Landscape Refinement for Elliptic Eigenvalue Problems

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Joint work with Stefano Giani

Landscape Inequalities

Elliptic Operator: $\mathcal{L}v = -\nabla \cdot (A\nabla v) + cv$ [Maximum Principle]

- Eigenvalue problem: $\mathcal{L}\psi = \lambda\psi$ in Ω, $\psi = 0$ on $\partial\Omega$
- Landscape problem: $\mathcal{L}u = 1$ in Ω, $u = 0$ on $\partial\Omega$

Theorem (Landscape Inequalities)

If $v \in C(\overline{\Omega})$ and $\mathcal{L}v \in L^\infty(\Omega)$, then

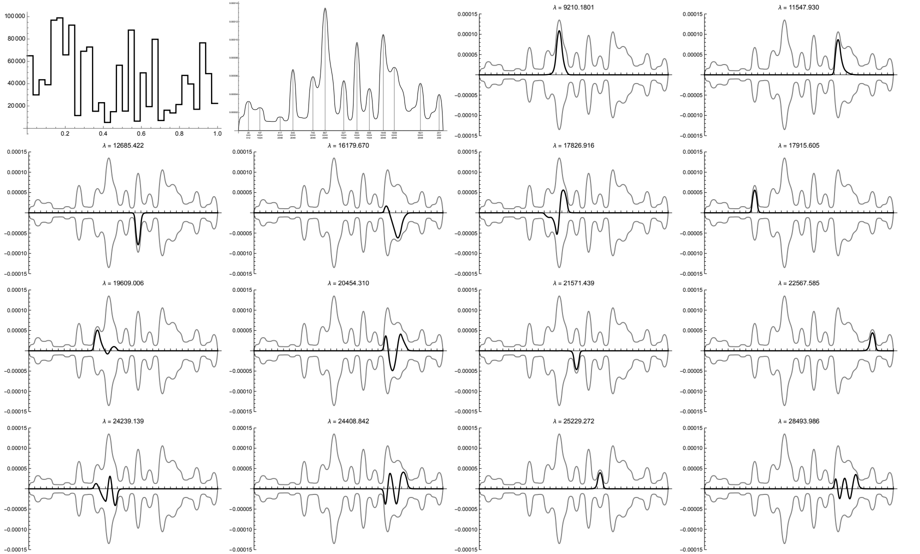
$$|v(x)| \leq \|v\|_{L^\infty(\partial\Omega)} + \|\mathcal{L}v\|_{L^\infty(\Omega)}\, u(x) \text{ for all } x \in \overline{\Omega}$$

More specifically, if (λ, ψ) is an eigenpair of $\mathcal{L}$,

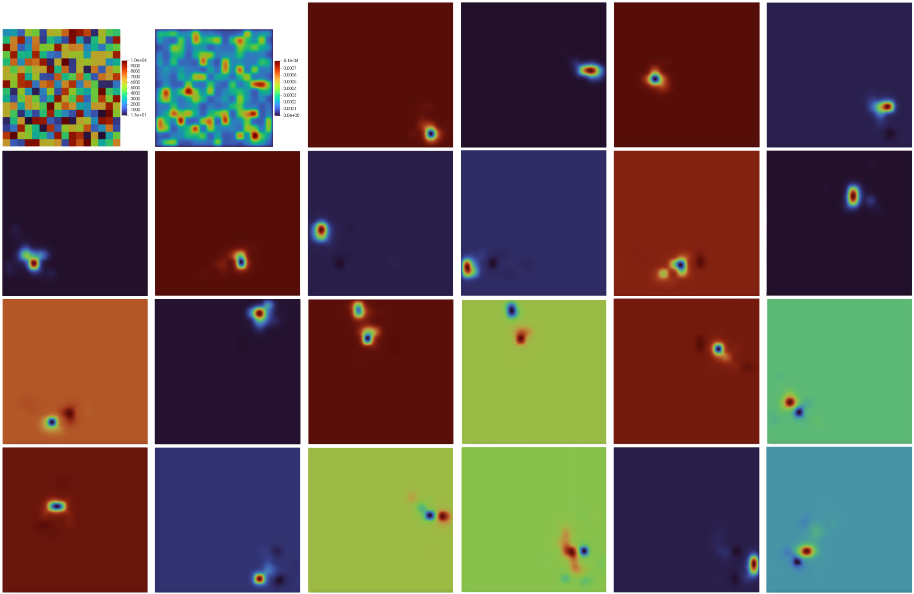
$$\frac{|\psi(x)|}{\lambda\|\psi\|_{L^\infty(\Omega)}} \leq u(x) \text{ for all } x \in \overline{\Omega}$$

