THE FERMAT-TORRICELLI PROBLEM IN THE LIGHT OF CONVEX ANALYSIS

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Abstract. In the early 17th century, Pierre de Fermat proposed the following problem: given three points in the plane, find a point such that the sum of its Euclidean distances to the three given points is minimal. This problem was solved by Evangelista Torricelli and was named the *Fermat-Torricelli problem.* A more general version of the Fermat-Torricelli problem asks for a point that minimizes the sum of the distances to a finite number of given points in $\mathbb{R}^n$. This is one of the main problems in location science. In this paper, we revisit the Fermat-Torricelli problem from both theoretical and numerical viewpoints using some ingredients of of convex analysis and optimization.

Key words. Fermat-Torricelli problem, Weiszfeld's algorithm, Kuhn's proof, convex analysis, subgradients

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1 Introduction

The Fermat-Torricelli problem asks for a point that minimizes the sum of the distances to three given points in the plane. This problem was proposed by Fermat and solved by Torricelli. Torricelli's solution states as follows: If one of the angles of the triangle formed by the three given points is greater than or equal to 120°, the corresponding vertex is the solution of the problem. Otherwise, the solution is a unique point inside of the triangle formed by the three points such that each side is seen at an angle of 120°. The first numerical algorithm for solving the general Fermat-Torricelli problem was introduced by Weiszfeld in 1937 [18]. The assumptions that guarantee the convergence along with the proof were given by Kuhn in 1972. Kuhn also pointed out an example in which the Weiszfeld's algorithm fails to converge; see [8]. The Fermat-Torricelli problem has attracted great attention from many researchers not only because of its mathematical beauty, but also because of its important applications to location science. Many generalized versions of the Fermat-Torricelli and several new algorithms have been introduced to deal with generalized Fermat-Torricelli problems as well as to improve the Weiszfeld's algorithm; see, e.g., [2, 9, 10, 12, 16, 17]. The problem has also been revisited several times from different viewpoints; see, e.g., [4, 5, 13, 19] and the references therein. In this picture, our goal is not to produce any new result, but to provide easy access to the problem from both theoretical and numerical aspects using some tools of convex analysis. These tools are presented in the paper by elementary proofs that are understandable for students with basic background in introduction to analysis.

The paper is organized as follows. In Section 2, we proof the existence and uniqueness of the optimal solution. We also present the proofs of properties of the optimal solution

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as well as its its construction using *convex subdifferential*. The advantage of using convex analysis when solving the Fermat-Torricelli problem has been observed in many books on convex and nonsmooth analysis; see, e.g., [3, 7, 15] and the references therein. Section 3 is devoted to revisiting Kuhn's proof of the convergence of the Weiszfeld's algorithm. In this section we follow the theme to proof the convergence given by Kuhn [8], but we include some ingredients from convex analysis to replace for some technical tools in order to make the proof more clear.

Throughout the paper, $I\!B$ denotes the closed unit ball of $\mathbb{R}^n$; $I\!B(\bar{x}; r)$ denotes the closed ball with center $\bar{x}$ and radius r.

2 Elements of Convex Analysis and Properties of Solutions

In this section, we review important concepts of convex analysis to study the classical Fermat-Torricelli problem as well as the problem in the general form. We also present element proofs for some properties of optimal solutions of the problem. More details of convex analysis can be found, for instance, in [14].

Let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^n$. Given a finite number of points a_i for $i = 1, \ldots, m$ in $\mathbb{R}^n$, define

$$\varphi(x) := \sum_{i=1}^{m} \|x - a_i\|. \tag{2.1}$$

Then the mathematical model of the Fermat-Torricelli problem is

$$\text{minimize } \varphi(x) \text{ subject to } x \in \mathbb{R}^n. \tag{2.2}$$

The *weighted version* of the problem can be formulated and treated by a similar way.

Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be an real-valued function. The *epigraph* of f is a subset of $\mathbb{R}^n \times \mathbb{R}$ defined by

$$\text{epi } f := \big\{(x, \alpha) \in \mathbb{R}^{n+1} \;\big|\; x \in \mathbb{R}^n \;\text{ and }\; \alpha \ge f(x)\big\}.$$

The function f is called *convex* if

$$f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda) f(y) \;\text{ for all }\; x, y \in \mathbb{R}^n \;\text{ and }\; \lambda \in (0,1).$$

If this equality becomes strict for $x \neq y$, we say that f is *strictly convex*. We can prove that f is a convex function on $\mathbb{R}^n$ if and only if its epigraph is a convex set in $\mathbb{R}^{n+1}$.

