2.3  Wave propagation: Phase and Group Velocity

2.3.A  Planes and Plane Waves

2.3.B  Traveling Plane Waves and Phase velocity

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2.3.A  Planes and plane waves

Let $\vec{r} = (x,y,z)$ and $\hat{n}$ be the spatial coordinates and a unit vector, respectively.

Notice,

$$\vec{r} \cdot \hat{n} = const$$

locates the points $\vec{r}$ that constitute a plane perpendicular to $\hat{n}$.

Different planes are obtained when using different values for the constant value (as seen in the figure below).
2.3.B Traveling Plane Waves and Phase velocity

Consider the two-variable vectorial function $\mathbf{E}$ of the form

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_o \, f(\mathbf{r} \cdot \mathbf{\hat{n}} - vt)$$

where $f$ is an arbitrary one-variable function and $\mathbf{E}_o$ is a constant vector.

(For example, $f$ could be the $\text{Cos}$ function.)

Notice, the points over a plane oriented perpendicular to $\mathbf{\hat{n}}$ and traveling with velocity $v$ define the locus of points where the phase of the wave $\mathbf{E}$ remains constant. For this reason, the wave $\psi$ is called a plane wave.
**Fig. 2** Schematic representation of a plane wave of electric fields. The figure shows the electric fields at two different planes, at a given instant of time. The fields lie oriented on the corresponding planes. The planes are perpendicular to the unit vector $\hat{n}$.

**Traveling Plane Waves (propagation in one dimension)**

$f(x - vt)$  
For any arbitrary function $f$, this represents a wave propagating to the right with speed $v$.  
$f$ could be COS, EXP, ... etc.

$f(x - vt)$ \(\overbrace{\text{phase}}\)  
Notice, a point $x$ advancing at speed $v$ will keep the phase of the wave $f$ constant.  
For this reason $v$ is called the *phase velocity* $v_{\text{ph}}$.

**Particular case: Traveling Harmonic Waves**

\[ \text{COS}(kx - \omega t) = \text{COS} \left[ k \left( x - \frac{\omega}{k} t \right) \right] \text{ and } e^{i(kx - \omega t)} \]

These are specific examples of waves propagating to the right with phase velocity $v_{\text{ph}} = \omega / k$.

In general $\omega = \omega(k)$.
That is, for different values of $k$, the corresponding waves travel with different phase velocities.

The specific relationship $\omega = \omega(k)$ depends on the specific physical system under analysis (waves in a crystalline array of atoms, light propagation, etc.).
2.3.C A Traveling Wave-package and its Group Velocity

Consider the expression 
\[ f(x) = \left( \frac{1}{\sqrt{2\pi}} \right) \int_{-\infty}^{\infty} F(k) e^{ikx} dk \]

as the representation of a pulse profile at \( t=0 \). Here \( e^{-ikx} \) is the profile of the harmonic wave \( e^{i(kx-\omega t)} \) at \( t = 0 \). The profile of the pulse at a later time will be represented by,

\[ \psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \ e^{i(kx-\omega t)} F(k) \]

Pulse composed by a group of traveling harmonic waves (1)

Since, in the general case, each component of the group travels with its own phase velocity,

\textit{would still it possible to associate a unique velocity to the propagating group of waves?}

The answer is positive; it is called \textit{group velocity}. Below we present an example that helps to illustrate this concept.

Case: Wavepacket composed of two harmonic waves

\textit{i) Analytical description)

For simplicity, let’s consider the case in which the packet of waves consists if only two waves of very similar wavelength and frequencies.

\[ \psi(x,t) = \cos[kx - \omega t] + \cos[(k + \Delta k)x - (\omega + \Delta \omega)t] \quad (2) \]

Using the identities \( \cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B) \)

and \( \cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B) \) one obtains

\[ \cos(A + B) + \cos(A - B) = 2\cos(A)\cos(B), \] which can be expressed as

\[ \cos(A) + \cos(B) = 2\cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right) \]

