The origin of the index of refraction: A view from a "phase-lagging" perspective

3.2.A An accelerated charge emits electromagnetic fields
Calculation of the field produced by a plane of oscillating charges.
Appendix-1 More detailed calculation of the electric field
(Calculation of the acceleration component perpendicular to the line of sight).
Appendix-2 The field produced by charges in a slab of finite thickness d.

3.2.B Phase lagging and the index of refraction
Description of the experimental setting under consideration:
Light passing through a dielectric slab of thickness "d".
3.2.B1 Approach-1: Calculation of the transmitted field:
Assuming light slows down while travelling inside the slab.
3.2.B2 Approach-2: Calculation of the transmitted field:
Assuming the field results from oscillating charges in the slab.
3.2.B3 Comparison between the two approaches.
The origin of the Refractive Index

It is approximately true that light does appear to travel at a speed \( c/n \) through a material whose index of refraction is \( n \), but the electromagnetic fields (i.e., light) are still produced by the motion of all the charges, including the charges moving in the material.

All these basic contributions to the electromagnetic field at \( P \) (from the source and from the atoms in the glass) travel at the ultimate velocity \( c \).

Our objective is: to understand how the apparently slower velocity of light in the glass comes about.
Preliminary calculations:
The field of a plane of oscillating charges

A. Accelerated charges produce electromagnetic fields

\[ E_y(p) = \frac{q}{4\pi\varepsilon_0 c^2} \frac{1}{r} a_y(t - \frac{r}{c}) \]

Electric field at \( p \) at the time \( t \)

Component of the acceleration of the charge, perpendicular to the line of sight

The acceleration is evaluated at a retarded time \( t - \frac{r}{c} \)
because the signal takes a time \( \frac{r}{c} \) to arrive from the charge location to the point \( p \)

\[ E(p) = \frac{q}{4\pi\varepsilon_0 c^2 r} a_y(t - \frac{r}{c}) \]

tangential component of the charge's acceleration
B. Electric field produced by a plane of oscillating charges.

- Plane full of charges,
- All charges oscillating synchronously, with the same amplitude \( x_0 \) and the same phase
- There are \( \eta \) charges per unit area of the plane,
- Each charge, having charge \( q \), moves around its own average position according to
  \[
  x = x_0 \cos \omega t = x_0 e^{i\omega t}
  \]
Let's consider that the oscillating charges on the glass plane sheet are excited by a light beam incident on, for example, a cross section of the glass.

The total electric field at "P" is the result of a contribution from all the charges in this cross section area.
Since the cross section area A is assumed to be far away from the observation point P, the omission of the component of the acceleration normal to the line of sight is justified. (A more detailed justification is provided in the attached Appendix-1 file.)

\[
E = \int_A \left( \frac{q \eta dA}{4 \pi \varepsilon_0 c^2} \frac{1}{r} \right) \omega^2 \chi_0 e^{i \omega (t - \frac{r}{c})}
\]

\[
E(\mathbf{r} + \mathbf{p}) = \int_0^\infty \left( \frac{q \eta 2\pi p dp}{4 \pi \varepsilon_0 c^2} \frac{1}{r} \right) \frac{\omega^2 \chi_0 e^{i \omega (t - \frac{r}{c})}}{r}
\]

\[2\rho \, dp = 2\rho \, d\rho\]

Notice \( \rho^2 + z^2 = r^2 \)

\[\int_{\rho=0}^{\infty} \left( \frac{q \eta r dr}{2 \varepsilon_0 c^2} \frac{1}{r} \right) \omega^2 \chi_0 e^{i \omega (t - \frac{r}{c})} \]
Let's see what happens if we assume, first, that the charge density on the glass plate $\eta$ is constant.

