EXPERIMENT 1
RLC SERIES CIRCUIT RESONANCE
(Complex impedance)

PURPOSE
To use an oscilloscope to make AC measurements of voltage and current.
To observe the frequency-dependence of the impedance and the phase in an AC circuit.
To use series resonance to determine the inductance of a coil using a known capacitor.

THEORETICAL CONSIDERATION

The total impedance for the series combination is,

\[ Z = R + j\omega L + \frac{1}{j\omega C} = R + j(\omega L - \frac{1}{\omega C}) \]  

(1)

If the input voltage is given by

\[ v = V_0 e^{j\omega t} \]  

(Driving voltage)

(2)

then the current is given by,

\[ i = \frac{v}{Z} = \frac{1}{R + j(\omega L - \frac{1}{\omega C})} v \]  

(3)

Fig.1 RLC series circuit.

Multiplying and dividing by the complex conjugate, this expression can be rewritten as

\[ i = \frac{1}{R + j(\omega L - \frac{1}{\omega C})} \left[ \frac{R - j(\omega L - \frac{1}{\omega C})}{R - j(\omega L - \frac{1}{\omega C})} \right] v = \frac{R - j(\omega L - \frac{1}{\omega C})}{R^2 + (\omega L - \frac{1}{\omega C})^2} v \]

\[ \begin{align*}
&= \frac{1}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} \left[ \frac{R}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} - j \frac{(\omega L - \frac{1}{\omega C})}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} \right] v
\end{align*} \]

Notice the quantity in brackets above has magnitude equal to 1, and can be expressed in terms of complex number of phase \( \Psi \):
\[ i = \frac{1}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} \left[ \cos(\psi) + j\cos(\psi) \right] v \]

\[ i = \frac{1}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} e^{j\psi} v \quad (4) \]

where

\[ \psi = \arctan \left[ -\frac{(\omega L - \frac{1}{\omega C})}{R} \right] \quad (5) \]

\( \psi \) is the phase the current is “lagging” with respect to the driving voltage. (Fig 2 shows a particular example in which \( \psi \) has a negative value).

Notice,

- at very low frequencies: \( \psi \to + \pi \)
- at \( \omega = \omega_o = \frac{1}{\sqrt{LC}} \) we have: \( \psi = 0 \)
- at very high frequencies \( \psi \to - \pi \)

Expression (4) can also be written as,

\[ i = \frac{V_o}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} e^{j(\omega t + \psi)} = I e^{j(\omega t + \psi)} \quad (6) \]

Notice in (1) and (4) that when \( \omega L = 1/(\omega C) \),

- The impedance is minimum
- The current is maximum

Hence \( \omega = \omega_o = \frac{1}{\sqrt{LC}} \) is called the resonance frequency.
**EXPERIMENTAL CONSIDERATIONS**

The oscilloscope is our basic measuring tool in alternating current (AC) circuits. We will be employing oscilloscopes that are equipped with two inputs, and which switch rapidly between them to display both. The method of switching is selectable (alternate or chopped), and which to use depends on the frequency you are observing.

Since the oscilloscope is basically a voltage measuring device, and we will want to measure both voltage and current amplitudes, it will be convenient to use several elements to make our “signal generator.”

The resistor \( R_0 \) is a known resistor (whose value is on the order of 100 \( \Omega \)). The ground of the oscilloscope will be connected to \( G \), while the two inputs will be to points A and B.

GA will measure the input voltage \( V_A \), while GB will measure \( V_R = I R_0 \) from which the current \( I \) can be deduced.

**Notes:** In atypical capacitor, labeled, for example 473G, it means \( C = 47 \times 10^3 \) picoFarads. (Suggestion: use \( C \sim 10 \) nF for this experiment.)

 Measure the frequency of the input voltage using the oscilloscope. (Do not trust the reading from the signal generator knob.)

**MEASUREMENTS**

a) The magnitude of the impedance \( Z \) of the test circuit is determined from the ratio of amplitudes of the two signals, \( V_A \) and \( I \).
   That is, plot \( |Z| \) as a function of \( \omega \).

b) The phase can be measured as a distance between the points where the two traces cross the horizontal axis, and converted to degrees by comparing the half (or full) wavelength as shown on the oscilloscope. See figure below.

Pay attention during the measurements to verify if \( \phi \) is positive or negative. That is, whether the \( V_R \) is lagging or ahead of \( V_A \).

Plot \( \phi = \phi(\omega) \)

\[
V_A = V_o \cos(\omega t); \quad V_R = V_o \cos(\omega t - \phi)
\]

The traces show \( V_R \) lagging \( V_A \) by \( \phi \).
c) During the course of measurements you should take enough data to make a graph of both impedance and phase as a function of frequency. How does the curve changes when using a higher value of the resistance $R_o$?