Accordingly, (38) can be expressed as,
\[ \psi(x,t) = 2\cos\left(\frac{\Delta k}{2}x - \frac{\Delta \omega}{2}t\right) \cos\left(k + \frac{\Delta k}{2}\right)x - \left(\omega + \frac{\Delta \omega}{2}\right)t \]

Since we are assuming that \( \Delta \omega \ll \omega \) and \( \Delta k \ll k \), we have
\[ \psi(x,t) = 2\cos\left(\frac{\Delta k}{2}x - \frac{\Delta \omega}{2}t\right) \cos\left(kx - \omega t\right) \quad (3) \]

**Modulation envelope**

Notice, the modulation envelope travels with velocity equal to
\[ v_g = \frac{\Delta \omega}{\Delta k}, \quad (4) \]
which is known as the group velocity.

In summary,
\[ \psi(x,t) = 2\cos\left(\frac{\Delta k}{2}x - \frac{\Delta \omega}{2}t\right) \cos\left(kx - \omega t\right) \quad (5) \]

The *phase velocity* is a measure of the velocity of the harmonic waves components that constitute the wave.
The *group velocity* is the velocity at which the positions of maximum interference propagate
More general, the *group velocity* is the velocity at which the “envelope” profile propagate (as will be observed better in the graphic analysis given below.)
**Graphical description**

- **EXAMPLE-1:** Visualization of the addition of two waves whose $k$’s and $\omega$’s are very similar in value.

Let,

\[
C(z, t) = \cos [k_1 z - \omega_1 t] = \cos [2z - 5t] \quad (6)
\]

\[
D(z, t) = \cos [k_2 z - \omega_2 t] = \cos [2.1z - 5.25t]
\]

Fig. 3a The profile of the individual waves $C(z, t)$ and $D(z, t)$ are plotted individually at $t=0.1$ over the $0 < z < 75$ range.
At $t=0.1$

**ADDITION of WAVES**

<table>
<thead>
<tr>
<th>$k_1$</th>
<th>$\Delta k$</th>
<th>$\omega_1$</th>
<th>$\Delta \omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.1</td>
<td>5</td>
<td>0.25</td>
</tr>
<tr>
<td>$V=\omega_1/k_1=2.5$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Fig. 3b** The profile of the individual waves $C(z, t)$ and $D(z, t)$ as well as the profile of the sum $C(z, t) + D(z, t)$ are plotted at $t=0.1$ over the $0<z<75$ range.

The two plots given above are repeated once more but over a larger range in order to observe the multiple regions of constructive and destructive interference.

**Fig. 4a**
Fig. 4b The profile of the individual waves $C(z, t)$ and $D(z, t)$ given in expression (41) above, as well as the profile of the sum $C(z, t) + D(z, t)$ are plotted at $t = 0.1$ over the $0 < z < 160$ range.

The wave $C(z, t) + D(z, t)$ (the individual waves given in expression (41) above) is plotted at 2 different times in order to observe the net motion of the interference profiles (group velocity).

\[
\begin{align*}
k_1 &= 2 \\
\omega_1 &= 5 \\
\Delta k &= 0.1 \\
\Delta \omega &= 0.25 \\
C &= \cos( k_1 z - \omega_1 t ) \\
D &= \cos( (k_1 + \Delta k) z - (\omega_1 + \Delta \omega) t )
\end{align*}
\]

GROUP VELOCITY $= 2.5$
Fig. 5 Verify by yourself, measuring in the graph above that the net displacement of the valley (~10) divided by the incremental time (4-0.1=3.9), that the envelope profile travels with a velocity equal to 10/3.9 ~ 2.5. This value coincides with $\Delta \omega / \Delta k = 0.25/01$.

In the example above:

**Phase velocity of the individual waves is 2.5**

**Group velocity is 2.5**

- **EXAMPLE-2:** Visualization of the addition of two waves whose $k$'s and $\omega$'s are very similar in value.

