An effective medium approach to the dynamic optical response of a graded index plasmonic nanoparticle

H. Y. Chung[1*], P. T. Leung[2] D. P. Tsai[1,3]

**[1]Department of Physics, National Taiwan University,
Taiwan 10617, R. O. C.
[2]Department of Physics, Portland State University, P. O. Box 751,
Oregon 97207-0751, U. S. A.
[3]Instrument Technology Research Center, National Applied Research Laboratories,
Hsinchu, Taiwan 300, R. O. C.**

Abstract

The optical response of graded index spherical particles is studied using an effective medium approach, where the homogenization of the graded particle is achieved by using a static effective dielectric function available in the literature. Full wave calculation using the standard Mie theory for this "homogenized system" shows that for a plasmonic particle, such an approximation can lead to highly-accurate results compared to the exact ones, especially for slowly and smoothly varying index profiles. An illustration is provided via an application of this method to the design of an optical cloak using a graded plasmonic coating based on the scattering cancelation scheme. This approach thus surpasses the various long-wavelength approximations currently available in the literature and provides an efficient numerical treatment of light scattering from these inhomogeneous particles without having to solve directly the Maxwell's equations with spatially varying dielectric functions.

*** Corresponding author at b89202048@ntu.edu.tw**

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Introduction

The recent explosive developments in the understanding and application of metamaterials have stimulated intensive interest in the study of the optical properties of various inhomogeneous systems [1]. Some examples of the intriguing applications of metamaterial optics include phenomena such as non-reciprocal transmission [2, 3], perfect lens [4], and cloaking [5]. While a powerful method based on the technique of transformation optics [6] has been recently applied with success to achieve broadband cloaking [7], an alternative approach has also been proposed based on scattering cancelation technique using single- or multi- shell plasmonic coatings [8, 9], as well as using plasmonic nanoparticles with graded refractive indices [10].

Being motivated by the intriguing optical properties of these graded plasmonic particles, it is the purpose of our present work to present an efficient effective medium approach for the calculation of the optical properties of these particles. We shall demonstrate the accuracy of this approximate scheme via comparison with exact numerical results from electrodynamic calculation, and shall provide a theoretical clarification for its accuracy and limitations. For simplicity, we shall limit ourselves to spherical particles with a radially-varying profile in their dielectric functions [10].

It is well-known that the fully electrodynamic scattering theory can be rather complicated even for a homogeneous sphere (the Mie theory [11]), and it took several decades for a complete development of theories for various radially heterogeneous spherical systems including the coated sphere (single shell) [12], the stratified multi-shell systems [13], and the graded index sphere with a continuously-varying radial index profile [14, 15]. Note that this last problem is also reformulated in a different approach using scalar potentials in the recent work appeared in [10]. However, all of the above theories were based on direct solutions of the Maxwell equations for the various heterogeneous systems via the so-called "extended Mie theory", which can be rather complicated for arbitrary radial profile functions.

In our present approach, we shall adopt a "homogenization procedure" by describing the graded-index sphere using an effective medium theory (EMT) in the static limit [16, 17]. Following this we shall apply the conventional full wave Mie theory to calculate the extinction cross sections from this "homogenized sphere" and compare the results with those from the exact solutions. The accuracy of the approach is then justified by resorting to the theoretical results in [14]. In a recent work [18], we have applied the modified long wave approximation (MLWA) [19] to extend the applicability of the static EMT [16, 17] for plasmonic nanoparticles of larger sizes, and have observed that the dynamic (finite wavelength) effects are minimal for the effective dielectric function calculation. Here we shall see that such insignificance of the dynamic effects (for the effective dielectric function calculation) prevails even for much greater particles with sizes comparable or even greater than the wavelength. This is confirmed from the high accuracy of our numerical results for these large spheres which surpass those from previous quasi-static or MLWA theories.

