Computational Tools for Exploring Eigenvector Localization

Jeff Ovall

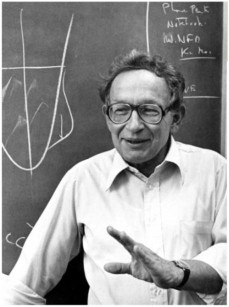
Maseeh Professor
Fariborz Maseeh Department of Mathematics & Statistics
Portland State University

jovall@pdx.edu

Simons Collaboration on Localization of Waves

DMS 2136228, *2208056*

Some Inspirational Quotes



Very few believed [localization] at the time, and even fewer saw its importance; among those who failed to fully understand it at first was certainly its author. It has yet to receive adequate mathematical treatment, and one has to resort to the indignity of numerical simulations to settle even the simplest questions about it.

—Philip W. Anderson, Nobel lecture, 8 December 1977

The purpose of computing is insight, not numbers.
R. Hamming (1962).

The purpose of computing is numbers — specifically, correct numbers.
L. Greengard (∼2000).

Lagendijk, van Tiggelen, Diederik, Physics Today 2009

Eigenvalue Problem

Eigenvalue Problem: Find (λ, ψ), $\psi \neq 0$

$$\mathcal{L}\psi = \lambda\psi \text{ in } \Omega \quad , \quad \psi = 0 \text{ on } \partial\Omega$$

Magnetic Schrödinger Operator:

$$\begin{aligned}\mathcal{L}v &= (-\mathrm{i}\nabla - \mathbf{A}) \cdot (-\mathrm{i}\nabla - \mathbf{A})v + V\,v \\ &= -\Delta v + \mathrm{i}\,(\nabla \cdot (\mathbf{A}v) + \mathbf{A} \cdot \nabla v) + \left(\|\mathbf{A}\|^2 + V\right) v\end{aligned}$$

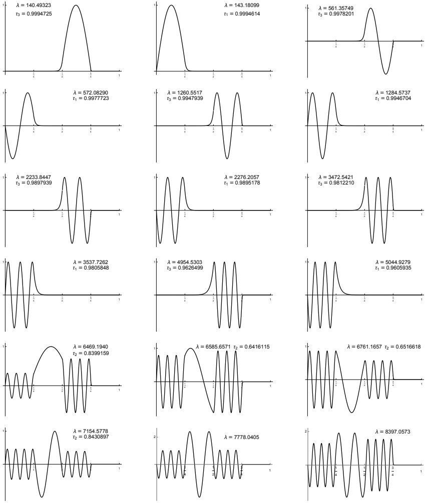
- Selfadjoint operator (real eigenvalues)
- Standard Schrödinger: $\mathbf{A} = \mathbf{0}$ — Rich mathematical theory, $\sim$ 15 years
- Magnetic Laplacian: $V = 0$

Eigenvector Localization:

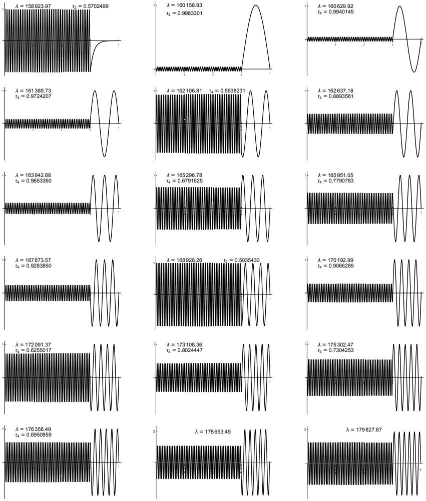
Coefficients ($\mathbf{A}$, V), domain geometry can cause strong spatial localization of some eigenvectors

1D Model Problem: $\mathcal{L} = -\Delta + V$, $V = (0, 80^2, 0, 400^2)$

Eigenvectors 1-16

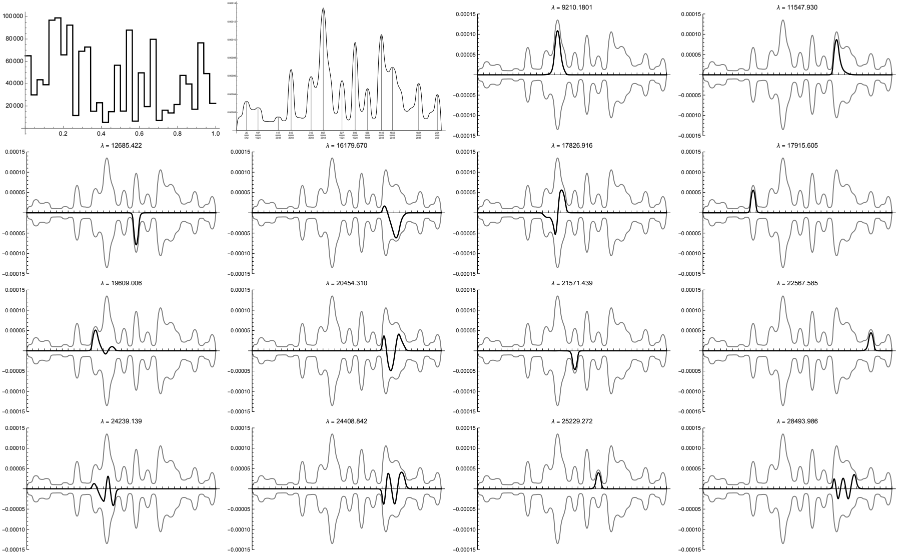


Eigenvectors 95-110

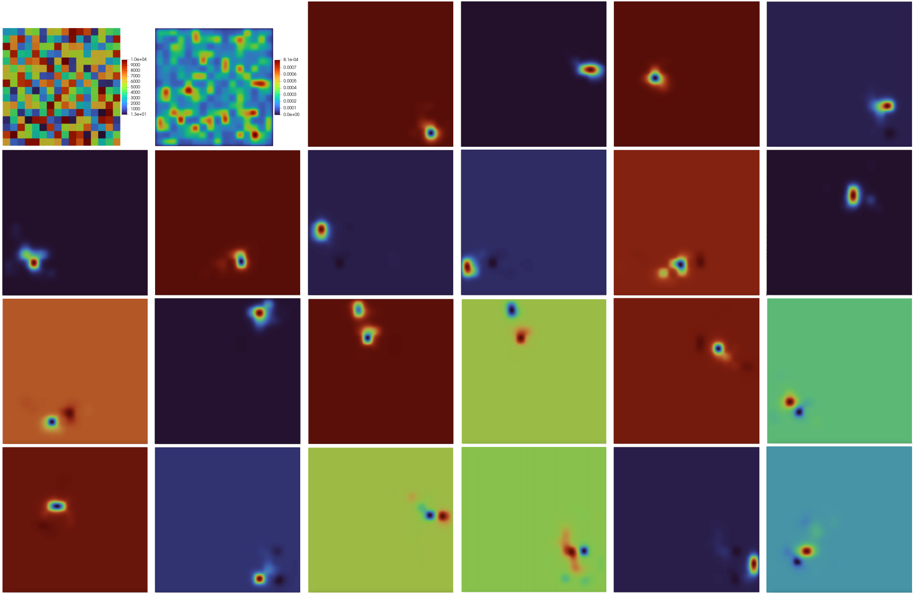


Landscape Inequality: 1D Illustration, $\mathcal{L} = -\Delta + V$

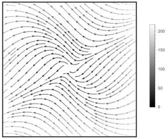
Potential | Landscape: $\mathcal{L}u = 1$ | $|\psi(x)|/(\lambda \, \|\psi\|_{L^\infty(\Omega)}) \leq u(x)$

