An analogy between electronic excitation in an atom and the motion of a mechanically forced oscillator.

\[ \text{me} = 9.1 \times 10^{-31} \text{kg} \]

But, how to choose \(\kappa\) ?

\[ \Delta E = \kappa \omega_0 \]

\[ \omega_{\text{res}} = \sqrt{\frac{\kappa}{\text{me}}} \]

\[ K = \text{me} \omega_0^2 \]
If $w_f = w_o$ the incident photon is absorbed by the atom

atom driven by the external radiation of frequency $w_f$

$w_f \rightarrow \Delta E = \hbar w_o$

$w_f = w_o \rightarrow \text{resonant absorption}$

input energy

output energy

$w_o, K$

accelerated change

loss energy
The emission of electromagnetic waves by the accelerated charge, can be considered as a channel of "dissipative" energy.

Light-atom interaction by a resonance absorption process can therefore be modelled by an equation similar to a harmonically forced damped oscillator.

\[ m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = F_0 \cos(\omega t) \]  \hspace{1cm} (2)

- \( k = m \omega_0^2 \) is related to the rate at which the accelerated electron re-emits light.
- \( b = b \left( \omega_0, m, \ldots \right) \) is the "position" of the electron.
- \( x \) is the position of the electron.

We'll give later a specific expression for \( b \).
Solving Eq (1) using complex variable.

First, we apply a trick:

Let \( y = y(t) \) be the solution of

\[
m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = F_0 \sin(\omega_f t) \quad (2)
\]

Multiply (2) by \( i \)

\[
m \frac{d^2 (iy)}{dt^2} + b \frac{d(iy)}{dt} + k(iy) = F_0 \left[ i \sin(\omega_f t) \right] \quad (3)
\]

(1) + (3) gives

\[
m \frac{d^2 z}{dt^2} + b \frac{dz}{dt} + k z = F_0 e^{i\omega_f t} \quad (4)
\]

where \( z(t) = x(t) + iy(t) \)

If we find a solution in (4), let's say \( z(t) \) then a solution to Eq (1) can be obtained by taking \( x(t) = \text{Real}\{z(t)\} \)
Solving Eq. 4

Since the driving force is harmonic of frequency \( w_f \), we guess the displacement \( z(t) \) will also be harmonic of frequency \( w_f \). Potentially, there may be phase difference between \( F_0 e^{i w_f t} \) and \( z(t) \). Thus, we propose a solution of the form,

\[
z(t) = A e^{i(w_f t + \phi)}
\]

where \( A \) and \( \phi \) can depend on \( w_f \) and other parameters.

Eq. 5 in Eq. 4 gives,

\[
(-w_f^2 + w_o^2 + i \frac{b}{m} w_f) A e^{i \phi} = \frac{F_0}{m}
\]

\[
A e^{i \phi} = \frac{-1}{(-w_f^2 + w_o^2 + i \frac{b}{m} w_f)} \frac{F_0}{m} = \frac{(-w_f^2 + w_o^2) - i \frac{b}{m} w_f}{(-w_f^2 + w_o^2)^2 + (\frac{b}{m} w_f)^2} \frac{F_0}{m}
\]
Notice, the complex number in the numerator can be expressed as,

\[ (-\omega_f^2 + \omega_0^2) + i \frac{b}{m} \omega_f = \]

\[ = \left[ (-\omega_f^2 + \omega_0^2)^2 + \left( \frac{b}{m} \omega_f \right)^2 \right]^{1/2} e^{i \tan^{-1} \frac{\frac{b}{m} \omega_f}{-\omega_f^2 + \omega_0^2}} \]

\[ \Rightarrow \]

\[ A e^{i\phi} = \frac{F_0/m}{\left[ (-\omega_f^2 + \omega_0^2)^2 + \left( \frac{b}{m} \omega_f \right)^2 \right]^{1/2}} e^{i \tan^{-1} \frac{\frac{b}{m} \omega_f}{-\omega_f^2 + \omega_0^2}} \]

\[ \text{5} \]

\[ A(\omega_f) = \frac{F_0/m}{\left[ (-\omega_f^2 + \omega_0^2)^2 + \left( \frac{b}{m} \omega_f \right)^2 \right]^{1/2}} \]

\[ \phi(\omega_f) = \tan^{-1} \frac{-\frac{b}{m} \omega_f}{-\omega_f^2 + \omega_0^2} \]
Summary.

\[ m \frac{d^2 z}{dt^2} + b \frac{dz}{dt} + k z = F_0 e^{i \omega_f t} \]

This equation admits solutions of the form

\[ z(t) = A e^{i (\omega_f t + \phi)} \]

where \( A = A(\omega_f) \) and \( \phi = \phi(\omega_f) \)
are given in expression (5)

Accordingly,

The solution to \( \ddot{z} \)

\[ m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + k x = F_0 \cos(\omega_f t) \]

is given by

\[ x(t) = \text{Re} \{ z(t) \} \]

\[ = A(\omega_f) \cos(\omega_f t + \phi) \]
**Graphic Analysis**

Phasor force: $F_0 e^{i \omega f t}$

Phasor position: $A e^{i (\omega f t + \phi)}$

Phasor velocity: $A \omega_f e^{i (\omega f t + \phi + \pi/2)}$

**Notice:**

Since $\phi$ is always negative, the position phasor always lags the force phasor.
Variation of $\phi$

At low $w_f$

\[ (-w_f^2 + w_0^2) \]

At higher $w_f$ but $w_f < w_0$

$w_f = w_0$

At $w_f > w_0$

\[ (-w_f^2 + w_0^2) \]
\[ A_{\text{max}} = \frac{F_0}{\kappa} \frac{1}{\alpha} (1 - \frac{1}{4Q^2}) \phi \]

For \( Q \gg 1 \)
\[ A_{\text{max}} \approx Q \frac{F_0}{\kappa} \]

\[ \omega' = \sqrt{\omega^2 - \frac{1}{2}(\frac{b}{m})^2} = \omega \sqrt{1 - \frac{1}{2Q^2}} \quad (Q = \frac{m}{b} \omega) \]

\( \omega' \approx \omega \) for high \( Q \)