Theory

The problem of scattering of a plane wave by a radially- inhomogeneous sphere was originally solved by Wyatt in 1962 [14] with the application of Maxwell's equations and the appropriate boundary conditions. Here we give a brief summary of Wyatt's theory.

For an incident plane wave of x-polarization propagating along the z direction,

$$\mathbf{E}^{(i)} = E_0 e^{i(kz-\omega t)} \mathrm{e}_x , \tag{1}$$

where $k = 2\pi/\lambda$ is the wave vector in the host medium (taken to be vacuum), the components of the scattered field (in spherical coordinates) can be expressed as follows [20]:

$$E_r^{(s)} = E_0 \frac{\cos\varphi}{(kr)^2} \sum_{\ell=1}^{\infty} \ell(\ell+1)\, {}^e B_\ell \zeta_\ell^{(1)}(kr) P_\ell^1(\cos\theta) , \tag{2}$$

$$E_\theta^{(s)} = -E_0 \frac{\cos\varphi}{kr} \sum_{\ell=1}^{\infty} \left\{ {}^e B_\ell \zeta_\ell^{(1)\prime}(kr) P_\ell^{1\prime}(\cos\theta)\sin\theta - i\, {}^m B_\ell \zeta_\ell^{(1)}(kr) P_\ell^1(\cos\theta) \frac{1}{\sin\theta} \right\}, \tag{3}$$

$$E_\varphi^{(s)} = -E_0 \frac{\sin\varphi}{kr} \sum_{\ell=1}^{\infty} \left\{ {}^e B_\ell \zeta_\ell^{(1)\prime}(kr) P_\ell^1(\cos\theta) \frac{1}{\sin\theta} - i\, {}^m B_\ell \zeta_\ell^{(1)}(kr) P_\ell^{1\prime}(\cos\theta)\sin\theta \right\}, \tag{4}$$

where P_ℓ^1 is the associated Legendre polynomial, $\zeta_\ell^{(1)}$ the Riccati-Hankel function of the first kind, and the prime indicates differentiation with respect to the argument in parentheses. Note that we have suppressed the time factor $e^{-i\omega t}$ for simplicity. With the appropriate boundary conditions, the scattering coefficients ${}^e B_\ell$ and ${}^m B_\ell$ for the transverse magnetic and transverse electric modes, respectively, can be obtained in the following form [14]:

$${}^e B_\ell = i^{\ell+1} \frac{2\ell+1}{\ell(\ell+1)} \frac{\varepsilon(x)\psi_\ell'(x)W_\ell(x) - W_\ell'(x)\psi_\ell(x)}{\varepsilon(x)\zeta_\ell^{(1)\prime}(x)W_\ell(x) - W_\ell'(x)\zeta_\ell^{(1)}(x)} \tag{5}$$

and

$${}^m B_\ell = i^{\ell+1} \frac{2\ell+1}{\ell(\ell+1)} \frac{\varepsilon(x)\psi_\ell(x)G_\ell'(x) - G_\ell(x)\psi_\ell'(x)}{\varepsilon(x)\zeta_\ell^{(1)}(x)G_\ell'(x) - G_\ell(x)\zeta_\ell^{(1)\prime}(x)} , \tag{6}$$

where $x = ka$ is the dimensionless size parameter and ψ_ℓ is the Riccati-Bessel function. We shall also introduce the dimensionless coordinate $\rho = kr$ in the following. The functions W_ℓ and G_ℓ are determined by the inhomogeneous dielectric function $\varepsilon(\rho)$ via the following pair of differential equations [14]:

$$\frac{d^2 W_\ell}{d\rho^2} - \frac{d\ln\varepsilon}{d\rho}\frac{dW_\ell}{d\rho} + \left[\varepsilon - \frac{\ell(\ell+1)}{\rho^2}\right] W_\ell = 0 \tag{7}$$

and

$$\frac{d^2 G_\ell}{d\rho^2} + \left[\varepsilon - \frac{\ell(\ell+1)}{\rho^2}\right] G_\ell = 0\,. \tag{8}$$