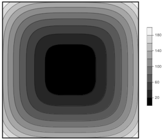


2D Model Problem: $\mathcal{L} = -\Delta + V$

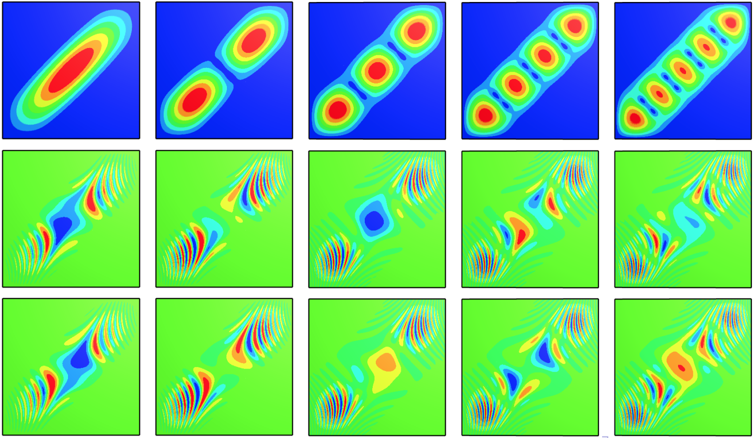


2D Localization Illustration: $\mathcal{L} = (-\mathrm{i}\nabla - \mathbf{A}) \cdot (-\mathrm{i}\nabla - \mathbf{A})$

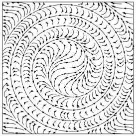




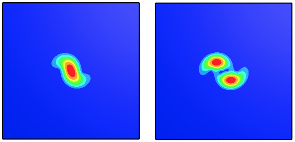
$$\mathbf{A} = -\alpha(x^2 + y^2, x^2 - y^2)$$
$$\alpha = 100$$

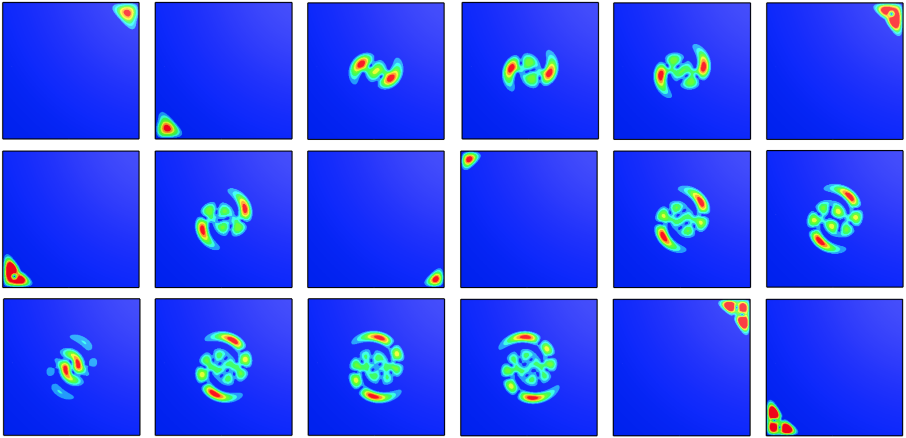


2D Localization Illustration: $\mathcal{L} = (-i\nabla - \mathbf{A}) \cdot (-i\nabla - \mathbf{A})$



$$\mathbf{A} = -\alpha(\cos f, \sin f)$$
$$f = 5\pi \sin(x^2 + y^2)$$
$$\alpha = 100$$

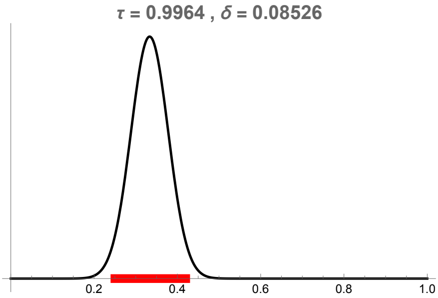


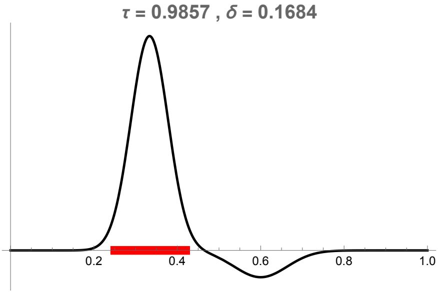


Two Complementary Measures

$$\tau(v,R) = \frac{\|v\|_{L^2(R)}}{\|v\|_{L^2(\Omega)}} \quad , \quad \delta(v,R) = \frac{\|v\|_{L^2(\Omega\setminus R)}}{\|v\|_{L^2(\Omega)}}$$

- For any $c \in \mathbb{C}$, $\tau(cv,R) = \tau(v,R)$ and $\delta(cv,R) = \delta(v,R)$
- $[\delta(v,R)]^2 + [\tau(v,R)]^2 = 1$





A Key Computational Task

Key Task:

Given a subdomain $R \subset \Omega$, a (small) tolerance $\delta^* > 0$ and a (large) interval $[a, b]$, find all eigenpairs (λ, ψ) of $\mathcal{L}$ such that

$$\lambda \in [a, b] \;\; \textit{and} \;\; \delta(\psi, R) \leq \delta^*$$

or determine that there are not any.

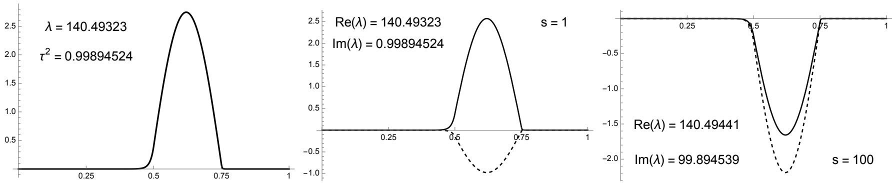
- R might be determined by landscape function (or similar)
  - Could miss regions of localization at high energy
- Multiple regions R and/or energy intervals $[a, b]$ can be explored independently
- "Encode" both constraints $\lambda \in [a, b]$ and $\delta(\psi, R) \leq \delta^*$ in a modified problem??