1D Localization Illustration, $\mathcal{L} = -\Delta + c$

Potential | Landscape: $\mathcal{L}u = 1$ | $-u(x) \leq \psi(x)/(\lambda \, \|\psi\|_{L^\infty(\Omega)}) \leq u(x)$



2D Localization Illustration, $\mathcal{L} = -\Delta + V$

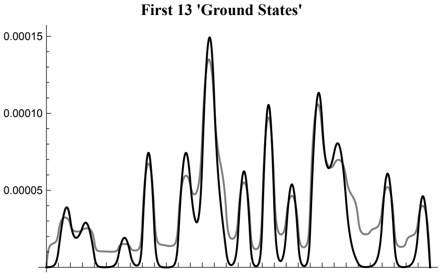


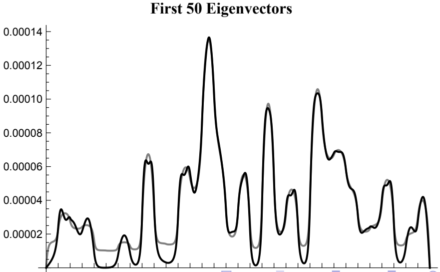
Eigenvector (Fourier) Expansion of Landscape

Theorem (Fourier Expansion of the Landscape Function)

Let (λ_n, ψ_n)*,* $n \in \mathbb{N}$*, be eigenpairs of* $\mathcal{L}$ *such that* $\{\psi_n : n \in \mathbb{N}\}$ *is an orthonormal Hilbert basis (a Fourier basis) of* $L^2(\Omega)$*. The landscape function has the Fourier expansion*

$$u = \sum_{n \in \mathbb{N}} c_n \, \psi_n \quad , \quad c_n = \left(\int_\Omega \psi_n \, dx \right) / \lambda_n \, .$$





“Landscape Refinement” for Eigenvalue Clusters?

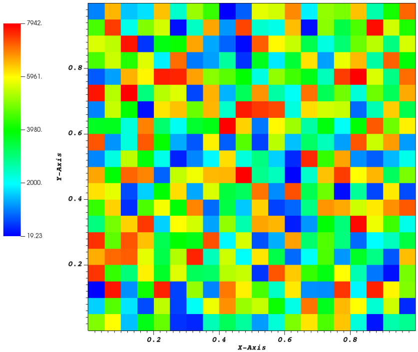
Conjecture (Landscape Refinement)

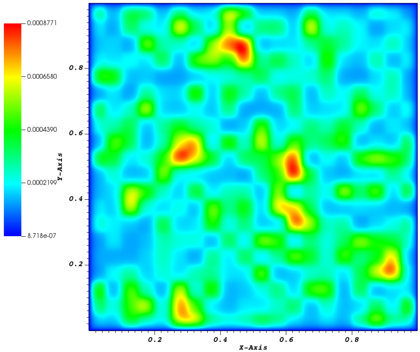
A sequence of finite element spaces “tuned” to approximate the landscape u well will also approximate clusters of eigenvalues/vectors well.

Error Estimation and Adaptivity for Eigenvalue/vector Clusters
Adaptivity should be driven by “approx. subspace error” as opposed to “approx. basis error”

- Giani, Grubišić, Hakula, Ovall: 2009
- Giani, Solin: 2012
- Gallistl: 2015
- Boffi, Gallistl, Gardini, Gastaldi: 2017
- Cancès, Dusson, Maday, Stamm, Vohralík: 2020
- Liu, Vejchodský: 2022

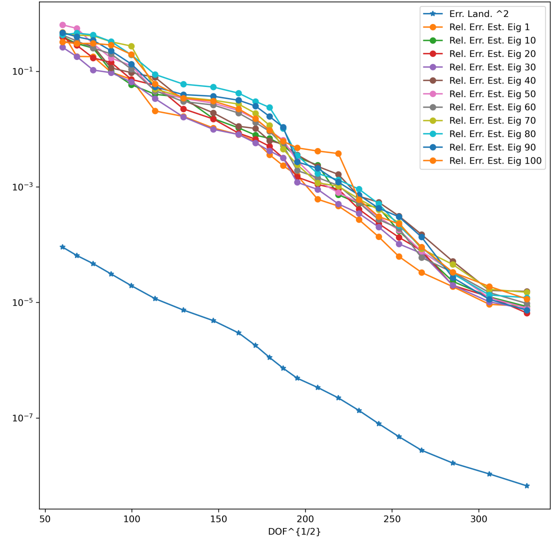
First Test: Discontinuous Reaction on Unit Square