Note that in the homogeneous limit with $d\varepsilon / d\rho \to 0$, both $W_\ell(\rho)$ and $G_\ell(\rho)$ reduce to the Riccati-Bessel function with the argument $\sqrt{\varepsilon}\rho$, i.e. $W_\ell(\rho) = G_\ell(\rho) = \psi_\ell(\sqrt{\varepsilon}\rho)$ and $W'_\ell(\rho) = G'_\ell(\rho) = \sqrt{\varepsilon}\psi'_\ell(\sqrt{\varepsilon}\rho)$. Note also that the above results are equivalent to those obtained recently using scalar potentials [10].

Although the above "extended Mie theory" has been available for several decades, the analytic solutions for the differential equations in (7) and (8) are not simple and may not be possible for an arbitrary index profile $\varepsilon(\rho)$. In addition, $\varepsilon(\rho)$ must be smoothly continuous (hence differentiable) which then excludes graded systems with piecewise continuous dielectric profiles [21]. Hence it will be of interest to explore alternative approaches which can go beyond the above limitations to a certain extent. Here we introduce a simple (approximate but accurate) procedure in two steps: (i) to "homogenize" the graded index sphere using EMT in the static limit [16, 17], and (ii) to execute full wave calculations for the “homogenized system” using the standard Mie theory [11]. We shall first provide some numerical studies using this approach confirming its accuracy for plasmonic or absorptive materials in general, and then provide a theoretical justification of it in Appendix I. Following this, we shall apply this methodology to study the extinction properties of various plasmonic nanoparticles with different index profiles described by a graded Drude model [18], including an application of plasmonic coating for the cloaking of a particle [5, 8, 9].

Numerical Results

In order to illustrate the applicability and limitation of our method, we first consider a hypothetical graded sphere with a linear dielectric profile of the form $\varepsilon(r) = cr/a$, where c is a complex constant in general and a the radius of the sphere.

Using a differential effective medium approach, we have previously applied the MLWA [19] to obtain a differential equation to be satisfied by the effective dielectric function $\varepsilon_s(a)$, for the dipole response of a radially-graded sphere in the form [18]:

$$\frac{d\varepsilon_s}{da} = -\frac{(\varepsilon_s - \varepsilon)}{3a\varepsilon}[(3 - x^2)(\varepsilon_s - \varepsilon) + 9\varepsilon], \qquad (9)$$

where $x = ka$ is the size parameter. In the limit $x \to 0$, Eq. (9) reproduces the same result as obtained in the literature from a quasi-static theory [16, 17]. We have clarified in [18] that the electrodynamic effects (i.e. $x \neq 0$) is minimal in Eq. (9) for all practical purposes. We further propose here that this insignificance of x is valid even beyond the MLWA and one can simply apply the quasi-static effective dielectric functions for all multipoles in the full wave electrodynamic calculations [see Appendix I].

For a graded dielectric function of the form $\varepsilon(r) = \eta r^m$, the zeroth order ($x = 0$) solution of (9) can be obtained in the following form [16, 17]:

$$\varepsilon_s(a) = za^m \,, \tag{10}$$

with

$$z = \frac{\eta}{2}[-(1+m) + \sqrt{m^2 + 2m + 9}] \,. \tag{11}$$

These results can be generalized to all higher multipole response of the sphere (see Appendix I and II). To illustrate the accuracy of our method, we apply the results obtained in the appendices to a linearly-graded sphere with $\varepsilon(r) = cr/a$, by setting $m = 1$ and $\eta = c/a = \varepsilon_{Ag}/L^m$ in Eq. (A-13), and then use these into the conventional Mie theory (i.e. with $W_\ell(x) = G_\ell(x) = \psi_\ell(\sqrt{\varepsilon_s}x)$ in Eqs. (5) and (6)) for a homogeneous sphere. We have in this way computed the extinction efficiency for a large sphere (a =1000 nm). Figure 1 shows the results using this approach and comparison is made with those obtained from the exact theory. Note that rather than solving directly the complicated differential equations in (7) and (8), the "exact results" in Fig. 1 are obtained via a multilayer modeling of the actual profile [22] with convergence guaranteed by using a sufficient number of layers. In all our computations here in this work, most of the graded-index particles we have considered do not require more than 15 layers to achieve convergence with high accuracy. From the results in Fig. 1, it is clearly demonstrated that except for a real positive dielectric function (Fig. 1a), our approach yields extremely accurate results in comparison with those from an exact calculation. Thus for either an absorptive (c complex) or a plasmonic (Re $c < 0$) sphere, our effective medium approach is highly accurate. This is reasonable since in these two cases penetration of the incident light into the bulk of the sphere is relatively insignificant, and hence the dynamic effects (such as retardation and radiation loss [19]) due to the appreciable extent of the particle are not manifested. The maximum relative errors for the EMT are limited to about 2% (for Fig. 1(b)) and 4% (for Fig. 1(c)), respectively; while EMT for Fig. 1(d) (both plasmonic and absorptive) is most accurate with a maximum error not exceeding 0.5%.

Having established the accuracy and limitation of our approach, we next apply it to a dispersive plasmonic particle characterized by a graded Drude function. We shall consider a "radially-graded" metallic (silver) sphere described by the following dielectric function: $\varepsilon(r) = \eta r^m$, $\eta = \varepsilon_{Ag} / a^m$, where a is the radius of the sphere, and ε_{Ag} is the Drude function in the form $\varepsilon_{Ag} = 1 - \omega_p^2 / \omega(1 + i\Gamma)$ with $\omega_p = 1.36 \times 10^{16}$ s^{-1}, $\Gamma = \Gamma_B + v_F / a$, $\Gamma_B = 2.56 \times 10^{13}$ s^{-1} and $v_F = 1.39 \times 10^{15}$ nm/s [23].

Figure 2 shows the extinction efficiency for a graded Drude sphere (a = 100 nm, $m = 1$) computed using our effective medium approach in comparison with those from the exact Mie results [14] (again we adopted a graduated-multilayer Mie approach here) as well as those from the static (Rayleigh) and MLWA approximations. The accuracy of our present homogenization approach is obvious from the comparison shown. For such a large sphere, the Rayleigh and MLWA are both invalid, although the latter does yield a red-shifted and broadened dipole resonance [18, 19] which is somewhat closer to the exact result. In particular, no higher multipolar resonance can be accounted for by these long-wavelength approximations. In contrast, our present effective medium approach yields excellent results with all the multipolar resonances correctly captured. It is of interest to note that more noticeable discrepancy between our present approach and the exact results occurs in the long wavelength regime (up to ~ 7% relative error for the EMT). This can be understood since the results in Fig. 1 have shown that the accuracy of our effective medium approach will be determined mainly by the imaginary part of the dielectric function in the presence of absorption. Thus more appreciable penetration of light into the sphere will occur at long incident wavelengths (due to the decrease in the imaginary part of the Drude function), which makes the extent of the sphere more manifested and the negligence of retardation effects in our "homogenization process" more significant.

Figure 3 shows the comparison of the results from the present approach with the exact results for various index profiles ($m=1$, 2, 3) for the same 100 nm sphere. It is seen that while the present homogenization approach is still very accurate (especially for the short wavelength region), the accuracy starts to get poorer for more rapidly varying profiles (larger value of m), with progressive maximum relative errors for the EMT ranging from 7% for Fig. 3(a) with $m=1$, to 14% for Fig. 3(b) ($m=2$) and 19% for Fig. 3(c) ($m=3$), respectively.