It is clear that the function φ given by (2.1) is a convex function.

Proposition 2.1 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then f has a local minimum at $\bar{x}$ if and only if f has an absolute minimum at $\bar{x}$.*

Proof: We only need to prove the implication since the converse is trivial. Suppose that f has a local minimum at $\bar{x}$. Then there exists $\delta > 0$ with

$$f(u) \ge f(\bar{x}) \text{ for all } u \in I\!B(\bar{x}; \delta).$$

For any $x \in \mathbb{R}$, one has that $x_k = (1 - \frac{1}{k})\bar{x} + \frac{1}{k}x \to \bar{x}$. Thus, $x_k \in I\!B(\bar{x};\delta)$ when k is sufficiently large. It follows that

$$f(\bar{x}) \leq f(x_k) \leq (1 - \frac{1}{k})f(\bar{x}) + \frac{1}{k}f(x).$$

This implies

$$\frac{1}{k}f(\bar{x}) \leq \frac{1}{k}f(x),$$

and hence $f(\bar{x}) \leq f(x)$. Therefore, f has an absolute minimum at $\bar{x}$. □

Proposition 2.2 *The solution set of the Fermat-Torricelli problem* (2.2) *is nonempty.*

Proof: Let $m := \inf\{\varphi(x) \mid x \in \mathbb{R}^n\}$. Then m is a nonnegative real number. Let (x_k) be a sequence such that

$$\lim_{k\to\infty} \varphi(x_k) = m.$$

By definition, there exists $k_0 \in \mathbb{N}$ satisfying

$$\|x_k - a_1\| \leq \varphi(x_k) \leq m + 1 \text{ for all } k \geq k_0.$$

This implies $\|x_k\| \leq m+1+\|a_1\|$. Thus, (x_k) is a bounded sequence, so it has a subsequence (x_{k_ℓ}) that converges to $\bar{x} \in \mathbb{R}^n$. Since φ is a continuous function,

$$\varphi(\bar{x}) = \lim_{\ell\to\infty} \varphi(x_{k_\ell}) = m.$$

Therefore, $\bar{x}$ is an optimal solution of the problem. □

For two different points $a, b \in \mathbb{R}^n$, the line containing a and b is the following set:

$$\mathcal{L}(a, b) := \{ta + (1 - t)b \mid t \in \mathbb{R}\}.$$

Proposition 2.3 *Suppose that a_i for $i = 1, \ldots, m$ do not lie on the same line (not collinear). Then the function φ given by* (2.1) *is strictly convex and the Fermat-Torricelli problem* (2.2) *has a unique solution.*

Proof: Define $\varphi_i(x) := \|x - a_i\|$ for $i = 1, \ldots, m$. Then $\varphi = \sum_{i=1}^m \varphi_i$. For any $x, y \in \mathbb{R}^n$ and $\lambda \in (0, 1)$, one has

$$\varphi_i(\lambda x + (1 - \lambda)y) \leq \lambda\varphi_i(x) + (1 - \lambda)\varphi_i(y) \text{ for } i = 1, \ldots, m.$$

This implies

$$\varphi(\lambda x + (1 - \lambda)y) \leq \lambda\varphi(x) + (1 - \lambda)\varphi(y). \tag{2.3}$$

On the contrary, suppose φ is not strictly convex. It means that there exist $\bar{x}, \bar{y} \in \mathbb{R}^n$ with $\bar{x} \neq \bar{y}$ and $\lambda \in (0, 1)$ for which (2.3) holds as equality. Then

$$\varphi_i(\lambda\bar{x} + (1 - \lambda)\bar{y}) = \lambda\varphi_i(\bar{x}) + (1 - \lambda)\varphi_i(\bar{y}) \text{ for } i = 1, \ldots, m.$$

Thus,

$$\|\lambda(\bar{x}-a_i)+(1-\lambda)(\bar{y}-a_i)\| = \|\lambda(\bar{x}-a_i)\| + \|(1-\lambda)(\bar{y}-a_i)\| \text{ for } i=1,\ldots,m.$$

