Algebraic Derivation of the Partial Correlation Coefficient

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The Linear Model

To begin we must define our linear model. This will be used to define our predictions and errors. In the regression framework we will deal with a line, specified as,

$$\hat{y} = b_0 + b_1 x, \tag{1}$$

where $\hat{y}$ is the predicted outcome, x is the predictor and the parameters b_0 and b_1 are the intercept and slope respectively.

The estimation of the parameters for this equation are based on minimizing the sum of square errors (Least Squares). We define errors as the difference between the model based prediction, $\hat{y}$ and the actual observed values, y. Using equation 1 to predict observed data, our errors are specified as,

$$\begin{aligned} y &= b_0 + b_1 x + \epsilon \\ \epsilon &= y - (b_0 + b_1 x) \end{aligned} \tag{2}$$

The slope can be estimated using the least squares criteria as,

$$b_1 = \frac{Cov_{xy}}{s_x^2} = \frac{\frac{1}{n}\Sigma(y_i - \bar{y})(x_i - \bar{x})}{\frac{1}{n}\Sigma(x_i - \bar{x})(x_i - \bar{x})} \tag{3}$$

Since we are using the means of both the outcome ($\bar{y}$) and predictor ($\bar{x}$) to estimate the slope, we know that the estimated line will pass through the sample means of both variables. Thus the intercept can be estimated as,

$$b_0 = \bar{y} - b_1 \bar{x}. \tag{4}$$

Standardized Estimates

If we instead standardize each variable such that,

$$z_y = \frac{(y_i - \bar{y})}{s_y} \tag{5}$$

and use them in the same equation, the estimates are similar,

$$z_y = \beta_0 + \beta_1 z_x, \tag{6}$$

however the expected values are based on the standardized variables. For example, the model implied estimate of β_0 is based on the sample means of both z_y and z_x, which by definition are 0, thus $\beta_0 = 0$. Additionally, the estimate of the slope, when represented using z scores becomes,

$$\begin{aligned} \beta_1 &= \frac{\frac{1}{n}\Sigma(z_{xi}-\bar{z}_x)(z_{yi}-\bar{z}_y)}{\frac{1}{n}\Sigma(z_{xi}-\bar{z}_x)(z_{xi}-\bar{z}_x)} \\ &= \frac{\frac{1}{n}\Sigma(z_{xi}-0)(z_{yi}-0)}{\frac{1}{n}\Sigma(z_{xi}-0)(z_{xi}-0)} \\ &= \frac{\frac{1}{n}\Sigma z_{xi}z_{yi}}{\frac{1}{n}\Sigma z_{xi}z_{xi}} \\ &= \frac{\frac{1}{n}\Sigma z_{xi}z_{yi}}{1} \\ &= r_{xy} \end{aligned} . \tag{7}$$

Thus, the prediction equation for a line using standardized scores becomes,

$$\hat{z}_y = r_{xy} z_x, \tag{8}$$

with a corresponding error definition of,

$$\epsilon_{zy} = z_y - \hat{z}_y = z_y - r_{xy} z_x. \tag{9}$$

Variance Explained

Using equation 9 above, we can derive the formula for the unexplained variance resulting form this prediction equation. To accomplish this we will square the error term as defined in equation 9.

$$\begin{aligned}
\tfrac{1}{n}\Sigma\epsilon_{zy}^2 &= \tfrac{1}{n}\Sigma(z_y - r_{xy}z_x)^2 \\
&= \tfrac{1}{n}\Sigma(z_y - r_{xy}z_x)(z_y - r_{xy}z_x) \\
&= \tfrac{1}{n}\Sigma[z_y^2 - r_{xy}z_xz_y - r_{xy}z_xz_y + r_{xy}^2z_x^2] \\
&= \tfrac{1}{n}\Sigma[z_y^2 - 2r_{xy}z_xz_y + r_{xy}^2z_x^2] \\
&= \tfrac{1}{n}\Sigma(z_y^2) - \tfrac{1}{n}\Sigma(2r_{xy}z_xz_y) + \tfrac{1}{n}\Sigma(r_{xy}^2z_x^2) \qquad (10) \\
&= (1) - 2r_{xy}\tfrac{1}{n}\Sigma(z_xz_y) + r_{xy}^2\tfrac{1}{n}\Sigma(z_x^2) \\
&= (1) - 2r_{xy}r_{xy} + r_{xy}^2(1) \\
&= (1) - 2r_{xy}^2 + r_{xy}^2 \\
&= 1 - r_{xy}^2
\end{aligned}$$