After clarifying the accuracy of our approach, we next apply it to study the possibility of achieving minimal interaction of these graded-index particles with light to achieve "optical transparency" (cloaking) using such inhomogeneous systems. In a recent work [10], this is achieved via a "scattering cancellation scheme" using these particles in the ideal limit when they are described by a lossless graded Drude function. Here we use a slightly different graded Drude function where loss is accounted for as described above [23].

We first demonstrate the usefulness of our approach in cloaking design using a simpler example with a hypothetical lossless plasmonic layer of linearly-graded index in the form: $\varepsilon_2(r)=-3r/b$. We consider a shell of thickness $(b-a)$ being used to coat and cloak a sphere of radius a and dielectric constant $\varepsilon_1=4$. The details for the above effective medium theory for a spherical shell are given in Appendix II. We shall set a criterion for achieving minimum extinction efficiency to correspond to a dipolar effective dielectric constant (for the shell, i.e.

$\beta_\ell^s, \ell = 1$, see Appendix II) such that the overall effective dielectric function ($\varepsilon_{s,1}$) of the core-shell system is as close to one as possible.

Figure 4 shows the calculations for various sphere radius a for a fixed outer shell radius of $b = 0.01\lambda$ where λ is the incident wavelength. We thus see that for a sphere size of about $0.47b$, the extinction efficiency becomes exceedingly small (< 10^{-10}) corresponding exactly to an overall effective dielectric function (as defined above) close to unity. Similar cloaking phenomenon has been demonstrated in [8, 9] using the conventional Mie theory with homogeneous plamonic shell as the cloaking material.

Finally, we study the effects due to the loss in a real metal. Figure 5 shows a similar calculation as in Fig. 4 but with a small damping parameter added to the dielectric function of the plasmonic shell. It can be seen that the effect due to loss on the scattering efficiency is very little with the “scattering minimum” only slightly shifted to a larger core radius, and our proposed criterion for “cloaking design” (i.e. by requiring the overall effective dielectric function to be close unity) remains largely valid. However, the total extinction efficiency of the coated particle will keep on decreasing in this case with the increase of the core size, which is understandable due to the decrease in metallic volume (and hence decrease in absorption) for a fixed external radius b.

Discussion and Conclusion

In this work, we have shown that one can account for the optical properties of a graded spherical particle in an efficient way with high accuracy in two steps: (1) homogenization using an electrostatic effective medium approach; and (2) application of the standard Mie theory to the homogenized particle. We have observed that such an approximation is accurate as long as the particle is absorptive, and can yield very good results even for particles much larger than the wavelengths. This hence provides a useful method to study the graded plasmonic particles, an alternative to the approach based on the direct solution of the Maxwell equations for inhomogeneous medium (i.e. Eqs. (7) and (8)). However, we have to admit that here we also have to solve a complicated differential equation in general (Eq. (9) with $x = 0$), although the complexity involved may be comparably less for some simple inhomogeneous dielectric functions, and only one instead of a pair of differential equations is needed to be solved.

It is perhaps not too surprising that such a scheme can work so well for an inhomogeneous graded particle, since it has been quite well-known already as an accurate description of full-wave optical phenomena for bulk heterogeneous composites using static effective medium models such as Maxwell-Garnett (MG) and Bruggeman theories for the effective dielectric function of the composites [24]. Here we have provided further clarification of the limitation and accuracy of our method in terms of the absorption property of the particle. Thus it will be of interest to re-examine the validity of the conventional effective medium theories (MG,…etc.) as applied to composites of transparent host and ingredients. In addition, it may be an interesting problem to investigate the real dynamic nature of any effective medium theory as applied to an extended heterogeneous system [25].

Besides technical simplicity, our proposed effective medium approach possesses two other distinct features. One is that the formalism allows one to treat spherical multilayered systems with piecewise-continuous index profiles in a rather straightforward way. The same will not be trivial in the approach which involves direct solution of the Maxwell's equations (e.q., Eqs. (7) and (8)) if the profile functions are not differentiable at all locations within the particle. Furthermore, as we have illustrated in the "cloaking-design" application, our method allows a more direct determination of the parameters critical to the cloaking condition (e.g. our criterion of setting the overall effective dielectric constant with the parameter β_1 to be unity). It will thus be of interest to extend the application of such an approach to study other metamaterial systems such as those involve “zero refractive index” photonic crystals [26].