If $\bar{x} \neq a_i$ and $\bar{y} \neq a_i$, then there exists $t_i > 0$ such that

$$t_i\lambda(\bar{x}-a_i) = (1-\lambda)(\bar{y}-a_i).$$

Thus, $\bar{x} - a_i = \gamma_i(\bar{y}-a_i)$, where $\gamma_i := \dfrac{1-\lambda}{t_i\lambda}$. Since $\bar{x} \neq \bar{y}$, one has $\gamma \neq 1$, and

$$a_i = \frac{1}{1-\gamma}\bar{x} - \frac{\gamma}{1-\gamma}\bar{y} \in \mathcal{L}(\bar{x},\bar{y}).$$

In the case where $\bar{x} = a_i$ or $\bar{y} = a_i$, it is obvious that $a_i \in \mathcal{L}(\bar{x},\bar{y})$. We have proved that $a_i \in \mathcal{L}(\bar{x},\bar{y})$ for $i = 1,\ldots,m$, which is a contradiction. □

An element $v \in \mathbb{R}^n$ is called a *subgradient* of a convex function $f\colon \mathbb{R}^n \to \mathbb{R}$ at $\bar{x} \in \mathbb{R}^n$ if it satisfies the inequality

$$f(x) \geq f(\bar{x}) + \langle v, x-\bar{x}\rangle \ \text{ for all } \ x \in \mathbb{R}^n,$$

where $\langle\cdot,\cdot\rangle$ stands for the usual scalar product in $\mathbb{R}^n$. The set of all subgradients of f at $\bar{x}$ is called the *subdifferential* of the function at $\bar{x}$ and is denoted by $\partial f(\bar{x})$.

Directly from the definition, one has the following subdifferential Fermat rule:

$$f \text{ has an absolute minimum at } \bar{x} \text{ if and only if } 0 \in \partial f(\bar{x}). \tag{2.4}$$

The proposition below shows that the subdifferential of a convex function at a given point reduces to the gradient at that point when the function is differentiable.

Proposition 2.4 *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is convex and Fréchet differentiable at $\bar{x}$. Then*

$$\langle \nabla f(\bar{x}), x-\bar{x}\rangle \leq f(x) - f(\bar{x}) \ \text{ for all } x \in \mathbb{R}^n. \tag{2.5}$$

Moreover, $\partial f(\bar{x}) = \{\nabla f(\bar{x})\}$.

Proof: Since f is Fréchet differentiable at $\bar{x}$, by definition, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$-\epsilon\|x-\bar{x}\| \leq f(x) - f(\bar{x}) - \langle \nabla f(\bar{x}), x-\bar{x}\rangle \leq \epsilon\|x-\bar{x}\| \text{ whenever } \|x-\bar{x}\| < \delta.$$

Define

$$\psi(x) = f(x) - f(\bar{x}) - \langle \nabla f(\bar{x}), x-\bar{x}\rangle + \epsilon\|x-\bar{x}\|.$$

Then $\psi(x) \geq \psi(\bar{x}) = 0$ for all $x \in I\!B(\bar{x};\delta)$. Since φ is a convex function, $\psi(x) \geq \psi(\bar{x})$ for all $x \in \mathbb{R}^n$. Thus,

$$\langle \nabla f(\bar{x}), x-\bar{x}\rangle \leq f(x) - f(\bar{x}) + \epsilon\|x-\bar{x}\| \text{ for all } x \in \mathbb{R}^n.$$

Letting $\epsilon \to 0$, one obtains (2.5).

Equality (2.5) implies that $\nabla f(\bar{x}) \in \partial f(\bar{x})$. Take any $v \in \partial f(\bar{x})$, one has

$$\langle v, x - \bar{x} \rangle \le f(x) - f(\bar{x}) \text{ for all } x \in \mathbb{R}^n.$$

The Fréchet differentiability of f also implies that for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\langle v - \nabla f(\bar{x}), x - \bar{x} \rangle \le \epsilon \|x - \bar{x}\| \quad \text{whenever } \|x - \bar{x}\| < \delta.$$

It follows that $\|v - \nabla f(\bar{x})\| \le \epsilon$, which implies $v = \nabla f(\bar{x})$ since $\epsilon > 0$ is arbitrary. Therefore, $\partial f(\bar{x}) = \{\nabla f(\bar{x})\}$. □

The subdifferential formula for the norm function in the next example plays a crucial role in our subsequent analysis to solve the Fermat-Torricelli problem.

