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NOTE

# A note on the 'system-free' expressions of Maxwell's equations

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**Abstract**

Expressions for Maxwell's equations independent of the unit system are presented and compared with those given in Jackson's book. Both the cases of electromagnetism in vacuum and in a medium are considered through the introduction of two sets of proportional constants: the set of empirical constants and that of 'conventional constants'. The latter set is needed to account for the different conventions adopted in different unit systems for the definitions of various electromagnetic quantities in the presence of a medium.

It is a well-known fact that one major hurdle for students in a class of electromagnetism (EM) is to get familiar with the adopted unit system, and to move from one unit system to another (e.g. SI to Gaussian). The complexity of this issue can be attested simply by referring to the recent publication of a whole text exclusively devoted to this issue [1]. This problem is further intensified with the appearance of the latest edition of Jackson [2], which adopts a *mixture* of unit systems with the first 10 chapters in SI and the last six in Gaussian units.

While there have been publications in the literature from time to time on the subject of transforming Maxwell equations from one unit system to another [3], it will be valuable if one can just formulate the equations in a 'system-free' approach independent of any unit system adopted. Indeed, this was done in Jackson's appendix in terms of four empirical constants [2] for the case of EM in vacuum. In this note, we shall present a different and slightly simplified formulation of these 'system-free' equations. In addition, we shall also discuss the case in a medium and shall see that such a formulation becomes too messy due to various definitions and conventions adopted previously in the different unit systems.

As is well-known, the origin of the difference in the various unit systems arises from the different choices of the proportional constants in expressing various empirical laws in the form of an equation. In the case of EM in vacuum, it is easy to see that one only needs to introduce *three* such constants: $k_{\mathrm{E}}$, $k_{\mathrm{B}}$, and $k_{\mathrm{EM}}$ (which stand for electric, magnetic, and electromagnetic constants) to express all the equations for both the fields (Maxwell's equations) and the source (Lorentz force law). Note that the choice of constant for Faraday's electromagnetic induction law is intimately related to that for Lorentz force law via the concept of 'motional electromotive force' [4], and one needs only one empirical constant for both these two laws. This can easily be shown by considering the induced emf on a constantly moving wire, which completes a

**Table 1.** Definitions of various empirical constants.

| | SI | Gaussian | esu | emu | Heaviside |
|---|---|---|---|---|---|
| $k_{\mathrm{E}}$ | $\frac{1}{4\pi\varepsilon_0}$ | 1 | 1 | $c^2$ | $\frac{1}{4\pi}$ |
| $k_{\mathrm{B}}$ | $\frac{\mu_0}{4\pi}$ | $\frac{1}{c}$ | $\frac{1}{c^2}$ | 1 | $\frac{1}{4\pi c}$ |
| $k_{\mathrm{F}}$ | 1 | $\frac{1}{c}$ | 1 | 1 | $\frac{1}{c}$ |

**Table 2.** Definitions of various 'conventional' constants.

| | SI | Gaussian | esu | emu | Heaviside |
|---|---|---|---|---|---|
| $k_D$ | $\varepsilon_0$ | 1 | 1 | $1/c^2$ | 1 |
| $k_M$ | 1 | $c$ | 1 | 1 | $c$ |
| $k_H$ | $1/\mu_0$ | 1 | $c^2$ | 1 | 1 |

circuit located in a uniform magnetic field. Hence, instead of $k_{\mathrm{E}}$, $k_{\mathrm{B}}$, and $k_{\mathrm{EM}}$, we may as well use $k_{\mathrm{E}}$, $k_{\mathrm{B}}$, and $k_{\mathrm{F}}$ to formulate the field equations where $k_{\mathrm{F}}$ is the 'force constant' defined through the equation $\vec{F} = q(\vec{E} + k_{\mathrm{F}}\vec{v} \times \vec{B})$. In addition, $k_{\mathrm{E}}$ is defined through Coulomb's law ($\vec{E} = k_{\mathrm{E}}\frac{q}{r^2}\hat{r}$) and $k_{\mathrm{B}}$ through the Biot–Savart law ($\overrightarrow{dB} = k_{\mathrm{B}}\frac{I\,\overrightarrow{dl} \times \hat{r}}{r^2}$), respectively. Using this set of three constants, it is not difficult to show that Maxwell's equations in vacuum can be expressed in the following way:

$$\begin{aligned}
&\vec{\nabla} \cdot \vec{E} = 4\pi k_{\mathrm{E}}\rho \\
&\vec{\nabla} \cdot \vec{B} = 0 \\
&\vec{\nabla} \times \vec{E} + k_{\mathrm{F}}\frac{\partial \vec{B}}{\partial t} = 0 \\
&\vec{\nabla} \times \vec{B} - \frac{k_{\mathrm{B}}}{k_{\mathrm{E}}}\frac{\partial \vec{E}}{\partial t} = 4\pi k_{\mathrm{B}}\vec{J}.
\end{aligned} \tag{1}$$

The expressions for the three constants in the five common unit systems are given in table 1.