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Appendix I

Here we provide a justification of our homogenization approach by introducing an effective Mie theory for the exact theory as given in Eqs. (5) – (8) [14]. To begin, we shall introduce for each multipole mode ℓ, an effective sphere with a homogeneous dielectric function β_ℓ such that

$$ {}^{e}B_\ell = i^{\ell+1}\frac{2\ell+1}{\ell(\ell+1)}\frac{\sqrt{\beta_\ell}\,\psi_\ell'(x)\psi_\ell(\sqrt{\beta_\ell}x)-\psi_\ell'(\sqrt{\beta_\ell}x)\psi_\ell(x)}{\sqrt{\beta_\ell}\,\zeta_\ell^{(1)\prime}(x)\psi_\ell(\sqrt{\beta_\ell}x)-\psi_\ell'(\sqrt{\beta_\ell}x)\zeta_\ell^{(1)}(x)}, \qquad \text{(A-1)} $$

where $\psi_\ell(x)$ is the Riccati-Bessel function satisfying the differential equation

$$\frac{d^2\psi_\ell(x)}{dx^2}+\left[1-\frac{\ell(\ell+1)}{x^2}\right]\psi_\ell(x)=0\,. \tag{A-2}$$

Next, by referring to Eqs. (5) and (A-1) and by defining $w_\ell(x)=W_\ell'(x)/W_\ell(x)$ and $f_\ell(x)=\psi_\ell'(\sqrt{\beta_\ell}x)/\psi_\ell(\sqrt{\beta_\ell}x)$, the equivalence between Eq. (5) and (A-1) then leads to the following relation:

$$\frac{\varepsilon(x)\psi_\ell'(x)-w_\ell(x)\psi_\ell(x)}{\varepsilon(x)\zeta_\ell^{(1)\prime}(x)-w_\ell(x)\zeta_\ell^{(1)}(x)}=\frac{\sqrt{\beta_\ell(x)}\psi_\ell'(x)-\psi_\ell(x)f_\ell(x)}{\sqrt{\beta_\ell(x)}\zeta_\ell^{(1)\prime}(x)-\zeta_\ell^{(1)}(x)f_\ell(x)}\,, \tag{A-3}$$

which implies

$$w_\ell(x)=\frac{\varepsilon(x)}{\sqrt{\beta_\ell(x)}}\,f_\ell(x)\,. \tag{A-4}$$

On the other hand, from equations similar to those in Eqs. (7), one can show that w_ℓ satisfies the following differential equation:

$$w_\ell'(x)+w_\ell^2(x)-\frac{\varepsilon'(x)}{\varepsilon(x)}w(x)+\left[\varepsilon(x)-\frac{\ell(\ell+1)}{x^2}\right]=0\,. \tag{A-5}$$

Note that we have here changed the variable in (7) from ρ to x. Now taking derivative of both sides of Eq. (A-4), i.e.

$$w_\ell'(x)=\left(\frac{\varepsilon'(x)}{\sqrt{\beta_\ell(x)}}-\frac{\varepsilon(x)\beta_\ell'(x)}{2\beta_\ell^{3/2}(x)}\right)f_\ell(x)+\frac{\varepsilon(x)}{\sqrt{\beta_\ell(x)}}\,f_\ell'(x)\,, \tag{A-6}$$

and employing Eqs. (A-4) to (A-6) to eliminate the w_ℓ dependence, we finally obtain:

$$\beta_\ell' = \frac{2\beta_\ell^{\ 3/2}}{\varepsilon f}\left[\frac{\varepsilon^2}{\beta_\ell} f_\ell^2 + \frac{\varepsilon}{\sqrt{\beta_\ell}} f_\ell' + \varepsilon - \frac{\ell(\ell+1)}{x^2}\right]. \quad \text{(A-7)}$$

Corresponding results can also be obtained from ${}^mB_\ell$ in Eq. (6). But since we shall be interested in the long wavelength limit of the effective dielectric function in the following, this shall not be considered further for it is ${}^eB_\ell$ that dominates in this limit [20].

