The Laplacian and Mean and Extreme Values

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Abstract. The Laplace operator is pervasive in many important mathematical models, and fundamental results such as the Mean Value Theorem for harmonic functions, and the Maximum Principle for super-harmonic functions are well-known. Less well-known is how the Laplacian and its powers appear naturally in a series expansion of the mean value of a function on a ball or sphere. This result is proven here using Taylor's Theorem and explicit values for integrals of monomials on balls and spheres. This result allows for non-standard proofs of the Mean Value Theorem and the Maximum Principle. Connections are also made with the discrete Laplacian arising from finite difference discretization.

1. THE SERIES EXPANSION OF THE MEAN VALUE. The Laplace operator (or Laplacian) in d dimensions,

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}\ , \tag{1}$$

appears in a variety of important mathematical models, such as the heat/diffusion equation, the wave equation, the Schrödinger equation and some forms of the Navier-Stokes equations. The Mean Value Theorem for harmonic functions ($\Delta u = 0$) on a ball or sphere, and the Maximum and Minimum Principles for super-harmonic ($\Delta u \geq 0$) and sub-harmonic ($\Delta u \leq 0$) functions on bounded domains, are well-known. It is perhaps less well-known that the Laplacian and its powers appear very naturally when the mean value of u is expanded in terms of a series with respect to the radius of the ball (or sphere). Our key result, Theorem 3, can be found, for example, in [**2**] for balls and spheres in $\mathbb{R}^3$. The expansion given in Theorem 3 is sometimes referred to as a Pizzetti series (cf. [**1**]), in reference to its earliest known occurrence [**4**]. The proof given in [**2**] uses Green's second identity, the fundamental solution for the Laplacian, and a clever recursion. In this section we provide a more straightforward proof of the result in $\mathbb{R}^d$, employing Taylor's Theorem and formulas for integrals of monomials on balls and spheres.

Definition (Multi-index Notation and Taylor Polynomials). Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_d) \in [\mathbb{N}_0]^d$, we define

$$|\alpha| = \alpha_1 + \cdots + \alpha_d \quad , \quad \alpha! = \alpha_1! \cdots \alpha_d!\ . \tag{2}$$

The α partial derivative of a function u, $D^\alpha u$, and the monomial $(x - \overline{x})^\alpha$ for fixed $\overline{x} = (\overline{x}_1, \cdots, \overline{x}_d)$ and variable $x = (x_1, \cdots, x_d)$, are given by

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \quad , \quad (x - \overline{x})^\alpha = (x_1 - \overline{x}_1)^{\alpha_1} \cdots (x_d - \overline{x}_d)^{\alpha_d}. \tag{3}$$

Assuming that $D^\alpha u(\overline{x})$ is defined for all multi-indices $|\alpha| \leq m$, the m^{th}-order Taylor Polynomial of u, centered at $\overline{x}$, is

$$T_m(x) = \sum_{|\alpha| \leq m} \frac{D^\alpha u(\overline{x})}{\alpha!} (x - \overline{x})^\alpha. \tag{4}$$

Theorem 1 (Taylor's Theorem). *Suppose $\Omega \subset \mathbb{R}^d$ is open and convex, and $x, \overline{x} \in \Omega$. If $u \in C^m(\Omega)$, then*

$$u(x) = T_m(x) + o(|x - \overline{x}|^m) \text{ as } |x - \overline{x}| \to 0, \tag{5}$$

where T_m is given by (4).

Here, and elsewhere, we overload the notation $|\cdot|$, using it to represent the order of a multi-index, the Euclidean norm of a point/vector in $\mathbb{R}^d$, and the volume of a ball, $|B|$, or surface area of a sphere, $|S|$.

Definition (Balls and Spheres). Given a point $\overline{x} \in \mathbb{R}^d$ and a radius $h > 0$, we have the ball and sphere

$$B = B(\overline{x}, h) = \{x \in \mathbb{R}^d : \ |x - \overline{x}| < h\}, \tag{6}$$

$$S = S(\overline{x}, h) = \{x \in \mathbb{R}^d : \ |x - \overline{x}| = h\}. \tag{7}$$

An elegant proof of the following result concerning integration of monomials on balls and spheres is given in [**3**].