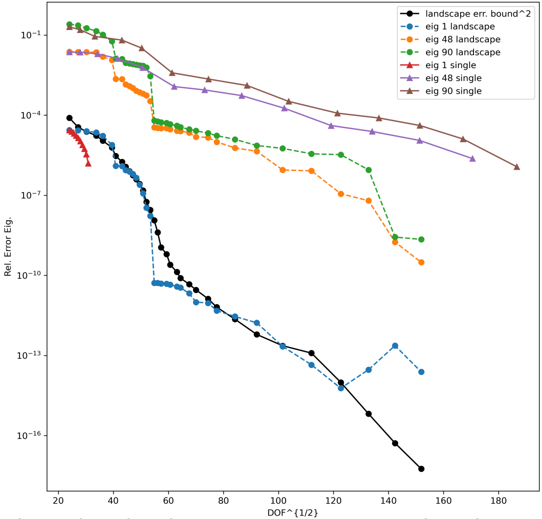
- Initial mesh 20×20 squares (same as reaction term), bi-quadratic elements
- Sequence of *hp*-adapted meshes, adadptivity drivent by approx. u; approx. u shown on finest mesh
- Reference eigenvalues (first 100) computed on very fine mesh

First Test: Discontinuous Reaction on Unit Square



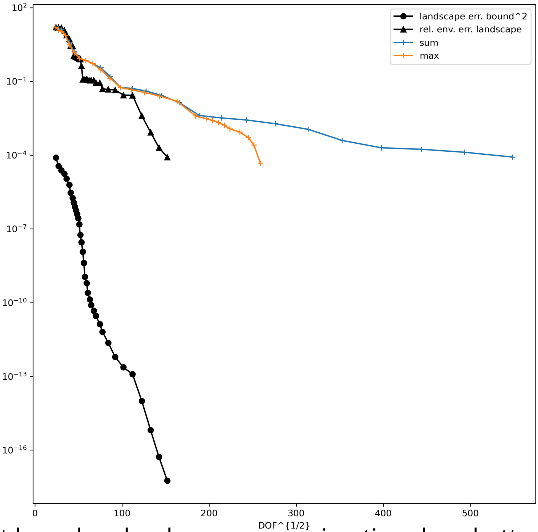
- Individual eigenvalues converge at (same) optimal rate

Second Test: Laplacian on Unit Square



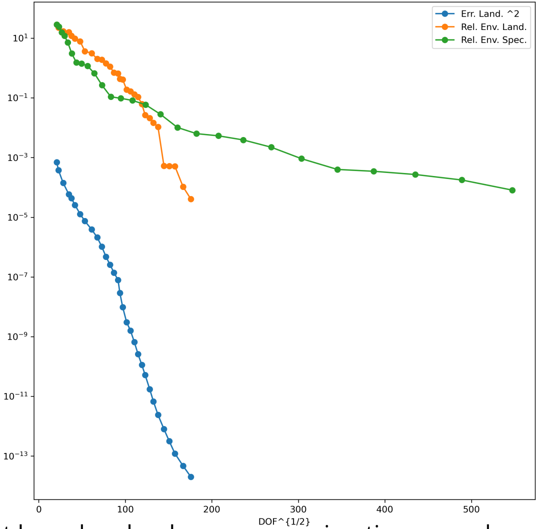
- Refinement based on landscape approximation does better job for (λ_j, ψ_j), $j = 48, 90$, than refinement based on either of these individual eigenpairs!

Second Test: Laplacian on Unit Square



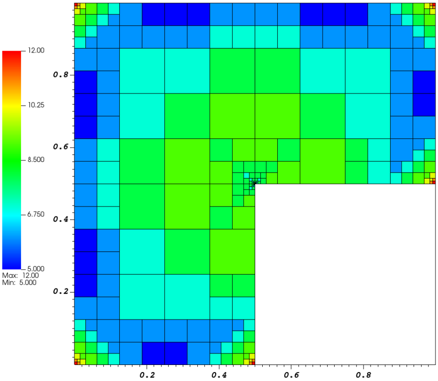
- Refinement based on landscape approximation does better job for first 100 eigenpairs than two common strategies based on collective eigenpair error estimates!

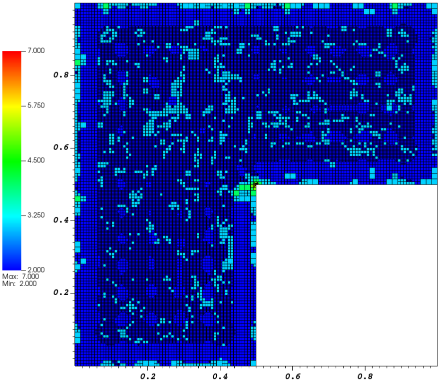
Third Test: Laplacian on L-Shape



- Refinement based on landscape approximation soon does better job for worst error in first 100 than refinement based on collective eigenpair error estimates!

Third Test: Laplacian on L-Shape (Final Meshes)





- Refinement based on landscape approximation (left) yields a much more sensible mesh for first 100 than refinement based on collective eigenpair error estimates!