Now consider the long wavelength limit with $\sqrt{\beta_\ell}x \ll 1$, the Riccati-Bessel function ψ_ℓ has the following Taylor expansion:

$$\psi_\ell(x) = \frac{x^{\ell+1}}{(2\ell+1)!!}\left[1 - \frac{x^2}{2(2\ell+3)} + \cdots\right]. \quad \text{(A-8)}$$

Hence in the lowest order approximation, $\psi_\ell(\sqrt{\beta_\ell}x) \approx \left(\sqrt{\beta_\ell}x\right)^{\ell+1} / (2\ell+1)!!$, which leads to

$f_\ell(x) \approx (\ell+1)/(\sqrt{\beta_\ell}x)$ and

$$\beta_\ell'(x) = -\frac{\beta_\ell - \varepsilon}{x\varepsilon}\left[\ell\beta_\ell + (\ell+1)\varepsilon\right] + \frac{\beta_\ell^2 x}{\ell+1}. \quad \text{(A-9)}$$

Note that (A-9) reduces back to the result obtained from the quasi-static approach [16, 17]:

$$\beta_\ell'(x) = -\frac{\beta_\ell - \varepsilon}{x\varepsilon}\left[\ell\beta_\ell + (\ell+1)\varepsilon\right], \quad \text{(A-10)}$$

if we ignore the term $\beta_\ell^2 x / (\ell+1)$ in (A-9). Furthermore, by considering only the dipole $(\ell = 1)$ term in (A-10), the result in Eq. (9) (with $x \to 0$) is recovered.

Appendix II

Here we provide an extension of the effective medium theory in Appendix I to the case of a core (radius a) covered by a graded-index spherical shell (radius b), where the core can have either a graded-index or just be homogeneous. We first recall the case when both the core and shell are homogeneous with uniform dielectric functions ε_a and ε_b, an effective dielectric function (or order ℓ)can be obtained in the electrostatic limit in the form [27]:

$$\beta_\ell = \frac{[\ell\varepsilon_a + (\ell+1)\varepsilon_b]b^{2\ell+1} + (\ell+1)(\varepsilon_a - \varepsilon_b)a^{2\ell+1}}{[\ell\varepsilon_a + (\ell+1)\varepsilon_b]b^{2\ell+1} - \ell(\varepsilon_a - \varepsilon_b)a^{2\ell+1}}\varepsilon_b \quad . \tag{A-11}$$

Note that by setting $\beta_\ell = 1$, (A-11) leads back to the following "cloaking condition" as obtained previously by Alu et al [8]:

$$(\varepsilon_1 - \varepsilon_2)[(\ell+1)\varepsilon_2 + \ell\varepsilon_h]a^{2\ell+1} = (\varepsilon_2 - \varepsilon_h)[\ell\varepsilon_1 + (\ell+1)\varepsilon_2]b^{2\ell+1} \quad . \tag{A-12}$$

Now let us consider a graded-index shell with a power law profile: $\varepsilon(r,\omega) = \varepsilon_{Ag}(\omega)r^m / L^m$.

