The Variance Sum Law

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Properties of the Expectation Operator $\mathbb{E}$

1. $\mathbb{E}(k) = k$ if k is a constant
2. $\mathbb{E}(kX) = k\mathbb{E}(X)$ if k is a constant and X is random
3. $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$ if X and Y are both random

Variance/Covariance Definitions

Variance is defined using the Expectation Operator, as

$$\sigma_X^2 = \mathbb{E}(X - \mathbb{E}(X))^2 \tag{1}$$

and the Covariance between X and Y is defined as,

$$\sigma_{XY} = \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))). \tag{2}$$

Variance Sum Law derivation

With these two definitions we can derive the variance of the sum or difference of two random variables via the Variance Sum Law.

Our initial question is what is the variance of the sum of two random variables? We can express this mathematically as,

$$\sigma_{X+Y}^2 = ?$$

$$
\begin{aligned}
\sigma^2_{X+Y} &= \mathbb{E}((X+Y) - \mathbb{E}(X+Y))^2 \\
&= \mathbb{E}((X+Y) - (\mathbb{E}(X) + \mathbb{E}(Y)))^2 \\
&= \mathbb{E}((X - \mathbb{E}(X)) + (Y - \mathbb{E}(Y)))^2 \\
& \\
&\text{define } a \text{ and } b \\
&= \mathbb{E}(\underbrace{(X - \mathbb{E}(X))}_{a} + \underbrace{(Y - \mathbb{E}(Y))}_{b})^2 \\
& \\
&\text{Substitute and solve} \\
&= \mathbb{E}(a+b)^2 \\
&= \mathbb{E}((a+b)(a+b)) \\
&= \mathbb{E}(a^2 + b^2 + 2ab) \\
& \\
&\text{back substitute and simplify} \\
&= \mathbb{E}[(X - \mathbb{E}(X))^2 + (Y - \mathbb{E}(Y))^2 + 2(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))] \\
&= \mathbb{E}(X - \mathbb{E}(X))^2 + \mathbb{E}(Y - \mathbb{E}(Y))^2 + \mathbb{E}(2(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))) \\
&= \mathbb{E}(X - \mathbb{E}(X))^2 + \mathbb{E}(Y - \mathbb{E}(Y))^2 + 2\mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))) \\
& \\
&= \sigma^2_X + \sigma^2_Y + 2\sigma_{XY}
\end{aligned}
$$

Variance of the difference of two random variables

For the difference of two random variables we follow the same steps.

Mathematically, this question is

$$\sigma^2_{X-Y} = ?$$

$$
\begin{aligned}
\sigma^2_{X-Y} &= \mathbb{E}((X-Y)-\mathbb{E}(X-Y))^2 \\
&= \mathbb{E}((X+[-Y])-\mathbb{E}(X+[-Y]))^2 \\
&= \mathbb{E}((X+[-Y])-(\mathbb{E}(X)+\mathbb{E}([-Y])))^2 \\
&= \mathbb{E}((X-\mathbb{E}(X))+([-Y]-\mathbb{E}([-Y])))^2 \\
\\
& \text{define } a \text{ and } b \\
&= \mathbb{E}(\underbrace{(X-\mathbb{E}(X))}_{a}+\underbrace{([-Y]-\mathbb{E}([-Y]))}_{b})^2 \\
\\
& \text{Substitute and solve} \\
&= \mathbb{E}(a+b)^2 \\
&= \mathbb{E}((a+b)(a+b)) \\
&= \mathbb{E}(a^2+b^2+2ab) \\
\\
& \text{back substitute and simplify} \\
&= \mathbb{E}[(X-\mathbb{E}(X))^2+([-Y]-\mathbb{E}([-Y]))^2+2(X-\mathbb{E}(X))([-Y]-\mathbb{E}([-Y]))] \\
&= \mathbb{E}(X-\mathbb{E}(X))^2+\mathbb{E}([-Y]-\mathbb{E}([-Y]))^2+\mathbb{E}(2(X-\mathbb{E}(X))([-Y]-\mathbb{E}([-Y]))) \\
&= \mathbb{E}(X-\mathbb{E}(X))^2+\mathbb{E}([-Y]-\mathbb{E}([-Y]))^2+2\mathbb{E}((X-\mathbb{E}(X))([-Y]-\mathbb{E}([-Y]))) \\
&= \mathbb{E}(X-\mathbb{E}(X))^2+\mathbb{E}([-Y]-\mathbb{E}([-Y]))^2+2\mathbb{E}((X-\mathbb{E}(X))[-1(Y-\mathbb{E}(Y))]) \\
&= \mathbb{E}(X-\mathbb{E}(X))^2+\mathbb{E}([-Y]-\mathbb{E}([-Y]))^2-2\mathbb{E}((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))) \\
\\
&= \sigma^2_X+\sigma^2_Y-2\sigma_{XY}
\end{aligned}
$$

General Form of the Variance Sum Law

Given the specifications above we can express the variance sum law in general form as,

$$\sigma^2_{X\pm Y}=\sigma^2_X+\sigma^2_Y\pm 2\sigma_{XY}$$