Random Slopes Example

This example again uses the HSB data with math achievement test scores as the outcome. SES is used as a predictor (covariate) as it was in the prior example (see the handout "ANCOVA Example (One Level-1 Predictor Assuming Homogeneous Slopes): SPSS, R, and HLM"). For this model, the slopes are allowed to vary across schools, so the relationship between SES and math achievement is not assumed to be the same in every school.

```
MIXED mathach WITH ses
  /METHOD = REML
  /PRINT = SOLUTION TESTCOV
  /FIXED =  ses | SSTYPE(3)
  /RANDOM = INTERCEPT  ses | SUBJECT(schoolid) COVTYPE(UN).
```

Mixed Model Analysis

Fixed Effects

Type III Tests of Fixed Effects [a]

Source	Numerator df	Denominator df	F	Sig.
Intercept	1	145.568	4450.850	<.001
ses	1	157.544	410.725	<.001

a. Dependent Variable: mathach.

Estimates of Fixed Effects [a]

Parameter	Estimate	Std. Error	df	t	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Intercept	12.665	.190	145.568	66.715	<.001	12.290	13.040
ses	2.394	.118	157.544	20.266	<.001	2.161	2.627

a. Dependent Variable: mathach.

Covariance Parameters

Estimates of Covariance Parameters [a]

Parameter		Estimate	Std. Error	Wald Z	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Residual		36.830	.629	58.524	<.001	35.617	38.084
Intercept + ses [subject = schoolid]	UN (1,1)	4.828	~~.672~~	~~7.184~~	~~<.001~~	~~3.675~~	~~6.343~~
	UN (2,1)	-.154	.299	-.517	.605	-.740	.431
	UN (2,2)	.413	~~.235~~	~~1.757~~	~~.079~~	~~.135~~	~~1.260~~

a. Dependent Variable: mathach.

Variance tests should be one-tailed, so halve p-value or use 90% CIs and p-value from separate run for variance tests – covariance test and CIs are correct though

Retest model to obtain the 90% confidence limits

```
MIXED mathach WITH ses
  /METHOD = REML
  /PRINT = SOLUTION TESTCOV
  /FIXED =  ses | SSTYPE(3)
  /RANDOM = INTERCEPT  ses | SUBJECT(schoolid) COVTYPE(UN).
  /CRITERIA=CIN(90).
```

Estimates of Covariance Parameters[a]

Parameter		Estimate	Std. Error	Wald Z	Sig.	90% Confidence Interval Lower Bound	90% Confidence Interval Upper Bound
Residual		37.409577	.636250	58.797	.000	36.377542	38.470890
Intercept + minority [subject = schoolid]	UN (1,1)	5.621392	.802254	7.007	.000	4.445245	7.108731
	UN (2,1)	.913528	~~.706185~~	~~1.294~~	~~.196~~	~~-.248044~~	~~2.075099~~
	UN (2,2)	3.241937	1.117177	2.902	.004	1.839239	5.714404

a. Dependent Variable: mathach.

R

```
> #random effects ANCOVA model (hetergeneous slopes)
> model2 <- lme(mathach ~ ses, random = ~ ses|schoolid, data = mydata, method="REML")
> summary(model2)
Linear mixed-effects model fit by REML
  Data: mydata
      AIC      BIC   logLik
  46652.4 46693.68 -23320.2

Random effects:
 Formula: ~ses | schoolid
 Structure: General positive-definite, Log-Cholesky parametrization
            StdDev    Corr
(Intercept) 2.1974167 (Intr)
ses         0.6425964 -0.109
Residual    6.0687860

Fixed effects:  mathach ~ ses
                Value Std.Error   DF  t-value p-value
(Intercept) 12.665023  0.189845 7024 66.71244       0
ses          2.393813  0.118121 7024 20.26576       0
 Correlation:
    (Intr)
ses -0.045

Standardized Within-Group Residuals:
        Min          Q1         Med          Q3         Max
-3.12272309 -0.73045840  0.02144271  0.75610145  2.94356058

Number of Observations: 7185
Number of Groups: 160
> #nlme provides standard deviations of the random effects by default, use VarCorr to obtain variances
> VarCorr(model2)
schoolid = pdLogChol(ses)
            Variance   StdDev    Corr
(Intercept)  4.8286403 2.1974167 (Intr)
ses          0.4129302 0.6425964 -0.109
Residual    36.8301638 6.0687860
> #obtain confidence intervals for fixed and random effects (in SD units), similar to SPSS values
> intervals(model2)
Approximate 95% confidence intervals

 Fixed effects:
               lower      est.     upper
(Intercept) 12.29287 12.665023 13.037177
ses          2.16226  2.393813  2.625366

 Random Effects:
  Level: schoolid
                          lower       est.     upper
sd((Intercept))       1.9171925  2.1974167 2.5185996
sd(ses)               0.3685976  0.6425964 1.1202737
cor((Intercept),ses) -0.6683121 -0.1092559 0.5286711

 Within-group standard error:
   lower     est.    upper
5.967913 6.068786 6.171364
> intervals(model2,.90,"var-cov")
Approximate 90% confidence intervals

 Random Effects:
  Level: schoolid
                          lower       est.     upper
sd((Intercept))       1.9597065  2.1974167 2.4639610
sd(ses)               0.4030519  0.6425964 1.0245088
cor((Intercept),ses) -0.6014832 -0.1092559 0.4430989

 Within-group standard error:
   lower     est.    upper
5.984017 6.068786 6.154756
```