Example 2.5 Let $p(x) = \|x\|$, the Euclidean norm function on $\mathbb{R}^n$. Then

$$\partial p(x) = \begin{cases} I\!\!B & \text{if } x = 0, \\ \left\{ \dfrac{x}{\|x\|} \right\} & \text{otherwise.} \end{cases}$$

Since the function p is Fréchet differentiable with $\nabla p(x) = \dfrac{x}{\|x\|}$ for $x \neq 0$, it suffices to prove the formula for $x = 0$. By definition, an element $v \in \mathbb{R}^n$ is a subgradient of p at 0 if and only if

$$\langle v, x \rangle = \langle v, x - 0 \rangle \le p(x) - p(0) = \|x\| \text{ for all } x \in \mathbb{R}^n.$$

For $x := v \in \mathbb{R}^n$, one has $\langle v, v \rangle \le \|v\|$. This implies $\|v\| \le 1$ or $v \in I\!\!B$. Moreover, if $v \in I\!\!B$, by the Cauchy-Schwarz inequality,

$$\langle v, x - 0 \rangle = \langle v, x \rangle \le \|v\| \|x\| \le \|x\| = p(x) - p(0) \text{ for all } x \in \mathbb{R}^n.$$

It follows that $v \in \partial p(0)$. Therefore, $\partial p(0) = I\!\!B$.

Solving the Fermat-Torricelli problem involves using the following simplified subdifferential rule for the sum of a nondifferentiable function and a differentiable function. A more general formula holds true when all of the functions involved are nondifferentiable.

Proposition 2.6 *Suppose that $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, 2$ are convex functions and f_2 is differentiable at $\bar{x}$. Then*

$$\partial(f_1 + f_2)(\bar{x}) = \partial f_1(\bar{x}) + \nabla f_2(\bar{x}). \tag{2.6}$$

Proof: Fix any $v \in \partial(f_1 + f_2)(\bar{x})$. For any $x \in \mathbb{R}^n$, one has

$$\langle v, x - \bar{x} \rangle \le f_1(x) - f_1(\bar{x}) + f_2(x) - f_2(\bar{x}) = f_1(x) - f_1(\bar{x}) + \langle \nabla f_2(\bar{x}), x - \bar{x} \rangle + o(\|x - \bar{x}\|).$$

For any $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 \le \langle \nabla f_2(\bar{x}) - v, x - \bar{x} \rangle + f_1(x) - f_1(\bar{x}) + \epsilon \|x - \bar{x}\| \text{ for } x \in I\!\!B(\bar{x}; \delta).$$

The convexity of f_1 implies that this is true for all x. Letting $\epsilon \to 0$, one has

$$0 \leq \langle \nabla f_2(\bar{x}) - v, x - \bar{x} \rangle + f_1(x) - f_1(\bar{x}) \text{ for } x \in \mathbb{R}^n.$$

By definition, $v - \nabla f_2(\bar{x}) \in \partial f_1(\bar{x})$, and hence $v \in \partial f_1(\bar{x}) + \nabla f_2(\bar{x})$. We have proved the inclusion $\subseteq$.

The opposite inclusion follows from the definition. □

Let us now use subgradients of the norm function to derive Torricelli's solution for the Fermat-Torricelli problem. Given two nonzero vectors u and v, define

$$\cos(u, v) = \frac{\langle u, v \rangle}{\|u\| \|v\|}.$$

For $\bar{x} \neq a_i$, let

$$v_i = \frac{\bar{x} - a_i}{\|\bar{x} - a_i\|}, \ i = 1, 2, 3.$$

Geometrically, v_i is the unit vector pointing in the direction from the vertex a_i to $\bar{x}$. Observe that the classical Fermat-Torricelli problem always has a unique solution even if three given points are on the same line. In the latter case, the middle point is the solution of the problem.

Proposition 2.7 *Consider the Fermat-Torricelli problem given by three points a_1, a_2, a_3.*
(i) *Suppose $\bar{x} \notin \{a_1, a_2, a_3\}$. Then $\bar{x}$ is the solution of the problem if and only if*

$$\cos(v_1, v_2) = \cos(v_2, v_3) = \cos(v_3, v_1) = -1/2.$$

(ii) *Suppose $\bar{x} \in \{a_1, a_2, a_3\}$, say $\bar{x} = a_1$. Then $\bar{x}$ is the solution of the problem if and only if*

