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NOTE

On the singularities of the electrostatic and magnetostatic dipole fields

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Abstract

The singular behaviour of the electrostatic dipole field is compared with that of the magnetostatic one, with pedagogical access to the difference between the two established without necessarily resorting to certain advanced differentiation identities involving delta functions. Such an approach may be more suitable for a regular curriculum in electrodynamics.

It is rather common in many standard texts on electromagnetism that statements are made on the similarity between the dipolar electrostatic and magnetostatic fields, by noting that the expressions (in Gaussian units)

$$\vec{E}^{(2)} = \frac{3(\vec{p}\cdot\vec{r})\vec{r} - r^2\vec{p}}{r^5}, \tag{1}$$

$$\vec{B}^{(2)} = \frac{3(\vec{m}\cdot\vec{r})\vec{r} - r^2\vec{m}}{r^5} \tag{2}$$

are completely identical in form to each other [1]. Here we have assumed that both the electric dipole ($\vec{p}$) and the magnetic dipole ($\vec{m}$) are located at the origin. On the other hand, it is also rather well known that the singular behaviours in both (1) and (2) have to be accounted for by introducing terms with the Dirac δ-function as follows:

$$\vec{E}^{(2)} = \frac{3(\vec{p}\cdot\vec{r})\vec{r} - r^2\vec{p}}{r^5} - \frac{4\pi}{3}\vec{p}\delta(\vec{r}), \tag{3}$$

$$\vec{B}^{(2)} = \frac{3(\vec{m}\cdot\vec{r})\vec{r} - r^2\vec{m}}{r^5} + \frac{8\pi}{3}\vec{m}\delta(\vec{r}). \tag{4}$$

As is well known, the significance of the δ-function term in (4) is manifested in the hyperfine interaction of the ground state of the H atom, giving rise to the famous 21 cm radiation in astronomy based on which most of the cosmological hydrogen was detected [2].

There exist in the literature at least two different ways of justifying the δ-function terms in (3) and (4). According to Jackson [2], it is by considering the volume integrals of the dipole fields over a spherical volume (radius R) enclosing the dipoles,

$$\int_{(r\leqslant R)} \vec{E}^{(2)}\,\mathrm{d}^3x = -\frac{4\pi}{3}\vec{p} \tag{5}$$

and

$$\int_{(r\leqslant R)} \vec{B}^{(2)}\,\mathrm{d}^3x = \frac{8\pi}{3}\vec{m} \tag{6}$$

that the introduction of the δ-function terms is justified. In another approach, one can also justify these terms by referring to the following differentiation identity [3]:

$$\partial_i\partial_j\left(\frac{1}{r}\right) = \frac{3x_ix_j - r^2\delta_{ij}}{r^5} - \frac{4\pi}{3}\delta_{ij}\delta(\vec{r}). \tag{7}$$

It is then a straightforward algebra to derive the expressions in (3) and (4) through the relationships between the fields and the respective dipole potentials as follows:

$$\vec{E}^{(2)} = -\vec{\nabla}\left(\frac{\vec{p}\boldsymbol{\cdot}\vec{r}}{r^3}\right) \tag{8}$$

and

$$\vec{B}^{(2)} = \vec{\nabla}\times\left(\frac{\vec{m}\times\vec{r}}{r^3}\right). \tag{9}$$

While both of these two approaches are correct, they are just purely mathematical, and it leaves one to wonder why the two δ-function terms appear so differently (in sign and in the coefficient) in the two dipole fields as in (3) and (4). Although it was later pointed out in [2] that the difference in the two δ-function terms in (3) and (4) arises mainly from the fact that the source for magnetic moments is from electric currents rather than magnetic charges, yet there is no explicit mathematical illustration to account for this physical origin of the difference.

The purpose of the present note is to provide a simple mathematical derivation to illustrate that the physical origin of this difference in the δ-function terms is indeed deeply rooted in the disparity between the existence of a monopole source for electric fields and the corresponding nonexistence of such a source for magnetic fields.

One straightforward way to illustrate this, as done in [3], is to resort to another advanced differentiation identity as follows:

$$\partial_i\left(\frac{3x_ix_j - r^2\delta_{ij}}{r^5}\right) = -\frac{8\pi}{3}\partial_j\delta(\vec{r}), \tag{10}$$

which can be obtained by setting the index $k = i$ in equation (7) of [3]. Using (10), one shows immediately that the dipole fields in (3) and (4) indeed satisfy the following conditions:

$$\vec{\nabla}\boldsymbol{\cdot}\vec{E}^{(2)} = -4\pi\,\vec{p}\boldsymbol{\cdot}\vec{\nabla}\delta(\vec{r}) \tag{11}$$

and

$$\vec{\nabla}\boldsymbol{\cdot}\vec{B}^{(2)} = 0. \tag{12}$$

Here we present an alternative approach to establish (11) and (12) without using any of the advanced differentiation identities [3] as that in (10). To demonstrate this explicitly, we first note that the electric dipole source can be represented by an equivalent volume charge distribution given by $\rho_{\mathrm{ED}} = -\vec{p}\boldsymbol{\cdot}\vec{\nabla}\delta(\vec{r})$.[1] One easy way to see this is to show that the following 'Coulomb integral' indeed leads to the correct electric dipole potential:

$$\int \frac{\rho_{\mathrm{ED}}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,\mathrm{d}^3x' = \frac{\vec{p}\boldsymbol{\cdot}\vec{r}}{r^3}. \tag{13}$$

[1] See e.g. problem 4.2 in [2].

Hence, by Gauss's law, we are led back to (11) as follows:

$$\vec{\nabla}\cdot\vec{E}^{(2)} = 4\pi\rho_{\mathrm{ED}} = -4\pi\,\vec{p}\cdot\vec{\nabla}\delta(\vec{r}). \tag{14}$$

Consequently, from (3) and (14), we conclude that the divergence of the first term in (3) must be given by

$$\vec{\nabla}\cdot\left(\frac{3(\vec{p}\cdot\vec{r})\vec{r} - r^2\vec{p}}{r^5}\right) = -\frac{8\pi}{3}\,\vec{p}\cdot\vec{\nabla}\delta(\vec{r}). \tag{15}$$

Hence, by replacing $\vec{p}$ by $\vec{m}$ in (15) and by using the result in (4), one obtains immediately the divergence-less condition for $\vec{B}^{(2)}$ as given in (12). As a result, one concludes from (11) and (12) that the difference in the δ-function terms in the expressions for the dipole fields in (3) and (4) is indeed deeply rooted in the disparity between the existence of a monopole source for electric fields and the corresponding nonexistence of such a source for magnetic fields.

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References

[1] See e.g. Landau L D and Lifshitz E M 1975 *The Classical Theory of Fields* (New York: Pergamon) p 104
[2] Jackson J D 1998 *Classical Electrodynamics* (New York: Wiley)
[3] Frahm C P 1983 *Am. J. Phys.* **51** 826