Theorem 2 (Integration of Monomials). *It holds that*

$$\int_B (x - \overline{x})^\alpha \, dx = \begin{cases} \frac{2\Gamma(\beta_1)\cdots\Gamma(\beta_d)}{\Gamma(\beta_1+\cdots+\beta_d)} \frac{h^{|\alpha|+d}}{|\alpha|+d} & \text{if all } \alpha_j \text{ are even,} \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

$$\int_S (x - \overline{x})^\alpha \, ds = \begin{cases} \frac{2\Gamma(\beta_1)\cdots\Gamma(\beta_d)}{\Gamma(\beta_1+\cdots+\beta_d)} h^{|\alpha|+d-1} & \text{if all } \alpha_j \text{ are even,} \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

where $\beta_j = (\alpha_j + 1)/2$, and $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt$ is the Gamma-function.

From the case $\alpha = (0, \dots, 0)$, we quickly obtain the well-known formulas for the volume of B and the surface area of S,

$$|B| = \frac{2\pi^{d/2}}{d\,\Gamma(d/2)} h^d \quad , \quad |S| = \frac{2\pi^{d/2}}{\Gamma(d/2)} h^{d-1}. \tag{10}$$

Let $\alpha = 2\sigma$, so all components are even, and take $|\alpha| = 2k$. After some simplification, using $\Gamma(z+1) = z\Gamma(z)$, $\Gamma(1/2) = \sqrt{\pi}$ and (10), we see that

$$\frac{1}{\alpha!} \int_B (x - \overline{x})^\alpha \, dx = \frac{|B| \, h^{2k}}{2^k \sigma! \prod_{j=1}^k (d + 2j)}, \tag{11}$$

$$\frac{1}{\alpha!} \int_S (x - \overline{x})^\alpha \, ds = \frac{|S| \, h^{2k}}{2^k \sigma! \prod_{j=0}^{k-1} (d + 2j)}. \tag{12}$$

We are now ready to state our main result concerning the series expansion of the mean values of u on B and S.

Theorem 3 (Series Expansion of Mean Values). *Suppose that $u \in C^{2p}(\Omega)$ for some open set $\Omega \subset \mathbb{R}^d$ containing $\overline{B}$. It holds that*

$$u_{ave}(B) = \frac{1}{|B|} \int_B u(x) \, dx = \sum_{k=0}^{p} \frac{\Delta^k u(\overline{x})}{2^k k! \prod_{j=1}^k (d + 2j)} h^{2k} + o(h^{2p}), \tag{13}$$

$$u_{ave}(S) = \frac{1}{|S|}\int_S u(x)\,ds = \sum_{k=0}^{p} \frac{\Delta^k u(\overline{x})}{2^k k! \prod_{j=0}^{k-1}(d+2j)}\, h^{2k} + o(h^{2p}), \tag{14}$$

as $h \to 0$.

Proof. Since the proof for S is essentially identical, we only prove the result for B. In particular, we see that

$$\int_B T_{2p}(x)\,dx = \sum_{k=0}^{p} \frac{|B|\Delta^k u(\overline{x})}{2^k k! \prod_{j=1}^{k}(d+2j)}\, h^{2k}.$$

Here we have used three basic results: only multi-indices of the form $\alpha = 2\sigma$ survive the integration, the identity (11), and the fact that

$$\Delta^k u = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}\right)^k u = \sum_{\substack{|\alpha|=2k\\ \alpha=2\sigma}} \frac{k!}{\sigma!}\, D^\alpha u, \tag{15}$$

which follows from the multinomial expansion of Δ^k. Noting that

$$\int_B u\,dx = \int_B T_{2p}\,dx + |B|\,o(h^{2p}),$$

and dividing by $|B|$, completes the proof. ∎

For convenience, we give the first three terms of the series for both mean values.

$$u_{ave}(B) = u(\overline{x}) + \frac{\Delta u(\overline{x})}{2(d+2)}\, h^2 + \frac{\Delta^2 u(\overline{x})}{8(d+2)(d+4)}\, h^4 + \cdots, \tag{16}$$

$$u_{ave}(S) = u(\overline{x}) + \frac{\Delta u(\overline{x})}{2d}\, h^2 + \frac{\Delta^2 u(\overline{x})}{8d(d+2)}\, h^4 + \cdots. \tag{17}$$

The series expansions of the mean values can be used to provide a simple proof of the Mean Value Theorem for harmonic functions. More specifically, if $\Delta u = 0$ in some open set $\Omega \supset \overline{B}$, then $\Delta^k u(\overline{x}) = 0$ for all $k > 0$ (we note that u is analytic on Ω). So $u_{ave}(B) = u_{ave}(S) = u(\overline{x})$. We have proved

Theorem 4 (Mean Value Theorem for Harmonic Functions). *If* $\Delta u = 0$ *in some open set* $\Omega \supset \overline{B}$*, then* $u_{ave}(B) = u_{ave}(S) = u(\overline{x})$.