Modifying the Problem: Heuristic Motivation

Intuition: If (λ, ψ) is an eigenpair of $\mathcal{L}$ with $\delta(\psi, R)$ small, then

$$\overbrace{(\mathcal{L} + \mathrm{i}\, s\chi_R)}^{\mathcal{L}_s}\psi = (\mathcal{L} + \mathrm{i}\, s)\psi - \mathrm{i}\, s\chi_{\Omega\setminus R}\psi$$
$$= (\lambda + \mathrm{i}\, s)\psi - \mathrm{i}\, s\chi_{\Omega\setminus R}\psi \approx (\lambda + \mathrm{i}\, s)\psi\ ,$$

- Should be an eigenpair (μ, ϕ) of $\mathcal{L}_s$ with $\mu \approx \lambda + \mathrm{i}\, s$, $\phi \approx \psi$
- Hunt for eigenpairs of $\mathcal{L}_s$ with $\Im\mu \approx s$



"Encoding" Theorem

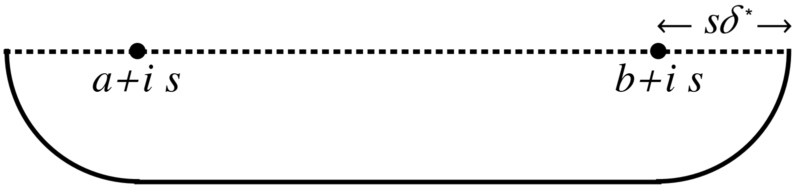
Theorem (Ovall/Reid, MCOM 2023)

Let (λ, ψ) *be an eigenpair of* $\mathcal{L}$*. For* $s > 0$ *sufficiently small, there is an eigenpair* $(\mu(s), \phi(s))$ *of* $\mathcal{L}_s$ *such that*

$$|\lambda + \mathrm{i}\, s - \mu(s)| \leq s\delta(\psi, R) \; .$$

In this case, if $\lambda \in [a, b]$ *and* $\delta(\psi, R) \leq \delta^*$*, then* $\mu(s) \in U = U(a, b, s, \delta^*)$ *(pictured below). Furthermore,*

$$\lim_{s\to 0} \frac{\mu(s) - \lambda}{\mathrm{i}s} = [\tau(\psi, R)]^2 \quad , \quad \lim_{s\to 0} \frac{\lambda + \mathrm{i}s - \mu(s)}{\mathrm{i}s} = [\delta(\psi, R)]^2 \; .$$

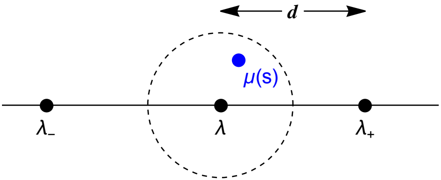


How Small is "Sufficiently Small"?

Theorem (Ovall/Reid, 2024)

Let (λ, ψ) be a simple eigenpair of $\mathcal{L}$, with $\delta = \delta(\psi, R)$, and λ. Let $d = \mathrm{dist}(\lambda, \mathrm{Spec}(\mathcal{L}) \setminus \{\lambda\})$. For $s \leq \frac{d}{2}\frac{1-\delta}{1+\delta}$, there is an eigenpair $(\mu(s), \phi(s))$ of $\mathcal{L}_s$ such that

$$|\lambda + \mathrm{i}s - \mu(s)| \leq s\delta(\psi, R) \ .$$



$$I - P = ((\mathcal{L} + \mathrm{i}s)(I - P) - \mu(s))^{-1}(I - P)(\mathcal{L} + \mathrm{i}s - \mu(s))$$

$$(I - P)\phi(s) = ((\mathcal{L} + \mathrm{i}s)(I - P) - \mu(s))^{-1}(I - P)(\mathrm{i}s\chi_{\Omega\setminus R}\phi(s))$$

$$\|(I - P)\phi(s)\|_{L^2(\Omega)} \leq \frac{s\,\|\phi(s)\|_{L^2(\Omega\setminus R)}}{\mathrm{dist}(\mu(s), \mathrm{Spec}(\mathcal{L} + \mathrm{i}s) \setminus \{\lambda + \mathrm{i}s\})} \leq \frac{s\,\|\phi(s)\|_{L^2(\Omega\setminus R)}}{d/2}$$

"Decoding" Theorem

Theorem (Ovall/Reid, MCOM 2023)

Let (μ, ϕ) *be an eigenpair of* $\mathcal{L}_s$ *with* $\|\phi\|_{L^2(\Omega)} = 1$ *and set* $\delta = \delta(\phi, R)$ *and* $\tau = \tau(\phi, R)$*. Then* $\Im\mu = s\tau^2$*, and*

$$\|(\mathcal{L} - \Re\mu)\phi\|^2_{L^2(\Omega)} = s^2\delta^2\tau^2 \ ,$$

$$\|(\mathcal{L} - \Re\mu)(\Re\phi)\|^2_{L^2(\Omega)} = s^2\left(\delta^4\|\Im\phi\|^2_{L^2(R)} + \tau^4\|\Im\phi\|^2_{L^2(\Omega\setminus R)}\right) \ .$$

Furthermore, if $\lambda \in \Lambda \subset \mathrm{Spec}(\mathcal{L})$ *satisfies* $|\lambda - \Re\mu| = \mathrm{dist}(\Re\mu, \mathrm{Spec}(\mathcal{L}))$*, then* $|\lambda - \Re\mu| \leq s\delta\tau$ *and*

$$\inf_{v \in E(\Lambda, \mathcal{L})} \frac{\|\phi - v\|_{L^2(\Omega)}}{\|\phi\|_{L^2(\Omega)}} \leq \frac{s\,\delta\tau}{\mathrm{dist}(\Re\mu\,,\mathrm{Spec}(\mathcal{L}) \setminus (\Lambda \cup \{\Re\mu\}))} \ .$$

Theorem (Ovall/Reid, MCOM 2023, Partial Restatement)

Let (μ,ϕ) be an eigenpair of $\mathcal{L}_s$, and set $\delta=\delta(\phi,R)$ and $\tau=\tau(\phi,R)$. Then