Let,

$$C(z, t) = \cos[k_1 z - \omega_1 t] = \cos[2z - 5t]$$

$$D(z, t) = \cos[k_2 z - \omega_2 t] = \cos[2.1z - 5.1t]$$

$$k_1 = 2 \quad \Delta k = 0.1$$

$$\omega_1 = 5 \quad \Delta \omega = 0.1$$

$$V_{\text{phase}} = \frac{\omega_1}{k_1} = 2.5 \quad V_{\text{group}} = \frac{\Delta \omega}{\Delta k} = 1$$

**GROUP VELOCITY = 1**

![Diagram](image)

Fig. 6 Verify by yourself, measuring in the graph above that the net displacement of the valley (~4) divided by the incremental
time \((4-0.1=3.9)\), that the envelope profile travels with a velocity equal to \(5/3.9 \sim 1\). This value coincides with \(\Delta \omega / \Delta k = 0.1/01\).

In the example above:

Phase velocity of the individual waves is 2.5
Group velocity is 2.5

Comparing Fig.5 and Fig.6 we get evidence that the wavepackets of the higher group velocity advances more than the one lower group velocity, despite the fact that the components have similar phase velocity.

2.3.D Phasor method to analyze waves. Understanding how of a wavepacket forms.

We illustrate first how to use the method of phasors to add waves.

What is phasor?

\[ e^{i\theta} \]

\[ \text{Fig.7 Phasor representation in the complex plane.} \]

Using \(\theta = kx-\omega t\), where \(t\) is the time variable, one obtains the following interpretation,
**Fig. 8** The horizontal components of a phasor represents a real harmonic wave.

**EXAMPLE**
Consider the wave described by:
\[ \Psi(x,t) = \cos(kx - \omega t) \]
where \( k=2 \) and \( \omega = 5 \)

Analyze the wave at a given fixed time (\( t = 0.1 \))

- **Real variable representation**

  At \( t=0.1 \)

- **Phasor representation**
EXAMPLE
Case: wavepacket composed of two waves

Let's consider the addition of two harmonic waves

\[ \cos(k_A x) + \cos(k_B x) \]

where \( k_A < k_B \), and \( k_B - k_A = \Delta k \).

To evaluate (8) we will work in the complex plane. Accordingly, to each wave we will associate a corresponding phasor,

\[ e^{ik_A x} + e^{ik_B x} \]
The projection of the (complex) phasors along the horizontal axis gives the real-value we are looking for, in expression (8).

At a given position \( x \), the phase-difference between the two wave profiles is equal to,

\[
\text{Phase difference} = k_B x - k_A x = (k_B - k_A) x
\]  

(9)

The following happens:

\(a\) The waves interfere constructively at \( x = 0 \equiv x_o \) (both waves have a phase equal to zero.)

\[\begin{align*}
\text{Cos}(k_B x) \quad \text{Cos}(k_A x)
\end{align*}\]

\(b\) As \( x \) increases a bit, the interference is not as perfect since the phase of the waves start to differentiate from each other \((k_B - k_A) x \neq 0\); consequently the sum of the waves should display an oscillatory behavior as \( x \) increases.

\[\begin{align*}
\text{Fig. 9 Analysis of wave addition by phasors in the complex plane. For clarity, the magnitude of one of the phasors has been drawn larger than the other one.}
\end{align*}\]
c) As $x$ increases, it will reach a particular value $x = x_1$ that makes the phase difference between the waves equal to $2\pi$. The value of $x_1$ is determined by the condition,

$$(k_B - k_A)x_1 = 2\pi$$

That is, the waves interfere constructively again at $x_1 = 2\pi / \Delta k$, where $k_B - k_A \equiv \Delta k$

**Fig. 10 Left:** In general the phasors do not coincide. **Right:** At a specific value of $x = x_1$, both phasors coincide, thus giving a maximum value to the sum of the waves (at that location.) The phasors diagram also makes clear that as $x$ keeps increasing, constructive interference will also occur at multiple values of $x_1$.