$$\cos\langle v_2, v_3 \rangle \leq -1/2.$$

Proof: (i) In this case, the function φ given by (2.1) is Fréchet differentiable at $\bar{x}$. Since φ is convex, $\bar{x}$ is the solution of the Fermat-Torricelli problem if and only if

$$\nabla \varphi(\bar{x}) = v_1 + v_2 + v_3 = 0.$$

Since $\|v_i\| = 1$ for $i = 1, 2, 3$, one has

$$\begin{aligned} \langle v_1, v_2 \rangle + \langle v_1, v_3 \rangle &= -1 \\ \langle v_2, v_1 \rangle + \langle v_2, v_3 \rangle &= -1 \\ \langle v_3, v_1 \rangle + \langle v_3, v_2 \rangle &= -1. \end{aligned}$$

Solving this system of equations yields

$$\langle v_i, v_j \rangle = \cos(v_i, v_j) = -1/2 \text{ for } i \neq j, i, j \in \{1, 2, 3\}.$$

Moreover, if $\langle v_i, v_j \rangle = -1/2$ for $i \neq j$, $i, j \in \{1, 2, 3\}$, then

$$\|v_1 + v_2 + v_3\|^2 = \sum_{i=1}^{3} \|v_i\| + \sum_{i,j=1, i \neq j}^{3} \langle v_i, v_j \rangle = 0.$$

It follows that $v_1 + v_2 + v_3 = 0$.

(ii) By the subdifferential Fermat rule (2.4) and the subdifferential sum rule (2.6), $\bar{x} = a_1$ is the solution of the Fermat-Torricelli problem if and only if

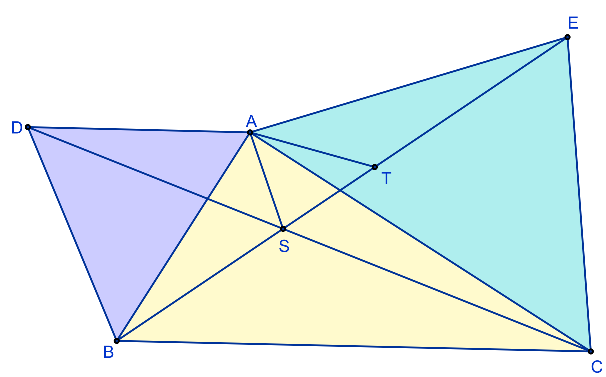
$$0 \in \partial\varphi(a_1) = I\!B + v_2 + v_3.$$

This is equivalent to $\|v_2 + v_3\|^2 \leq 1$ or $\|v_2\|^2 + \|v_3\|^2 + 2\langle v_2, v_3\rangle \leq 1$. Since v_2 and v_3 are unit vectors, we obtain

$$\langle v_2, v_3\rangle = \cos(v_2, v_3) \leq -1/2.$$

The proof is now complete. □

Example 2.8 Let us discuss the construction of the solution of the Fermat-Torricelli problem in the plane. Consider the Fermat-Torricelli problem given by three points A, B, and C as in the figure above. If one of the angles of the triangle ABC is greater than or equal to $120°$, then the corresponding vertex is the solution of the problem by Proposition 2.7 (ii). Let us consider the case where non of the angles of the triangle is greater than or equal to $120°$. Construct two equilateral triangles ABD and ACE and let S be the intersection of DC and BE as in the figure. Two quadrilaterals $ADBC$ and $ABCE$ are convex, and hence S lies inside the triangle ABC. It is clear that two triangles DAC and BAE are congruent (SAS). A rotation of $60°$ about A maps the triangle DAC to the triangle BAE. The rotation maps CD to BE, so $\angle DSB = 60°$. Let T be the image of S though this rotation. Then T belongs to BE. It follows that $\angle AST = \angle ASE = 60°$. Moreover, $\angle DSA = 60°$, and hence $\angle BSA = 120°$. It is now clear that $\angle ASC = 120°$ and $\angle BSC = 120°$. By Proposition 2.7 (i), the point S is the solution of the problem.



3 The Weiszfeld's Algorithm

In this section, we revisit Kuhn's proof [8] of the convergence of the Weiszfeld's algorithm [18] for solving the Fermat-Torricelli problem (2.2). With some additional ingredients of convex analysis, we are able to provide a more clear picture of Kuhn's proof. Throughout this section, we assume that a_i for $i = 1, \ldots, m$ are not collinear.

