) COVTYPE(UN).

\( ^a \) Dependent Variable: mathach.
Because p-values are halved for variances but not for covariances a mixture of the two p-values is needed. I did the adjustments using a spreadsheet (available on the class website).

\[ D_0 - D_1 = 46645.163 - 46640.398 = 4.765 \]  This value does not exceed the 5.14 chi-square cutoff value from the mixture ("chi-bar") chi-square critical value in Snijders & Bosker (2012, p. 99), so the test of the slope and covariance between slope and intercept, tested together, is not significant.

**R**

```r
> library(lme4)
> #install.packages("lmerTest")
> library(lmerTest)  # lmerTest generates Satterthwaitte df with summary.function
> model1 <- lmer(mathach ~ ses + (ses|schoolid), data = mydata, REML = TRUE)
> summary(model1)
```

**Linear mixed model fit by REML**

**Formula:**  

\[
\text{mathach} \sim \text{ses} + (\text{ses} | \text{schoolid})
\]

**Data:** mydata

**REML criterion at convergence:** 46640.4

**Scaled residuals:**

\[
\begin{align*}
\text{Min} & \quad 10 \\
\text{1Q} & \quad 0.73046 \\
\text{Median} & \quad 0.02144 \\
\text{3Q} & \quad 0.75610 \\
\text{Max} & \quad 2.94356
\end{align*}
\]

**Random effects:**

\[
\begin{array}{llll}
\text{Groups} & \text{Name} & \text{Variance} & \text{Std. Dev.} & \text{Corr} \\
\text{schoolid} (\text{Intercept}) & 4.8286 & 2.1974 & \\
\text{ses} & 0.4129 & 0.6426 & -0.11 \\
\text{Residual} & 36.8302 & 6.0688 & \\
\end{array}
\]

**Number of obs:** 7185, **groups:** schoolid, 160

**Fixed effects:**

\[
\begin{array}{lllllll}
\text{Estimate} & \text{Std. Error} & \text{df} & \text{t value} & \text{Pr(>|t|)} \\
(\text{Intercept}) & 12.6650 & 0.1898 & 145.5500 & 66.71 & <0.0000000000000002 \\
\text{ses} & 2.3938 & 0.1181 & 157.5300 & 20.27 & <0.0000000000000002 \\
\end{array}
\]

**Correlation of Fixed Effects:**

\[
\begin{align*}
\text{(Intr)} & \quad -0.045 \\
\text{ses} & \quad \\
\end{align*}
\]

Although the profile likelihood confidence intervals can be obtained with the `confint()` function (demonstrated previously) to test the covariance for significance, the `anova` function from the `lmerTest` package can be used to perform a likelihood ratio test to compare two nested models. Note that it retests the models using full maximum likelihood, which is not necessary when just random effects differ but should be ok with larger sample sizes.

```r
> library(lmerTest)
> # sets the covariance equal 0 (diagonal matrix of random effects)
> model2 <- lmer(mathach ~ ses + (1|schoolid) + (0+ses|schoolid), data=mydata)
> summary(model2)
```

**Random effects:**

\[
\begin{array}{llll}
\text{Groups} & \text{Name} & \text{Variance} & \text{Std. Dev.} \\
\text{schoolid} (\text{Intercept}) & 4.853 & 2.2029 \\
\text{schoolid} \cdot \text{ses} & 0.424 & 0.6511 \\
\text{Residual} & 36.822 & 6.0681 \\
\end{array}
\]

**Number of obs:** 7185, **groups:** schoolid, 160

```r
> # conducts a LR comparison of the unconstrained and constrained models
> anova(model, model2)
```

**Refitting model(s) with ML (instead of REML)**

**Data:** mydata

**Model s:**

\[
\begin{array}{llllllllll}
\text{df} & \text{AIC} & \text{BIC} & \text{logLik} & \text{deviance} & \text{Pr(>Chi sq)} \\
\text{model 2} & 5 & 46647 & 46681 & -23318 & 46637 \\
\text{model 1} & 6 & 46648 & 46690 & -23318 & 46636 & 0.2762 & 1.0000 & 0.5992 \\
\end{array}
\]