Remark. Re-expressing the results of Theorem 3 in the case $p = 1$, we see that that

$$\Delta u(\overline{x}) = \lim_{h\to 0} 2(d+2)\frac{u_{ave}(B) - u(\overline{x})}{h^2} = \lim_{h\to 0} 2d\,\frac{u_{ave}(S) - u(\overline{x})}{h^2}. \tag{18}$$

It follows simply from this that, for $u \in C^2$ in some open set Ω, if the average value of u on any ball (sphere) in Ω is equal to the value of u at the center of the ball (sphere), then u is harmonic in Ω.

2. THE MAXIMUM AND MINIMUM PRINCIPLES. We provide a proof of the Maximum Principle that applies Theorem 3 in the case $p = 1$ at a key point in the argument.

Theorem 5 (Maximum and Minimum Principles). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary $\partial\Omega$, and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$.*

1. *If $\Delta u \geq 0$ in Ω, then* $\max\limits_{x\in\overline{\Omega}} u(x) = \max\limits_{x\in\partial\Omega} u(x)$.
2. *If $\Delta u \leq 0$ in Ω, then* $\min\limits_{x\in\overline{\Omega}} u(x) = \min\limits_{x\in\partial\Omega} u(x)$.

Proof. We prove only the first of these, by contradiction. Let $M_0 = \max_{x\in\overline{\Omega}} u(x)$ and $M = \max_{x\in\partial\Omega} u(x)$. Suppose that $M < M_0$, and let $x_0 \in \Omega$ be a point for which $u(x_0) = M_0$. Choose $R > 0$ large enough so that $B(x_0, R) \supset \overline{\Omega}$, and define

$$v = u + \frac{M_0 - M}{2dR^2}\,|x - x_0|^2. \tag{19}$$

Note that $\Delta v = \Delta u + (M_0 - M)/R^2 > 0$; there is no need to introduce v in the argument if we assume that $\Delta u > 0$ in Ω. We also see that $v(x_0) = u(x_0) = M_0$ and

$$v(x) \leq u(x) + \frac{M_0 - M}{2d} \leq M + \frac{M_0 - M}{2d} < M_0 \text{ for } x \in \partial\Omega, \tag{20}$$

so v must attain is maximum over $\overline{\Omega}$ at some interior point $x_1 \in \Omega$. Since $\Delta v(x_1) > 0$, employing Theorem 3 for $p = 1$ and $h > 0$ sufficiently small, we see that $v_{ave}(B) > v(x_1)$ for $B = B(x_1, h)$. But this contradicts the fact that $v(x_1)$ is the maximal value of v; the mean value on any ball cannot exceed the maximal value within that ball. ∎

3. CONNECTIONS WITH THE DISCRETE LAPLACIAN. Recalling (18), we here consider the case in which we fix some (small) h instead of taking the limit, and replace the true average on the sphere with a discrete average at the points on the sphere that intersect the cartesian axes. Letting e_i denote the standard coordinate vectors in $\mathbb{R}^d$, a simple consequence of Taylor's Theorem in $\mathbb{R}$ is that

$$u(\overline{x} - he_i) + u(\overline{x} + he_i) = 2u(\overline{x}) + \frac{\partial^2 u(\overline{x})}{\partial x_i^2}\,h^2 + o(h^2), \tag{21}$$

as $h \to 0$, for any $u \in C^2$ in a neighborhood of $\overline{x}$. Summing both sides for $1 \leq i \leq d$, and then dividing by the number of "boundary points", namely $2d$, we obtain

$$\frac{1}{2d}\sum_{i=1}^{d}[u(\overline{x} - he_i) + u(\overline{x} + he_i)] = u(\overline{x}) + \frac{h^2}{2d}\,\Delta u(\overline{x}) + o(h^2), \tag{22}$$

which is obviously a "discrete" version of (14) when $p = 1$, in which the true average is replaced by the standard discrete average of these $2d$ values.

The (finite difference) discrete Laplacian is defined by

$$\Delta_h u(\overline{x}) = \frac{1}{h^2}\sum_{i=1}^{d}[u(\overline{x} - he_i) - 2u(\overline{x}) + u(\overline{x} + he_i)], \tag{23}$$

and Taylor's Theorem guarantees that $\Delta_h u(\overline{x}) = \Delta u(\overline{x}) + o(1)$ as $h \to 0$. Denoting the lefthand-side of (22) by $\hat{u}_{ave}(S)$, we have

$$\hat{u}_{ave}(S) = u(\overline{x}) + \frac{h^2}{2d}\,\Delta_h u(\overline{x}), \tag{24}$$

a result that is commonly used to prove a Discrete Maximum Principle in the analysis of finite difference methods.

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