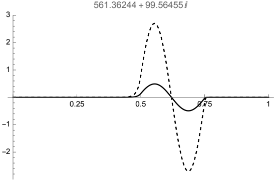
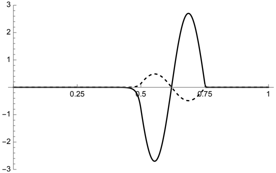
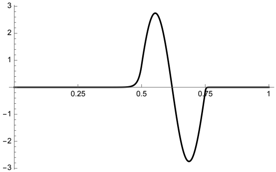
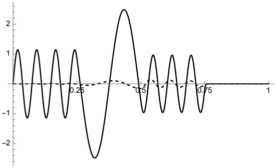
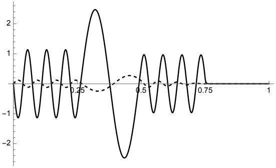
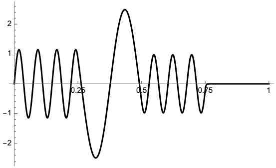
$$\|(\mathcal{L}-\Re\mu)(\Re\phi)\|^2_{L^2(\Omega)} = s^2\left(\delta^4\|\Im\phi\|^2_{L^2(R)} + \tau^4\|\Im\phi\|^2_{L^2(\Omega\setminus R)}\right) .$$

- Assuming $\|\phi\|_{L^2(\Omega)}=1$, we have $\|c\phi\|_{L^2(\Omega)}=1$ for any $c\in\mathbb{C}$ with $|c|=1$
- For any $c\in\mathbb{C}$, $\delta(c\phi,R)=\delta(\phi,R)$ and $\tau(c\phi,R)=\tau(\phi,R)$
- Further normalize ϕ, $\phi\longleftarrow c\phi$,

$$c=\operatorname*{arg\,min}_{b\in\mathbb{C}\,,\,|b|=1}\left(\delta^4\|\Im(b\phi)\|^2_{L^2(R)}+\tau^4\|\Im(b\phi)\|^2_{L^2(\Omega\setminus R)}\right)$$

- Minimization can be recast as 2×2 hermitian eigenvalue problem

$$\|(\mathcal{L} - \Re\mu)(\Re\phi)\|^2_{L^2(\Omega)} = s^2 \left(\delta^4 \|\Im\phi\|^2_{L^2(R)} + \tau^4 \|\Im\phi\|^2_{L^2(\Omega \setminus R)} \right)$$

Original ϕ	Renormalized ϕ	ψ
 42.141598	 0.0059877762	
 15.114742	 5.0622032	

Algorithm Template

procedure LOCALIZE(a, b, δ^*, R, s)

  Get eigenpairs (μ, ϕ) of $\mathcal{L}_s$ with $\mu \in U(a, b, s, \delta^*)$ ▷ First filter

  for each (μ, ϕ) **do**

    Re-normalize: $\phi \longleftarrow c\phi$

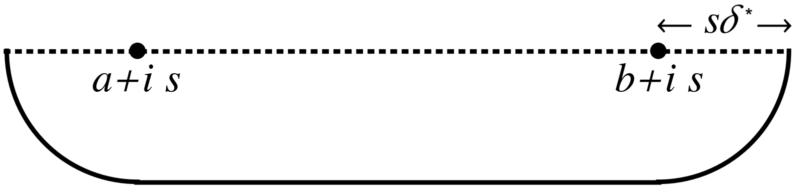
    Post-process: $(\Re\mu, \Re\phi) \rightsquigarrow (\tilde{\lambda}, \tilde{\psi})$ OR $(\Re\mu, \phi) \rightsquigarrow (\tilde{\lambda}, \tilde{\psi})$

    Final check: $\delta(\tilde{\psi}, R) < \delta^*$ and $\tilde{\lambda} \in [a, b]$? ▷ Second filter

  end for

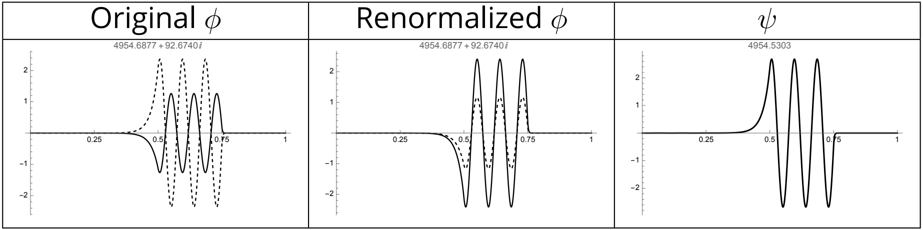
  return accepted $(\tilde{\lambda}, \tilde{\psi})$

end procedure



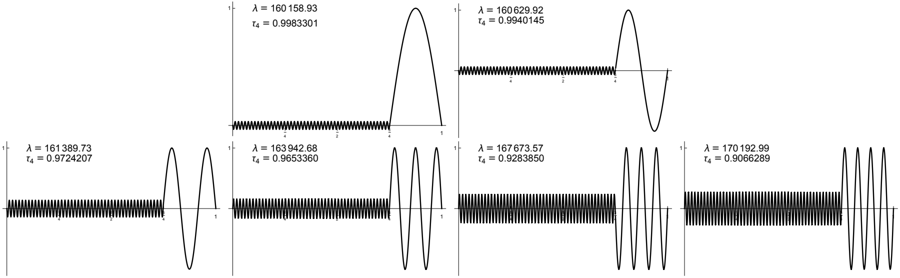
- 130 eigenvalues of $\mathcal{L}$ in $[a,b] = [0, 220\,000]$
- 5 eigenvectors localized in $R = (1/2, 3/4)$ with tolerance $\delta^* = 0.2$

$s = 1$			$s = 100$				
$\Re\mu$	$\Im\mu$	$\delta(\phi, R)$	$\Re\mu$	$\Im\mu$	$\delta(\phi, R)$	λ	$\delta(\psi, R)$
140.49323	0.99894524	0.0324770	140.49441	99.894539	0.0324748	140.49323	0.0324770
561.35749	0.99564487	0.0659934	561.36244	99.564551	0.0659886	561.35749	0.0659934
1260.5517	0.98961499	0.1019069	1260.5640	98.961665	0.1018987	1260.5517	0.1019069
2233.8447	0.97969199	0.1425062	2233.8708	97.969580	0.1424928	2233.8447	0.1425062
3472.5421	0.96279456	0.1928871	3472.5974	96.280425	0.1928620	3472.5421	0.1928871
4954.5303	0.92669479	0.2707494	4954.6877	92.674005	0.2706658	4954.5303	0.2707494