The gradient of the function φ given by (2.1) is

$$\nabla\varphi(x) = \sum_{i=1}^{m} \frac{x - a_i}{\|x - a_i\|},\ x \notin \{a_1, a_2, \ldots, a_m\}.$$

Solving the equation $\nabla\varphi(x) = 0$ gives

$$x = \frac{\sum_{i=1}^{m} \dfrac{a_i}{\|x - a_i\|}}{\sum_{i=1}^{m} \dfrac{1}{\|x - a_i\|}} =: F(x).$$

For continuity, define $F(x) := x$ for $x \in \{a_1, a_2, \ldots, a_m\}$.

Weiszfeld introduce the following algorithm: choose a starting point $x_0 \in \mathbb{R}^n$ and define

$$x_{k+1} = F(x_k) \text{ for } k \in \mathbb{N}.$$

He also claimed that if $x_0 \notin \{a_1, a_2, \ldots, a_m\}$, where a_i for $i = 1, \ldots, m$ are not collinear, then (x_k) converges to the unique optimal solution of the problem. A correct statement and the proof of the convergence were given by Kuhn in 1972.

The proposition below guarantees that the function value decreases after each iteration; see [8, Subsection 3.1].

Proposition 3.1 *If $F(x) \neq x$, then $\varphi(F(x)) < \varphi(x)$.*

Proof: It is clear that x is not a vertex, since otherwise, $F(x) = x$. Moreover, $F(x)$ is the unique minimizer of the following strictly convex function:

$$g(z) = \sum_{i=1}^{m} \frac{\|z - a_i\|^2}{\|x - a_i\|}.$$

Since $F(x) \neq x$, one has $g(F(x)) < g(x) = \varphi(x)$.

Moreover,

$$\begin{aligned} g(F(x)) &= \sum_{i=1}^{m} \frac{\|F(x) - a_i\|^2}{\|x - a_i\|} \\ &= \sum_{i=1}^{m} \frac{(\|x - a_i\| + \|F(x) - a_i\| - \|x - a_i\|)^2}{\|x - a_i\|} \\ &= \varphi(x) + 2(\varphi(F(x)) - \varphi(x)) + \sum_{i=1}^{m} \frac{(\|F(x) - a_i\| - \|x - a_i\|)^2}{\|x - a_i\|}. \end{aligned}$$

It follows that

$$2\varphi(F(x)) + \sum_{i=1}^{m} \frac{(\|F(x) - a_i\| - \|x - a_i\|)^2}{\|x - a_i\|} < 2\varphi(x).$$

Therefore, $\varphi(F(x)) < \varphi(x)$. □

The next two propositions show the behavior of the *algorithm mapping* F near a vertex and deal with the case where a vertex is the solution of the problem. Let us first present a necessary and sufficient condition for a vertex to be the optimal solution of the problem. It can be used to easily derive the result in [8, Subsection 2.1].

Define

$$R_k := \sum_{i=1, i\neq k}^{m} \frac{a_i - a_k}{\|a_i - a_k\|}.$$

Proposition 3.2 *The vertex a_k is the optimal solution of the problem if and only if*

$$\|R_k\| \leq 1.$$

Proof: By the subdifferential Fermat rule (2.4) and the subdifferential sum rule from Proposition 2.6, the vertex a_k is the optimal solution of the problem if and only if

$$0 \in \partial\varphi(a_k) = -R_k + I\!B(0,1).$$

This is equivalent to $\|R_k\| \leq 1$. □

Proposition 3.2 allows us to simplify the proof of the following result in [8, Subsection 3.2].

Proposition 3.3 *Suppose that a_k is not the optimal solution. Then there exists $\delta > 0$ such that $0 < \|x - a_k\| \leq \delta$ implies that there exists a positive integer s with*

$$\|F^s(x) - a_k\| > \delta \text{ and } \|F^{s-1}(x) - a_k\| \leq \delta.$$

Proof: For any x, which is not a vertex, one has

$$F(x) = \frac{\sum_{i=1}^{m} \dfrac{a_i}{\|x - a_i\|}}{\sum_{i=1}^{m} \dfrac{1}{\|x - a_i\|}}.$$

Then

$$F(x) - a_k = \frac{\sum_{i=1, i\neq k}^{m} \dfrac{a_i - a_k}{\|x - a_i\|}}{\sum_{i=1}^{m} \dfrac{1}{\|x - a_i\|}}.$$