This term represent the error variance in z_y by using z_x as a predictor.

The third variable problem

As seen above, when dealing with 2 variables, regression is equal to correlation. Problems arise when we are interested in multiple predictors for our regression equation. Notice that we are still relying on the equation for a line, however in this case the predictors are multi-variable. Keeping with standardized scores the equation can be specified as,

$$\hat{z}_y = \beta_1 z_{x1} + \beta_2 z_{x2} + \ldots + \beta_p z_{xp} = \Sigma_{j=1}^{p}\beta_j z_{xj}, \qquad (11)$$

where p is the number of unique predictors in the equation.

In order to isolate the influences of any given predictor, within the context of the other predictors, we must partial out an shared variance among the predictors as well as in the outcome. We accomplish this by defining a regression slope as in equation 3, using the unexplained portions, or errors, of the variables of interest. For the following we will designate our outcome as y and our two predictors as x and w. We are interested in deriving the partial regression for y regressed on x, controlling for w.

Our outcome error is defined as,

$$\epsilon_{zy} = z_y - r_{yw}z_w, \qquad (12)$$

and our predictor error is defined as,

$$\epsilon_{zx} = z_x - r_{xw}z_w. \qquad (13)$$

With both of these errors defined relative to the third variable w, we can proceed to derive the partial regression estimate.

$$\beta_{yx\cdot w} = \frac{Cov_{\epsilon zy,\epsilon zx}}{S^2_{\epsilon zx}}, \tag{14}$$

where $\beta_{yx\cdot w}$ represents the regression of y on x controlling for w.

Fortunately, the bottom term has already been derived for us in equation 10, as $1 - r^2_{xw}$, thus we turn our attention to the derivation of the covariance between ϵ_{zy} and ϵ_{zx} as defined in equations 12 and 13.

$$\begin{aligned}
Cov_{\epsilon zy,\epsilon zx} &= \tfrac{1}{n}\Sigma\epsilon_{zy}\epsilon_{zx} \\
&= \tfrac{1}{n}\Sigma(z_y - r_{yw}z_w)(z_x - r_{xw}z_w) \\
&= \tfrac{1}{n}\Sigma[z_y z_x - r_{xw}z_w z_y - r_{yw}z_w z_x + r_{yw}z_w r_{xw}z_w] \\
&= \tfrac{1}{n}\Sigma(z_y z_x) - \tfrac{1}{n}\Sigma(r_{xw}z_w z_y) - \tfrac{1}{n}\Sigma(r_{yw}z_w z_x) + \tfrac{1}{n}\Sigma(r_{yw}r_{xw}z_w^2)\ . \\
&= \tfrac{1}{n}\Sigma(z_y z_x) - r_{xw}\tfrac{1}{n}\Sigma(z_w z_y) - r_{yw}\tfrac{1}{n}\Sigma(z_w z_x) + r_{yw}r_{xw}\tfrac{1}{n}\Sigma(z_w^2) \\
&= r_{yx} - r_{xw}r_{yw} - r_{yw}r_{xw} + r_{yw}r_{xw}(1) \\
&= r_{yx} - r_{yw}r_{xw}
\end{aligned} \tag{15}$$

Substituting our derived values into equation 3 we get,

$$\beta_{yx\cdot w} = \frac{r_{yx} - r_{yw}r_{xw}}{1 - r^2_{xw}}. \tag{16}$$