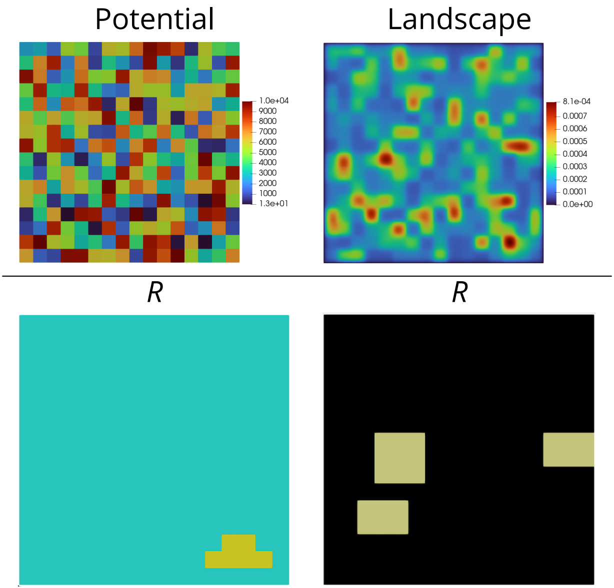


- 130 eigenvalues of $\mathcal{L}$ in $[a,b] = [0, 220\,000]$
- 2 eigenvectors localized in $R = (3/4, 1)$ with tolerance $\delta^* = 0.2$

$s=1$			$s=100$				
$\Re\mu$	$\Im\mu$	$\delta(\phi,R)$	$\Re\mu$	$\Im\mu$	$\delta(\phi,R)$	λ	$\delta(\psi,R)$
160158.93	0.99666301	0.0577667	160158.92	99.667632	0.0576513	160158.93	0.0577667
160629.92	0.98806481	0.1092483	160629.94	98.809982	0.1090879	160629.92	0.1092483
161389.73	0.94560213	0.2332335	161390.21	94.613050	0.2320981	161389.73	0.2332336
163942.68	0.93187361	0.2610103	163942.71	93.203093	0.2607088	163942.68	0.2610104
167673.57	0.86189881	0.3716197	167673.93	86.221341	0.3711962	167673.57	0.3716197
170192.99	0.82197599	0.4219289	170192.51	82.233071	0.4215084	170192.99	0.4219290

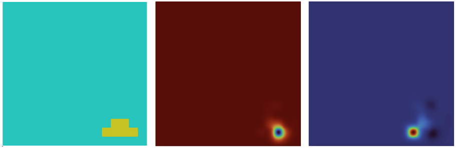


2D Model Problem: $\mathcal{L} = -\Delta + V$

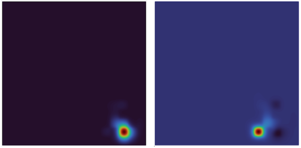


- R below, $\delta^* = 0.4$ ($\tau^* \approx 0.9165$), $[a, b] = [0, 4689]$, $s = 1$
- Two eigenvectors satisfy localization criterion

Region and Two Eigenvectors



Two Eigenvectors Found During First Filter: ψ_1, ψ_{18}



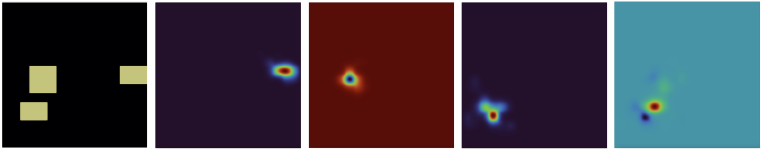
0.95136237 0.96078396

Algorithm Illustration

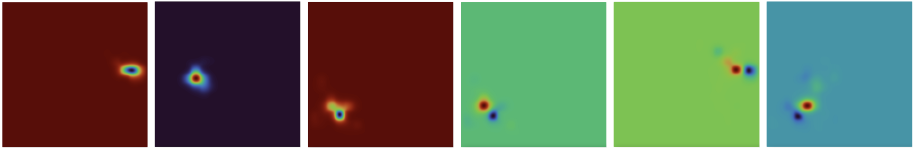
2D Model Problem

- R below, $\delta^* = 0.36$ ($\tau^* \approx 0.933$), $[a, b] = [0, 3100]$, $s = 1$
- Four eigenvectors satisfy localization criterion (two more close)

Region and Four Eigenvectors

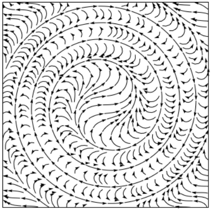


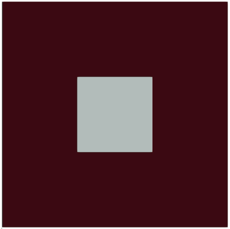
Six Eigenvectors Found During First Filter: $\psi_2, \psi_3, \psi_5, \psi_{16}, \psi_{19}, \psi_{22}$



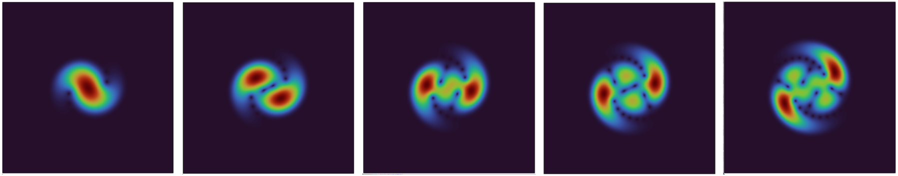
0.98086234 0.99869034 0.97697991 0.92034546 0.92118741 0.93467020

$s = 1, \delta^* = 0.3 \; (\tau^* \approx 0.954)$



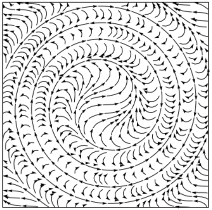


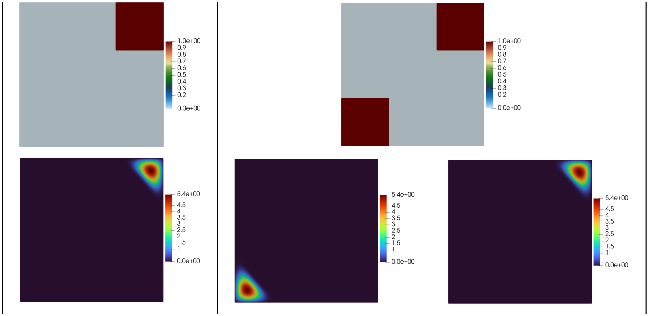
Five Eigenvectors Found During First Filter: $\psi_1, \psi_2, \psi_3, \psi_6, \psi_7$



0.99859423 0.98448747 0.93255720 0.88415904 0.87048385

$$s = 1,\ \delta^* = 0.3\ (\tau^* \approx 0.954)$$





Magnetic Schrödinger: Conjugation Lemma

Magnetic Schrödinger Operator (Notational Change):

$$H(\mathbf{A}, V) = (-\mathrm{i}\nabla - \mathbf{A}) \cdot (-\mathrm{i}\nabla - \mathbf{A}) + V \quad , \quad H(\mathbf{A}) = H(\mathbf{A}, 0)$$

Lemma (Conjugation Lemma (Gauge Invariance))

Suppose that $\mathbf{A} = \nabla a + \mathbf{F}$ *in* Ω. *Then,*

$$e^{-\mathrm{i}a} H(\mathbf{A}, V) e^{\mathrm{i}a} = H(\mathbf{F}, V) \ .$$

Furthermore, (λ, ψ) *is an eigenpair of* $H(\mathbf{A}, V)$ *iff* $(\lambda, e^{-\mathrm{i}a}\psi)$ *is an eigenpair of* $H(\mathbf{F}, V)$.