Thus,

$$\lim_{x \to a_k} \frac{F(x) - a_k}{\|x - a_k\|} = \frac{\sum_{i=1, i\neq k}^{m} \dfrac{a_i - a_k}{\|x - a_i\|}}{1 + \sum_{i=1, i\neq k}^{m} \dfrac{\|x - a_k\|}{\|x - a_i\|}} = R_k.$$

By Proposition 3.2,

$$\lim_{x \to a_k} \frac{\|F(x) - a_k\|}{\|x - a_k\|} = \|R_k\| > 1.$$

Thus, there exist $\epsilon > 0$ and $\delta > 0$ such that

$$\frac{\|F(x) - a_k\|}{\|x - a_k\|} > (1 + \epsilon) \text{ whenever } 0 < \|x - a_k\| < \delta.$$

and $a_i \notin I\!B(a_k, \delta)$ for $i \neq k$. The conclusion then follows easily. □

We finally present Kuhn's statement and proof for the convergence of the Weiszfeld's algorithm; see [8, Subsection 3.4].

Theorem 3.4 *Let (x_k) be the sequence formed by the Weiszfeld's algorithm. Suppose that $x_k \notin \{a_1, a_2, \ldots, a_m\}$ for $k \geq 0$. Then (x_k) converges to the optimal solution $\bar{x}$ of the problem.*

Proof: In the case where $x_k = x_{k+1}$ for some $k = k_0$, one has that x_k is a constant sequence for $k \geq k_0$. Thus, it converges to x_{k_0}. Since $F(x_{k_0}) = x_{k_0}$ and x_{k_0} is not a vertex, x_{k_0} is the solution of the problem. So we can assume that $x_{k+1} \neq x_k$ for every k. By Proposition 3.1, the sequence $(\varphi(x_k))$ is nonnegative and decreasing, so it converges. It follows that

$$\lim_{k \to \infty} (\varphi(x_k) - \varphi(x_{k+1})) = 0. \tag{3.7}$$

By definition, for $k \geq 1$, $x_k \in \text{co}\,\{a_1, \ldots, a_m\}$, which is a compact set. Then (x_k) has a convergent subsequence (x_{k_ℓ}) to a point $\bar{z}$. It suffices to prove that $\bar{z} = \bar{x}$. By (3.7),

$$\lim_{\ell \to \infty} (\varphi(x_{k_\ell}) - \varphi(F(x_{k_\ell})) = 0.$$

By the continuity, $\varphi(\bar{z}) = \varphi(F(\bar{z}))$, which implies $F(\bar{z}) = \bar{z}$. If $\bar{z}$ is not a vertex, one has $\bar{z}$ is the solution of the problem, so $\bar{z} = \bar{x}$. Let us consider the case where $\bar{z}$ is a vertex, say a_1. Suppose by contradiction that $\bar{z} \neq \bar{x}$. Choose δ sufficiently small such that the property in Proposition 3.3 holds and $I\!B(a_1; \delta)$ does not contain $\bar{x}$ and a_i for $i = 2, \ldots, m$. Since $x_{k_\ell} \to a_1 = \bar{z}$, we can assume without loss of generality that the sequence is contained in $I\!B(a_1; \delta)$.

For $x = x_{k_1}$, choose q_1 such that $x_{q_1} \in I\!B(a_1; \delta)$ and $F(x_{q_1}) \notin I\!B(a_1; \delta)$. Choose an index $k_\ell > q_1$ and apply Proposition 3.3, we find $q_2 > q_1$ such that $x_{q_2} \in I\!B(a_1; \delta)$ and $F(x_{q_2}) \notin I\!B(a_1; \delta)$. Repeating this procedure, we find (x_{q_ℓ}) with $x_{q_\ell} \in I\!B(a_1; \delta)$ and $F(x_{q_\ell})$ is not in this ball. Extracting a further subsequence, we can assume that $x_{q_\ell} \to \bar{q}$. By the procedure that has been used, one has $F(\bar{q}) = \bar{q}$. If $\bar{q}$ is not a vertex, then it is the solution, which is a contradiction because the solution $\bar{x}$ is not in $I\!B(a_1; \delta)$. Thus, $\bar{q}$ is a vertex, which must be a_1. Then

$$\lim_{\ell \to \infty} \frac{\|F(x_{q_\ell}) - a_1\|}{\|x_{q_\ell} - a_1\|} = \infty.$$

This is a contradiction according to Proposition 3.3. □

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