Next let us consider the modifications to the two 'source equations' in the presence of a medium. By introducing the polarized charge density $\rho_{\mathrm{P}} = -\vec{\nabla} \cdot \vec{P}$ into Gauss's law, we obtain

$$\vec{\nabla} \cdot (\vec{E} + 4\pi k_{\mathrm{E}}\vec{P}) = 4\pi k_{\mathrm{E}}\rho. \tag{2}$$

Further complication now arises due to the lack of a consistent definition of the displacement vector ($\vec{D}$) in various systems. For example, while $\vec{D}$ is identified with $\vec{E}$ in vacuum in Gaussian units, it is *defined* to be distinguished from $\vec{E}$ even in vacuum ($\vec{D} = \varepsilon_0\vec{E}$) in SI units[1]. In order to account for this artificial discrepancy, we have to introduce another constant, $k_D$, to keep track of the different definitions of $\vec{D}$ in different systems as follows:

$$\vec{D} = k_D(\vec{E} + 4\pi k_{\mathrm{E}}\vec{P}), \tag{3}$$

and the modified Gauss law takes the form

$$\vec{\nabla} \cdot \vec{D} = 4\pi k_D k_{\mathrm{E}}\rho. \tag{4}$$

Expressions for $k_D$ in various systems are given in table 2.

[1] Some authors claim that to have $\vec{D}$ and $\vec{E}$ as well as $\vec{B}$ and $\vec{H}$ to be differentiated (even in the vacuum case) is an added advantage for the SI system (e.g. Vanderlinde in [4]). We cannot agree with this. These vectors should be differentiated only in the presence of polarization and magnetization of a medium. It is not wrong to have them identified with each other in vacuum.

The situation will be still more complicated in the case of the magnetic medium since even the definition of the magnetic dipole moment ($\vec{m}$) varies from system to system. For example, while $\vec{m}$ for a constant current loop is simply the product of the current and area in the SI system, it is defined with an extra factor $1/c$ ($c$ being the speed of light in vacuum) in the Gaussian system. Just to account for this we have to introduce an extra constant $k_M$ (see table 2) in the relation between the magnetization current and the magnetization vector as follows:

$$\vec{J}_M = k_M \vec{\nabla} \times \vec{M}. \tag{5}$$

To obtain a complete modification of Ampère's law, we have to introduce one extra constant $k_H$ (see table 2) to take care of the different definitions of the $\vec{H}$ vector in various systems, just like the situation with the $\vec{D}$ vector in the electric case. Thus we have

$$\vec{H} = k_H(\vec{B} - 4\pi k_{\mathrm{B}} k_M \vec{M}), \tag{6}$$

and the modified Ampère's law is finally obtained as follows:

$$\vec{\nabla} \times \vec{H} - \left(\frac{k_{\mathrm{B}} k_H}{k_{\mathrm{E}} k_D}\right) \frac{\partial \vec{D}}{\partial t} = 4\pi k_{\mathrm{B}} k_H \vec{J}. \tag{7}$$

Thus the complete set of 'system-free' Maxwell equation in a medium is given by the two sourceless equations in (1) together with equations (4) and (7).

In summary, we have introduced a set of 'system-free' Maxwell equations in vacuum using *three* empirical constants as displayed in equation (1) which is slightly simpler than the one found in Jackson [2]. Jackson introduces the magnetic constant through the Ampère force law which thus mixes the two constants $k_{\mathrm{B}}$ and $k_{\mathrm{F}}$ as defined here. Hence Jackson ends up requiring *four* constants for the complete formulation of the equations, and only reduces to three constants with further dimensional analysis making use of the wave equation. In addition, the connection between the electromagnetic constant $k_{\mathrm{EM}}$ and the force constant $k_{\mathrm{F}}$ is not utilized in Jackson's approach.

We would like to further comment on the 'empirical nature' of these constants. Strictly speaking there is only one fundamental constant needed in the whole Maxwell equation in vacuum which is the speed of light $c$.[2] This is particularly clear in the Gaussian system by noting that only the two 'curl equations' are independent equations, and each of which contains only $c$ as the empirical constant. Together with the continuity equation for the conservation of charge, the two 'divergence equations' can then be derived. In the 'system-free' case, one can also see this by deriving the wave equation from equation (1) which yields the identity: $\frac{k_{\mathrm{F}} k_{\mathrm{B}}}{k_{\mathrm{E}}} = \frac{1}{c^2}$. This can also be checked explicitly from table 1. From this relation it is clear that out of the three constants $k_{\mathrm{E}}$, $k_{\mathrm{B}}$, $k_{\mathrm{F}}$, two are actually fixed *by convention* in the various unit systems. For example, one can fix $k_{\mathrm{E}}$ by defining the unit for charge in Coulomb's law, and $k_{\mathrm{B}}$ by defining the strength of the magnetic field in the Biot–Savart law. $k_{\mathrm{F}}$ is then automatically fixed by the above relation[3].

For the case with a medium, the situation becomes much more complicated due, unfortunately, to the conventional distinction in the definitions of the magnetic dipole moment as well as the $\vec{D}$ and $\vec{H}$ vectors in various systems. Because of this, we have to introduce three more '*conventional* constants' $k_D$, $k_M$ and $k_H$ (in contrast to *empirical* constants) to formulate the whole system of equations in a 'system-free' format. This thus makes the result not as appealing as that obtained in the vacuum case, and it may be preferable as well to work with one specific unit system in practice, and then learn to transform from one to another system as is done in the literature [3].

[2] We thank an anonymous referee for pointing this out to us.
[3] A similar argument is also presented by Jackson in [2].

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