- Helmholtz decomp., $\nabla \cdot \mathbf{F} = 0$, not necessary for lemma
- An eigenvector of $H(\mathbf{A}, V)$ is localized in some region R iff an eigenvector of $H(\mathbf{F}, V)$ is localized there

A Computational Implication of Conjugation Lemma

Lemma

Suppose that $\mathbf{A} = \nabla a + \mathbf{F}$ *in* Ω. *Then* (λ, ψ) *is an eigenpair of* $H(\mathbf{A}, V)$ *iff* $(\lambda, e^{-ia}\psi)$ *is an eigenpair of* $H(\mathbf{F}, V)$.

- Perhaps easier to compute eigenvectors of $H(\mathbf{F}, V)$ [less oscillatory?]
- Strategy
  1. Compute a Helmholtz decomposition: $\mathbf{A} = \nabla a + \mathbf{F}$
  2. Compute some eigenpairs of $H(\mathbf{F}, V)$
  3. If desired (why?), "remodulate" eigenvectors of $H(\mathbf{F}, V)$ to obtain eigenvectors of $H(\mathbf{A}, V)$
- Ideally, computations for $H(\mathbf{F}, V)$ can be done more cheaply (e.g. coarser finite element space, better preconditioner)

Strategy for Computing Eigenpairs

1. Compute a Helmholtz decomposition: $\mathbf{A} = \nabla a + \mathbf{F}$
2. Compute some eigenpairs of $H(\mathbf{F}, V)$

A Practical Helmholtz Decomposition

1. Find $a \in H^1(\Omega)$ such that $\mathbf{A} = \nabla a + \mathbf{F}$, $\nabla \cdot \mathbf{F} = 0$

$$\Delta a = \nabla \cdot \mathbf{A} \text{ in } \Omega \quad , \quad \nabla a \cdot \mathbf{n} = \mathbf{A} \cdot \mathbf{n} \text{ on } \partial\Omega$$

   - a unique up to additive constant
   - Equivalent to $\operatorname*{arg\,min}_{a \in H^1(\Omega)} \|\mathbf{A} - \nabla a\|_{L^2(\Omega)}$

2. Set $\mathbf{F} = \mathbf{A} - \nabla a$

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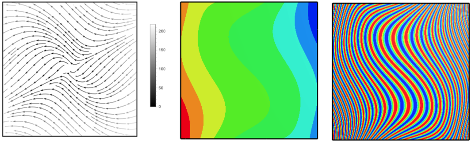
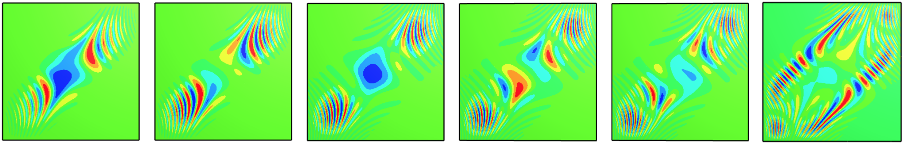
Comparing Eigenvectors of $H(A)$ and $H(F)$ $\mathbf{A} = \nabla a + \mathbf{F}$

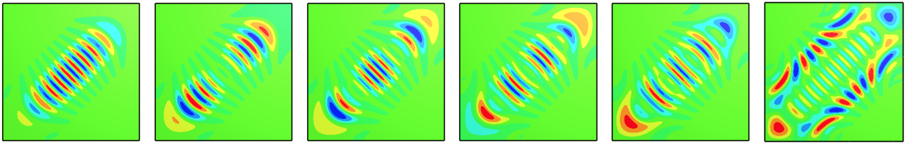
$\mathbf{A}$ a $\Re e^{-ia}$



$H(\mathbf{A})$: $\Re\psi_j$, $1 \leq j \leq 5, j = 20$



$H(\mathbf{F})$: $\Re\phi_j$, $1 \leq j \leq 5, j = 20$



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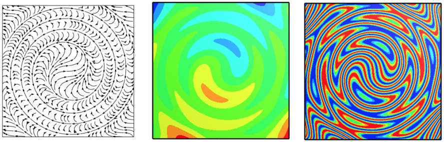

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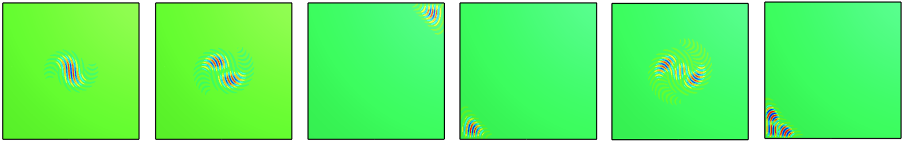

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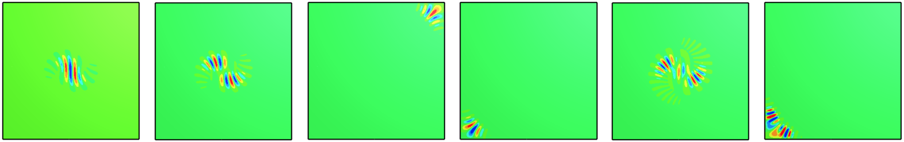

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A Landscape Inequality for $H(A)$

Theorem (Ovall/Quan/Reid/Steinerberger, 2024)

Let (λ, ψ) *be an eigenpair of* $H(\mathbf{A})$, *where* $\mathbf{A} = \nabla a + \mathbf{F}$ *and* $\nabla \cdot \mathbf{F} = 0$. *We have*

$$\frac{|\psi(x)|}{\lambda \|\psi\|_{L^\infty(\Omega)}} \leq \sqrt{u(x)} \left(\int_0^\infty \int_\Omega |\mathbb{E}(t,x,y)|^2 K_\Omega(t,x,y)\, dy\, dt \right)^{1/2}$$

where $-\Delta u = 1$ *in* Ω, $u = 0$ *on* $\partial\Omega$, *and*

$$\mathbb{E}(t,x,y) = \mathbb{E}_{\omega(0)=x,\omega(t)=y}\, e^{-\mathrm{i}\int_0^t \mathbf{F}(\omega(s))\cdot d\omega(s)} \,.$$

Furthermore,

$$\int_0^\infty \int_\Omega |\mathbb{E}(t,x,y)|^2 K_\Omega(t,x,y)\, dy\, dt \leq u(x)\,.$$