Variance tests should be one-tailed, so halve p-value or use 90% CIs from separate run for variance tests – covariance test and CIs are correct

HLM

(variance tests are correct in the output)

The outcome variable is MATHACH

Summary of the model specified
Step 2 model
Level-1 Model

$MATHACH_{ij} = \beta_{0j} + \beta_{1j}*(SES_{ij}) + r_{ij}$

Level-2 Model

$\beta_{0j} = \gamma_{00} + u_{0j}$
$\beta_{1j} = \gamma_{10} + u_{1j}$

Mixed Model

$MATHACH_{ij} = \gamma_{00}$
$+ \gamma_{10}*SES_{ij} + u_{0j} + u_{1j}*SES_{ij} + r_{ij}$

Final Results - Iteration 21
Iterations stopped due to small change in likelihood function

$\sigma^2 = 36.82835$

τ

INTRCPT1, β_0	4.82978	-0.15399
SES, β_1	-0.15399	0.41828

τ (as correlations)

INTRCPT1, β_0	1.000	-0.108
SES, β_1	-0.108	1.000

Random level-1 coefficient	Reliability estimate
INTRCPT1, β_0	0.797
SES, β_1	0.179

The value of the log-likelihood function at iteration 21 = -2.331928E+04

Final estimation of fixed effects:

Fixed Effect	Coefficient	Standard error	*t*-ratio	Approx. *d.f.*	*p*-value
For INTRCPT1, β_0					
INTRCPT2, γ_{00}	12.664935	0.189874	66.702	159	<0.001
For SES slope, β_1					
INTRCPT2, γ_{10}	2.393878	0.118278	20.240	159	<0.001

Final estimation of fixed effects
(with robust standard errors)

Fixed Effect	Coefficient	Standard error	*t*-ratio	Approx. *d.f.*	*p*-value
For INTRCPT1, β_0					
INTRCPT2, γ_{00}	12.664935	0.189251	66.921	159	<0.001
For SES slope, β_1					
INTRCPT2, γ_{10}	2.393878	0.117697	20.339	159	<0.001

Final estimation of variance components

Random Effect	Standard Deviation	Variance Component	*d.f.*	χ^2	*p*-value
INTRCPT1, u_0	2.19768	4.82978	159	905.26472	<0.001
SES slope, u_1	0.64675	0.41828	159	216.21178	0.002
level-1, r	6.06864	36.82835			

Statistics for current covariance components model

Deviance = 46638.560929
Number of estimated parameters = 4

Example Write-up.
A multilevel regression model using robust standard errors was tested to investigate the association between SES and math achievement, allowing the slopes to vary across schools. The average math achievement score was 12.66 for those with average SES. Math achievement also significantly varied across schools, τ_0^2 = 4.829, χ^2(159) = 905.26, p < .001. The model indicated that SES was significantly related to math achievement on average, γ_{10} = 2.39, SE_γ = .118, γ_{10}^* = .27, p < .001, but slopes significantly varied across schools, τ_1^2 = .418, χ^2(159) = 216.21, p = .002, suggesting that the SES does not have the same relationship to achievement in every school.

ANCOVA Allowing Slope Heterogeneity
(This analysis can also be interpreted as an ANCOVA in which the assumption of homogeneous slopes is relaxed. The significant variance of the slopes suggests that the assumption of homogeneous slopes in the ANCOVA is violated. The following write-up would be equally acceptable and would be used if the focus was on the group differences after controlling for a covariate, perhaps added onto the write-up from the prior handout on ANCOVA with homogeneous slopes.) To investigate whether the homogeneous slopes assumption was supported by the data, an additional model was tested allowing the association between SES and math achievement to vary across schools. This model also indicated that SES was significantly related to math achievement on average, γ_{10} = 2.39, SE_γ = .118, γ_{10}^* = .27, p < .001. Results indicated that slopes did significantly vary across schools, τ_1^2 = .418, χ^2(159) = 216.21, p = .002, suggesting that the SES does not have the same relationship to achievement in every school and that the assumption of homogeneous slopes was violated.[1] The between-school variance in math achievement controlling for ethnicity remained significant, τ_0^2 = 4.829, χ^2(159) = 905.26, p < .001, even after allowing for slope heterogeneity.[2]

Note on standardized slopes
Standardized values for slopes are rarely reported for multilevel regression models, but there is no reason not to report them. The primary reason seems to be that none of the software programs compute the standardized coefficients automatically. Without standardized coefficients (and R^2 values, which we will discuss later), there is little information about magnitude of effect the way multilevel models are currently reported. The computation of the standardized slope is the same as with OLS regression (Snijders & Bosker, 2012, p. 53) and I've just used the values for the standard deviations (*S.D.*) reported in the descriptive results from the HSB study handout "Raudenbush & Bryk Example Data Set: High School & Beyond" and use γ_{10}^* for the standardized coefficient:

$$\gamma_{10}^* = \gamma_{10}\left[\frac{S.D.(X)}{S.D.(Y)}\right] = 2.39\left(\frac{.78}{6.88}\right) = .27$$

[1] Another way to test the slope variance or the homogeneity of slopes assumption is to compare fit of nested of the two models (with and without homogeneous slopes). We will discuss this further later in the course.
[2] The SES variable is standardized so the mean is 0 and thus the intercept values is interpreted as when SES is at its mean. We will discuss predictor scaling later.