In order to compare this to theory, you will need to determine the value of the inductance $L$. This is best done early in the exercise by locating the resonance frequency, where the impedance is minimum and where the phase between $V_A$ and $V_R$ is zero. This is obviously the frequency region where a majority of your data needs to be taken. The quality of your plot of impedance and phase is more dependent upon taking data that spans the region of interest than it is on the volume of data alone.

By rearranging the circuit elements and shifting the ground it is possible to compare the signals across $C$ and $R$, and across $L$ and $R$.

d) At resonance condition, measure the voltage across the three elements (you will need to swap $C$ and $L$, one at a time, with the element $R$). Do these values add up?

e) Choose a frequency at which $I$ is $\sim 50\%$ of $I_{\text{resonance}}$. Verify experimentally and corroborate theoretically the validity of Kirchhoff law (measure the complex values for $V_c$, $V_L$, $V_R$)

OVERCOMING the SHORTCOMINGS ENCOUNTERED in the EXPERIMENT

Problem: Variability of the input voltage amplitude

Correction: By the normalization method.

While sweeping the frequency of the input voltage $V_A$ around the resonance frequency, it is observed that the amplitude of the driving voltage also changes. Ideally, it would be desirable that this amplitude remains constant.

The instability of the $V_A$’s amplitude makes somewhat inaccurate the procedure of locating the resonance frequency by monitoring the frequency at which the current is maximum. (When you do this, you may find that at the frequency where the current is maximum, $V_A$ and $I$ are not in phase.) Still this is still a good preliminary step, since it helps to identify the frequency range where we have to take a closer look.

Next, sweep the frequency a bit, until you find that the phase between $V_A$ and $I$ is zero.

The suggested following step is to manually normalize the input voltage $V_A$. By this we mean to choose a fixed amplitude value for $V_A$, let’s call it $V_{A,\text{fixed}}$.

For a given frequency, record the current amplitude and phase.

Move to another frequency (you may notice that the amplitude of $V_A$ has changed.) Turn the knob that controls the amplitude of $V_A$ until you obtain back the predetermined fixed value $V_{A,\text{fixed}}$. Then record the current amplitude and phase.

And so on, repeat the procedure for each frequency around the resonance frequency.
Phase lagging in the RLC-series circuit

\[ V - Ri = \frac{q}{C} - L \frac{di}{dt} = 0 \quad \text{Kirchhoff's law} \]

\[ V = Ri + \frac{q}{C} + L \frac{di}{dt} \]

\[ V = \frac{q}{C} + R \frac{dq}{dt} + L \frac{d^2 q}{dt^2} \quad \text{where } i = \frac{d q}{dt} \tag{1} \]

For \( V = V_0 e^{j\omega t} \) (given), we propose the solution \( q = q_0 e^{j\omega t} \)

where \( q_0 \) is a complex number

\[ V_0 e^{j\omega t} = \left[ \frac{q_0}{C} + j \omega R q_0 - L \omega^2 q_0 \right] e^{j\omega t} \]

\[ = \left[ \frac{i}{j\omega C} + j \omega L + R \right] j \omega q_0 e^{j\omega t} \]

Notice, this is the current

\[ i = \frac{d q}{dt} \]

Thus

\[ V_0 e^{j\omega t} = \left[ R + j \left( \omega L + \frac{1}{j\omega C} \right) \right] i \tag{3} \]
Or, \( i = \frac{v}{Z} = \frac{1}{R + j(\omega L - \frac{1}{\omega C})} v \), which gives,

\[
i = \frac{1}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} e^{j\psi} v
\]

where

\[
\psi = \arctan \left[ \frac{(\omega L - \frac{1}{\omega C})}{R} \right]
\]

Notice,

at very low frequencies: \( \psi \to + \pi \)

at \( \omega = \omega_o = \frac{1}{\sqrt{LC}} \) we have: \( \psi = 0 \)

at very high frequencies \( \psi \to - \pi \)

Thus we have,

\[
v = V_o e^{j\omega t} \quad \text{(driving voltage)}
\]

\[
i = I e^{j(\omega t + \psi)} = I e^{j\omega t} e^{j\psi} \quad \text{(current)}
\]

\[
i = \frac{dq}{dt} \text{ implies } q = \frac{I}{j\omega} e^{j(\omega t + \psi)} = -\frac{I}{\omega} e^{j(\omega t + \psi)} = \frac{I}{\omega} e^{j(\omega t + \psi - \pi/2)} \equiv Q e^{j(\omega t + \psi - \pi/2)}
\]

\[
q = Q e^{j(\omega t + \psi - \pi/2)} = Q e^{j(\omega t + \psi)} Q e^{-j\pi/2} \quad \text{(charge)}
\]

\( q \) lags the current \( i \) by \( \pi/2 \)

Using the phasors representation,
At $\omega \sim 0$

At $\omega \sim \omega_0$

At $\omega \to \infty$

At $\omega \sim 0$

At $\omega \sim \omega_0$

At $\omega \to \infty$