Path Integrals Involving $\mathbf{F}$

Linearization of F at x_0:

$$\mathbf{F}(x) = \underbrace{\mathbf{F}(x_0) + J(x_0)(x - x_0)}_{\mathbf{F}_L(x)} + \mathbf{R}(x)$$

$$\mathbf{F}_L(x) = \underbrace{\mathbf{F}(x_0) + \frac{1}{2}\left(J(x_0) + J(x_0)^T\right)(x - x_0)}_{\mathbf{F}_1(x)} + \underbrace{\frac{1}{2}\left(J(x_0) - J(x_0)^T\right)(x - x_0)}_{\mathbf{F}_2(x)}$$

$$\mathbf{F}_1(x) = \nabla f(x) \quad , \quad f(x) = \left(F(x_0) + \frac{1}{2}J(x_0)(x - x_0)\right) \cdot (x - x_0)$$

Path(s), Integrals (2D): $\omega = \omega(s)$**,** $s \in [0, t]$**:** $\omega(0) = x_0$**,** $\omega(t) = y$

$$\int_0^t \mathbf{F}(\omega(s)) \cdot d\omega(s) = \int_0^t \mathbf{F}_L(\omega(s)) \cdot d\omega(s) + \int_0^t \mathbf{R}(\omega(s)) \cdot d\omega(s)$$

$$\int_0^t \mathbf{F}_L(\omega(s)) \cdot d\omega(s) = f(y) + \mathrm{curl}\, F(x_0)\, \underbrace{\frac{1}{2}\int_0^t (\omega(s) - x_0)^{\perp} \cdot d\omega(s)}_{C(\omega)}$$

Path Integrals Involving **F**

$$\frac{|\psi(x)|}{\lambda\|\psi\|_{L^\infty(\Omega)}} \leq \sqrt{u(x)}\left(\int_0^\infty \int_\Omega |\mathbb{E}(t,x,y)|^2 K_\Omega(t,x,y)\,dy\,dt\right)^{1/2}$$

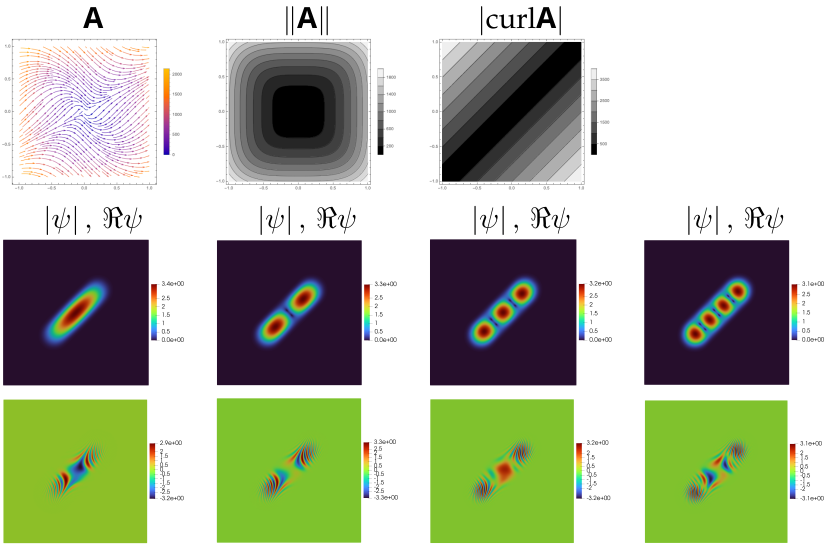
$$|\mathbb{E}(t,x_0,y)|^2 = \left|\mathbb{E}_{\omega(0)=x_0,\omega(t)=y}\left(e^{-\mathrm{i}\,C(\omega)\,\mathrm{curl}\,\mathbf{F}(x_0)}\,e^{-\mathrm{i}\int_0^t \mathbf{R}(\omega(s))\cdot d\omega(s)}\right)\right|$$

A Loose (but not meaningless) Statement:

If R is "small" "near" x_0*, then* $\mathrm{curl}\,\mathbf{F}(x_0) = \mathrm{curl}\,\mathbf{A}(x_0)$ *plays a key role in determining whether* $|\mathbb{E}(t,x_0,y)| \approx 1$.

- The curl appears in semi-classical limits (cf. Helffer), but we see it naturally as the first relevant term in the Taylor expansion
- Early in spectrum, one can expect localization of eigenvectors near places where $\mathrm{curl}\,\mathbf{F} = \mathrm{curl}\,\mathbf{A}$ (magnetic field!) is "small"

2D Localization Illustration, $H(A) = (-i\nabla - \mathbf{A}) \cdot (-i\nabla - \mathbf{A})$



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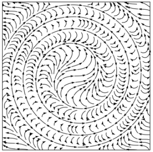
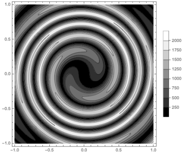
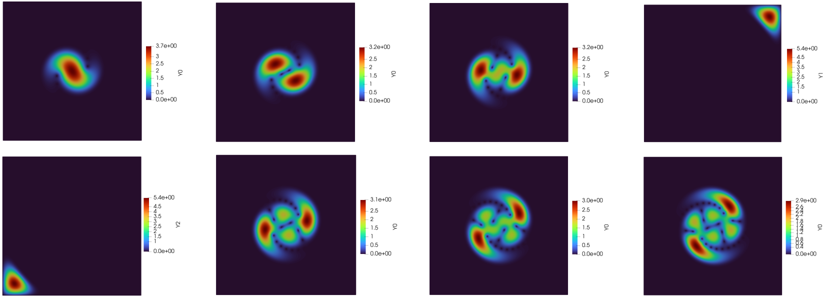
A



$|\mathrm{curl}\mathbf{A}|$





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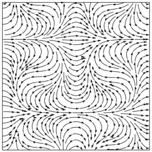
$\mathbf{A}$



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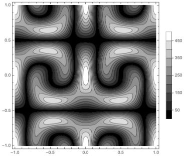
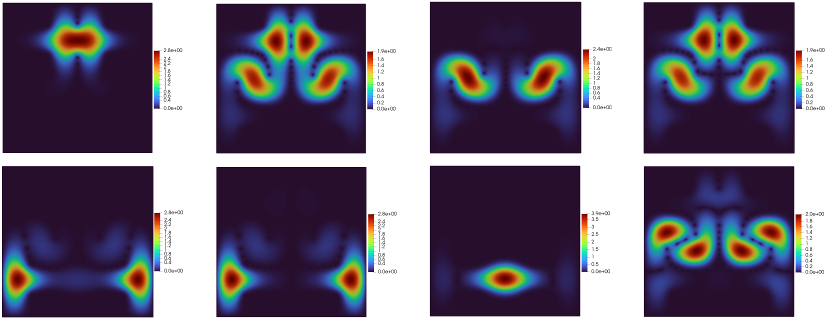