Then, the integral \( \int_{r=2}^{\infty} e^{-i \frac{\omega}{c} r} \eta dr \) can be interpreted as the sum of many small complex numbers, each of magnitude $dr$ and with an angle that increases by $-\frac{\omega}{c} dr = \Delta \theta$, starting from $\theta_0 = -\frac{\omega}{c} z$. 

\[
E(\text{at } P) = \frac{q \omega^2 \rho_0}{2 \varepsilon_0 c^2} \int_{r=2}^{\infty} \eta dr \ e^{-i \omega(t - \frac{r}{c})}
\]

\[
\frac{q \omega^2 \rho_0}{2 \varepsilon_0 c^2} \int_{r=2}^{\infty} e^{-i \frac{\omega}{c} r} \eta dr
\]
As we keep adding the little complex mumurs \( \Delta r \) \( \omega \Delta r \), a circumference of radius \( R \) is obtained. Let's calculate \( R \)

\[
R \sin \beta = \frac{\Delta r}{2}
\]

\[
R = \frac{\Delta r/2}{\sin \beta}
\]

But, what is the value of \( \beta \)?

Answer: \( 2\beta = \Delta \theta \)

Accordingly,

\[
R = \frac{\Delta r/2}{\sin (\frac{\Delta \theta}{2})}
\]
For small $\Delta \theta$ (small $\frac{\Delta r}{c \Delta \theta}$ or $\Delta r << \frac{c}{\omega} = \frac{a}{2\pi}$)

\[
R = \frac{\Delta r/2}{\sin \Delta \theta} = \frac{\Delta r/2}{\Delta \theta/2} = \frac{\Delta r}{\omega/c \Delta r} \implies R = \frac{c}{\omega} = \frac{\lambda}{2\pi}
\]

\[
\int_{-\infty}^{\infty} e^{-i \frac{\omega}{c} r} dr = \text{Sum}
\]

We realize that adding more and more complex numbers $dr e^{-i \frac{\omega}{c} r}$ results in going around the circumference of radius $R = \frac{c}{\omega}$. Thus, we do not get a definite answer!

But $\int_{-\infty}^{\infty} \eta dr$ may have a definite value

If we assume that $\eta$ (the charge density) tapers off as $r$ increases, let's see why.

But $\int_{-\infty}^{\infty} \eta dr$ may have a definite value.
The dashed line corresponds to the case where the adding complex number have the same magnitude $\Delta r$. The solid line corresponds to the addition of complex numbers whose magnitude decreases progressively.

For small $\Delta r$ we have:

$$\text{sum} = \text{Re} e^{-i \frac{\omega}{c} z - \frac{2 \pi}{2}}$$

$$= \frac{\mathbf{0}}{\omega} e^{-i \frac{\pi}{2}} e^{-i \frac{\omega}{c} z}$$
Thus,
\[ \int_{r=2}^{\infty} \eta \, dr = \eta_0 \frac{1}{\omega} e^{-\frac{2\pi}{\omega}} e^{-i\frac{\omega}{\varepsilon} z} \]
\[ = \eta_0 (-i\frac{\varepsilon}{\omega}) e^{-i\frac{\omega}{\varepsilon} z} \]
\[ \eta_0 = \eta(r=2) \]

Using (5) in expression (4)

\[ E(at P) = -\frac{\eta_0 x_0}{2\varepsilon_0} (i\frac{\omega}{\varepsilon}) e^{i\omega(t-\frac{z}{\varepsilon})} \]

where \( \eta_0 \) is the surface charge density
In Appendix -2 we show that the electric field produced by an infinitesimally-thin plate, at point along the central axis and away from plane, is independent of the position of the plate.

Accordingly, if we had not an infinitesimal-thin plate, but a thick plate of thickness “\(d\)”, then the total electric field at \(P\) will be a superposition of the fields produced by many infinitesimal sheets. The only difference will come from the term \(\eta\).

For a sheet of charges:

\[ \eta (\Delta A) \text{ was the total number of charges in the region of interest} \]

\( (\eta \text{ being the surface charge density}) \)

For a slab of thickness “\(d\)”

\[ N (d\ \Delta A) \text{ will be the total charges from the region of interest} \]

\( (N \text{ being the volumetric charge density.}) \)

Thus, in the expression for the field produced by a sheet of charges we have to replace \(\eta\) by \(Nd\)

\[ E \text{ (at } P\text{)} = -\frac{q\ Nd\ x_0}{2\varepsilon_0} (i\ \frac{\omega}{c})e^{i\omega(t-z/c)} \]