It is expected then that the wave-pattern (the sum of the two waves) observed around $x = 0$ will repeat again at around $x = x_1$. 
d) Notice that additional regions of constructive interference will occur at positions $x = x_n$ satisfying $(k_B - k_A)x_n = n \cdot 2\pi$ or $x_n = n \cdot 2\pi / \Delta k$ ($n = 1, 2, 3, \ldots$). The phasors diagram, therefore, makes clear that as $x$ keeps increasing, additional discrete values ($x_1, x_2, x_3$, etc) will be found to produce additional maxima.

![Fig. 11 Wavepacket composed of two harmonic waves](image)

It becomes clear from the analysis above that a packet composed of only two single harmonic waves of different wavelength can hardly represents a localized pulse. Rather, it represents a train of pulses. Let’s try to understand qualitatively (using the method of phasors) how can we end up with a single compact pulse.

**Case: A wavepacket composed of several harmonic waves**

When adding several harmonic waves $\sum_{i=1}^{M} A_i \cos(k_i x)$, with $M > 2$, the condition for having repeated regions of constructive interference still can occurs. In effect,

- First, there will be of course a constructive interference around $x=0$.
- Second, we expect the existence of a position $x=x_1$ that will make each of the quantities $(k_i-k_j)x_1$ equal to a multiple of $2\pi$. 
\[(k_i - k_j)x_1 = (\text{integer})_{ij} \cdot 2\pi \tag{10}\]

(for all the \(ij\) combinations, with \(i\) and \(j = 1, 2, 3, \ldots, M\)).

When this happens, it would mean that all the corresponding phasors coincide, thus giving a maximum of amplitude.

- Third, additional regions of maximum interference will occur at multiple integers of \(x_1\).

\[\text{Fig. 12 Wavepacket composed of a large number of harmonic waves (to be compared with Fig. 4.13 above.) The train of pulses are more separated from each other.}\]

Notice also that,

the greater the number \(M\) of harmonic components in the packet (with wavevectors \(k_i\) within the same range \(\Delta k\) shown in the figure above),

the more stringent becomes for all the \(M\) waves to satisfy at once the condition (10) for constructive interference.

This means, a greater value of \(x\) may be needed any time an extra number of harmonic waves are included in the packet.

Since the other maxima of interference occur at multiple values of \(x_1\), we expect, therefore, that the greater number of \(k\)-values (within the same range \(\Delta k\)), the more separated from each other will be the regions of constructive interference. This is shown in Fig. 4.14.

**Case: Wavepacket composed of an infinite number of harmonic waves**

Adding more and more wavevectors \(k\) (still all of them within the same range \(\Delta k\) show in the figure below)) will make the value of \(x_1\) to become greater and greater. As we consider a continuum variation of
$k$, the value of $x_1$ will become infinite. That is, we will obtain just one pulse.

**Fig. 4.13** Wavepacket composed of wavevectors $k$ within a continuum range $\Delta k$ produces a single pulse.

What about the variation of the pulse-size as the number of wavevectors (all with values within the range $\Delta k$) increases?

Fig. 13 above already suggests that the size should decrease. In effect, as the number of harmonic waves increases, the multiple addition of waves tends to average out to zero, unless $x = 0$ or for values of $x$ very small; that is the pulse becomes narrower.

Thus, we now can understand better the property stated in a previous paragraphs above,

*the more localized the function the broader its spectral response; and vice versa.*

In effect, notice in the previous figure that if we were to increase the range $\Delta k$, the corresponding range $\Delta x$ of values of the $x$ coordinate for which all the harmonic wave component can approximately interfere constructively would be reduced; and vice versa.
In short:

\[ \Delta x \sim \frac{1}{\Delta k} \]  

(11)

This is a general property of the Fourier analysis of waves (in principle, it has nothing to do with Quantum Mechanics.

Comment: One way to describe QM is within the framework of Fourier analysis. In this context, some of the mathematical terms are identified (via the de Broglie hypothesis) with the particle’s physical variables, which, accordingly, become subjected to the relationship indicated in (11). The realization that physical variables are subjected to the relationship (11) constitutes one of the cornerstones of Quantum Mechanics.