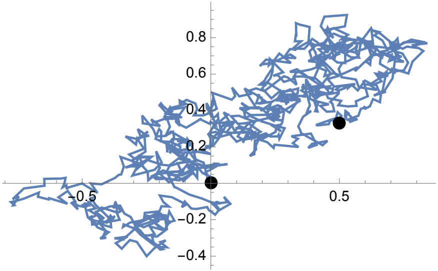


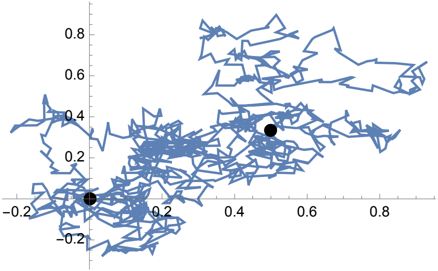
Main Result for Magnetic Laplacian

Theorem (Ovall/Quan/Reid/Steinerberger, 2024)

Let (λ, ψ) *be an eigenpair of* $H(\mathbf{A})$, *where* $\mathbf{A} = \nabla a + \mathbf{F}$ *and* $\nabla \cdot \mathbf{F} = 0$, *and* $|\psi(x_0)| = \|\psi\|_{L^\infty(\Omega)}$. *For any* $t > 0$, *it holds that*

$$\int_\Omega \left| \mathbb{E}_{\omega(0)=x_0, \omega(t)=y}\, e^{-i \int_0^t \mathbf{F}(\omega(s)) \cdot d\omega(s)} \right| K_\Omega(t, x_0, y)\, dy \ge e^{-\lambda t}$$





Proposition 2.9: Broderix, Hundertmark, Leschke, Rev. Math. Phys. (2000)

$$\left[e^{-t\,H(\mathbf{A},V)}w\right](x) = \mathbb{E}_{\omega(0)=x}\left(e^{-S_t(\mathbf{A},V\,;\,\omega)}\chi_\Omega(\omega,t)\,w(\omega(t))\right)$$

- Path integrals:

$$S_t(\mathbf{A},V\,;\,\omega) = \mathrm{i}\int_0^t \mathbf{A}(\omega(s))\cdot d\omega(s) + \frac{\mathrm{i}}{2}\int_0^t (\nabla\cdot\mathbf{A})(\omega(s))\,ds + \int_0^t V(\omega(s))\,ds$$

- Expectation and Heat Kernel on Ω:

$$\mathbb{E}_{\omega(0)=x}\left(e^{-S_t(\mathbf{A},V\,;\,\omega)}\chi_\Omega(\omega,t)\,w(\omega(t))\right) = \int_\Omega \left(\mathbb{E}_{\omega(0)=x,\omega(t)=y}e^{-S_t(\mathbf{A},V\,;\,\omega)}\right)K_\Omega(t,x,y)\,w(y)\,dy$$

- Eigenpair Simplification: For an eigenpair (λ,ψ) of $H(\mathbf{A},V)$,

$$\left[e^{-t\,H(\mathbf{A},V)}\psi\right](x) = e^{-\lambda t}\psi(x)$$

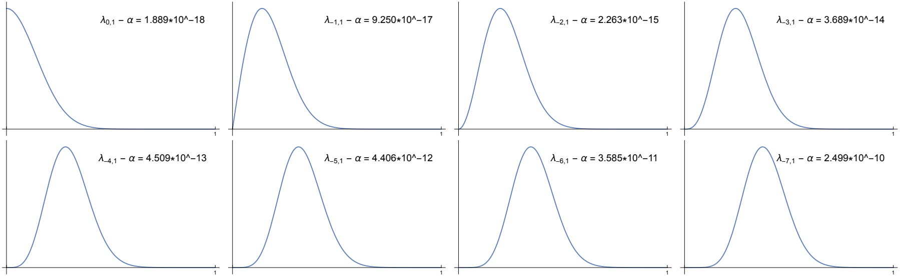
$H(A) = (\alpha/2)(-y, x)$, $\operatorname{curl} A = \alpha$

Eigenvalues/Vectors: $s = (\alpha/2)\, r^2$, $n \in \mathbb{Z}$, $m \in \mathbb{N}$

$$\lambda_{n,m} = \mathrm{Root}\left(m, {}_1F_1\left(\frac{1}{2}\left(n + |n| + 1 - \frac{t}{\alpha}\right), |n| + 1, \frac{\alpha}{2}\right)\right)$$

$$\psi_{n,m} = e^{in\theta} e^{-s/2} s^{|n|/2} {}_1F_1\left(\frac{1}{2}\left(n + |n| + 1 - \frac{\lambda_{n,m}}{\alpha}\right), |n| + 1, s\right)$$

Cluster of Eigenvalues Near Ground State: $\lambda_{0,1} \geq \alpha = 100$



Baur, Weidl, arXiv 2024

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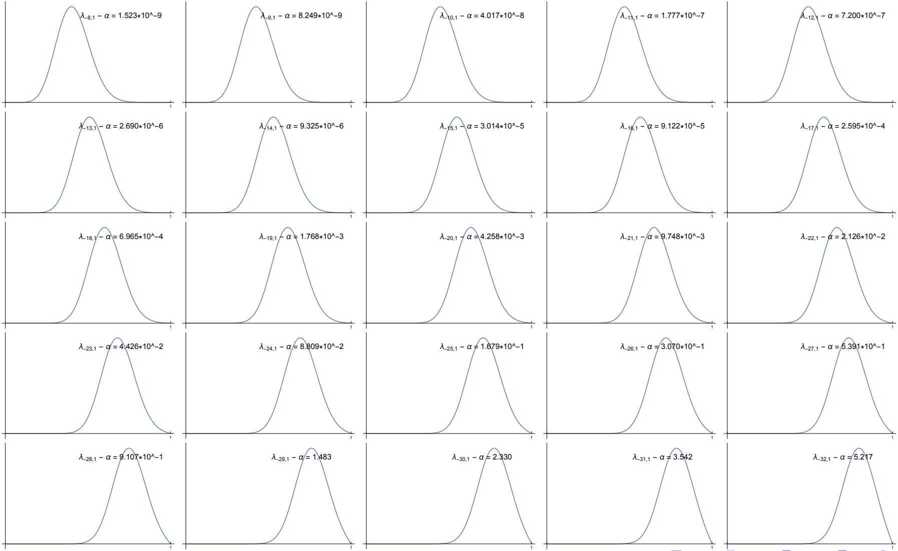

Adaptive Refinement Driven By Landscape, $\mathcal{L} = -\Delta + V$