**Likelihood Ratio Test of the Slope Variance and Intercept-Slope Covariance Together**

```r
> library(lmerTest)
> # rand function reports LR comparison to intercept only model using mixture chi-square
> rand(model1)
```

**Analysis of Random effects Table:**

\[
\begin{array}{llllll}
\text{Chi.sq} & \text{Chi.Df} & \text{Pr(>|Chi sq|)} \\
\text{ses:schoolid} & 4.77 & 2 & 0.09 \\
\end{array}
\]
HLM
The HLM package provides a method of likelihood ratio tests by requesting a comparison of a current model with a prior model. The first step is to test the full model, which is not shown but was exactly as the model demonstrated earlier with random slopes for SES specified and the covariance between intercept and slope estimated by default.

Likelihood Ratio Test of the Intercept-Slope Covariance
Under the Other Settings menu and then Estimation Settings one can check the box Diagonalize Tau, which sets all covariances equal to zero. When that model is estimated, the covariance between the intercept and slope will be set to 0. HLM automatically generates the likelihood ratio test when this box is checked.

\[
\begin{align*}
\tau^2 & \quad \text{INTRCPT1, } \beta_0 \quad 4.85319 \quad 0.00000 \\
\text{SES, } \beta_1 & \quad 0.00000 \quad 0.42574 \\
\end{align*}
\]

Variance-Covariance components test

\[
\chi^2 \text{ statistic} = 0.26428 \\
\text{Degrees of freedom} = 3 \\
\text{p-value} = >.500
\]

Likelihood Ratio Test of the Slope Variance and Intercept-Slope Covariance Together
To obtain the likelihood ratio test below, I first tested the model with non-varying slopes. The deviance and number of parameters from this model \((D_0)\) was:

Deviance = 46643.331427 \\
Number of estimated parameters = 2

I then enter these values into the Hypothesis Testing dialog box under the Other Settings menu when I test a new model \((D_1)\) that allows the SES slopes to vary across schools.

Test of Slope Variance Using the HLM Chi-square Parameter Test
Here are the results:

\[
\sigma^2 = 36.82835
\]

\[
\tau
\]

\[
\begin{align*}
\text{INTRCPT1, } \beta_0 & \quad 4.82978 \quad -0.15399 \\
\text{SES, } \beta_1 & \quad -0.15399 \quad 0.41828 \\
\end{align*}
\]

Final estimation of variance components

<table>
<thead>
<tr>
<th>Random Effect</th>
<th>Standard Deviation</th>
<th>Variance Component</th>
<th>d.f.</th>
<th>(\chi^2)</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRCPT1, u_0</td>
<td>2.19768</td>
<td>4.82978</td>
<td>159</td>
<td>905.26472</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>SES slope, u_1</td>
<td>0.64675</td>
<td>0.41828</td>
<td>159</td>
<td>216.21178</td>
<td>0.002</td>
</tr>
<tr>
<td>level-1, r</td>
<td>6.06864</td>
<td>36.82835</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The result of the model with varying slopes had a significant intercept variance, \(\chi^2 = 905.26472, p < 0.001\) and slope variance \(\chi^2 = 216.21178, p = 0.002\). p-values are not halved in HLM. No test of the covariance between the intercept and the slope is printed.

Statistics for current covariance components model
Deviance = 46638.560929 \\
Number of estimated parameters = 4

Variance-Covariance components test

\[
\chi^2 \text{ statistic} = 4.77050 \\
\text{Degrees of freedom} = 2 \\
\text{p-value} = 0.090
\]

The chi-square represents the \(D_0 - D_1\) difference, which this test indicates is non-significant. No mixture adjustment is needed as the HLM tests already makes a comparable adjustment.