In order for the result in (A-11) to be applicable in this case, we have to first determine the effective dielectric functions for each of the two homogenized spheres with radius $r = a$ and $r = b$, respectively, using the differential equation established in Eq. (A-10). The results thus obtained are as follows:

$$\beta_\ell^a = \frac{\varepsilon_{Ag}}{2\ell}[-(1+m) + \sqrt{(m+1)^2 + 4\ell(\ell+1)}]\frac{a^m}{L^m} , \tag{A-13}$$

and

$$\beta_\ell^b = \frac{\varepsilon_{Ag}}{2\ell}[-(1+m) + \sqrt{(m+1)^2 + 4\ell(\ell+1)}]\frac{b^m}{L^m} . \tag{A-14}$$

Now if we assume the effective dielectric function of the shell (i.e. for $a < r < b$) to be some value denoted by β_ℓ^S, and use (A-11) with the following replacements:

$$\varepsilon_a \to \beta_\ell^a, \quad \varepsilon_b \to \beta_\ell^S, \quad \beta_\ell \to \beta_\ell^b,$$

we can then solve for β_ℓ^S and finally obtain the following results:

$$\beta_\ell^S = -\xi \pm \sqrt{\xi^2 + \eta}\,, \tag{A-15}$$

where

$$\xi = \frac{1}{2(b^{2\ell+1} - a^{2\ell+1})}\left[\left(\frac{\ell}{\ell+1}\beta_\ell^a - \beta_\ell^b\right)b^{2\ell+1} - \left(\frac{\ell}{\ell+1}\beta_\ell^b - \beta_\ell^a\right)a^{2\ell+1}\right], \tag{A-16}$$

and

$$\eta = \frac{\ell}{\ell+1}\beta_a\beta_b\,. \tag{A-17}$$

Note the $\pm$ sign in Eq. (A-15) is chosen to satisfy the condition: $\min(\beta_\ell^a, \beta_\ell^b) < \beta_\ell^S < \max(\beta_\ell^a, \beta_\ell^b)$.

The results given in (A-13) – (A-17) with $m = 1$are then used in the calculations for the illustration of cloaking design in Figures 4 and 5.

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23. Note that our graded-Drude function is slightly different from that used in Ref. [10] and damping is included in our model.

24. See, e.g., V. Janicki, J. Sancho-Parramon, H. Zorc, “Refractive index profile modelling of dielectric inhomogeneous coatings using effective medium theories”, Thin Solid Films **516**, 3368-3373 (2008).

25. See, e.g., T. C. Choy, *Effective Medium Theory: Principles and Applications* (Clarendon, Oxford, 1999).

26. See, e.g., J. Schilling, “ The quest for zero refractive index”, Nature Photonics **5**, 449-451 (2011).

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Figure captions

1. Illustration of the accuracy and limitation of our proposed effective medium approach: plots of the extinction efficiencies of a graded sphere with dielectric function $\varepsilon(r) = cr/a$, where (a) $c = 2.5$, (b) $c = 2.5 + i$, (c) $c = -2.5$ and (d) $c = -2.5 + i$. The radius of the sphere is fixed at $a = 1000$ nm. (Color online)

2. Extinction efficiency for a graded Drude sphere with $\varepsilon = \varepsilon_{Ag} r^m / a^m$ with $m = 1$, and a radius fixed at $a = 100$ nm. Results from our EMT are compared with those from the exact graded-Mie theory as well as those from the static (Rayleigh) and MLWA approximations. (Color online)

3. Extinction efficiency for a graded Drude sphere with $\varepsilon = \varepsilon_{Ag} r^m / a^m$ of different power indices:(a) $m = 1$, (b) $m = 2$ and (c) $m = 3$. The radius is fixed at $a = 100$ nm. (Color online)

4. Extinction efficiency for a dielectric sphere of radius a coated by a graded-index shell. The dielectric constant of the sphere is $\varepsilon_1 = 4$, and that of the shell has a linear profile $\varepsilon_2 = -3r/b$, where the outer radius of the shell is fixed to $b = 0.01\lambda$. The effective dielectric function for the dipole resonance mode $\varepsilon_{s,1}$ is shown by the dashed line.

5. Same as in Fig. 4, except that loss is introduced for the shell with $\varepsilon_2 = (-3 + 0.1i)r/b$. (Color online).

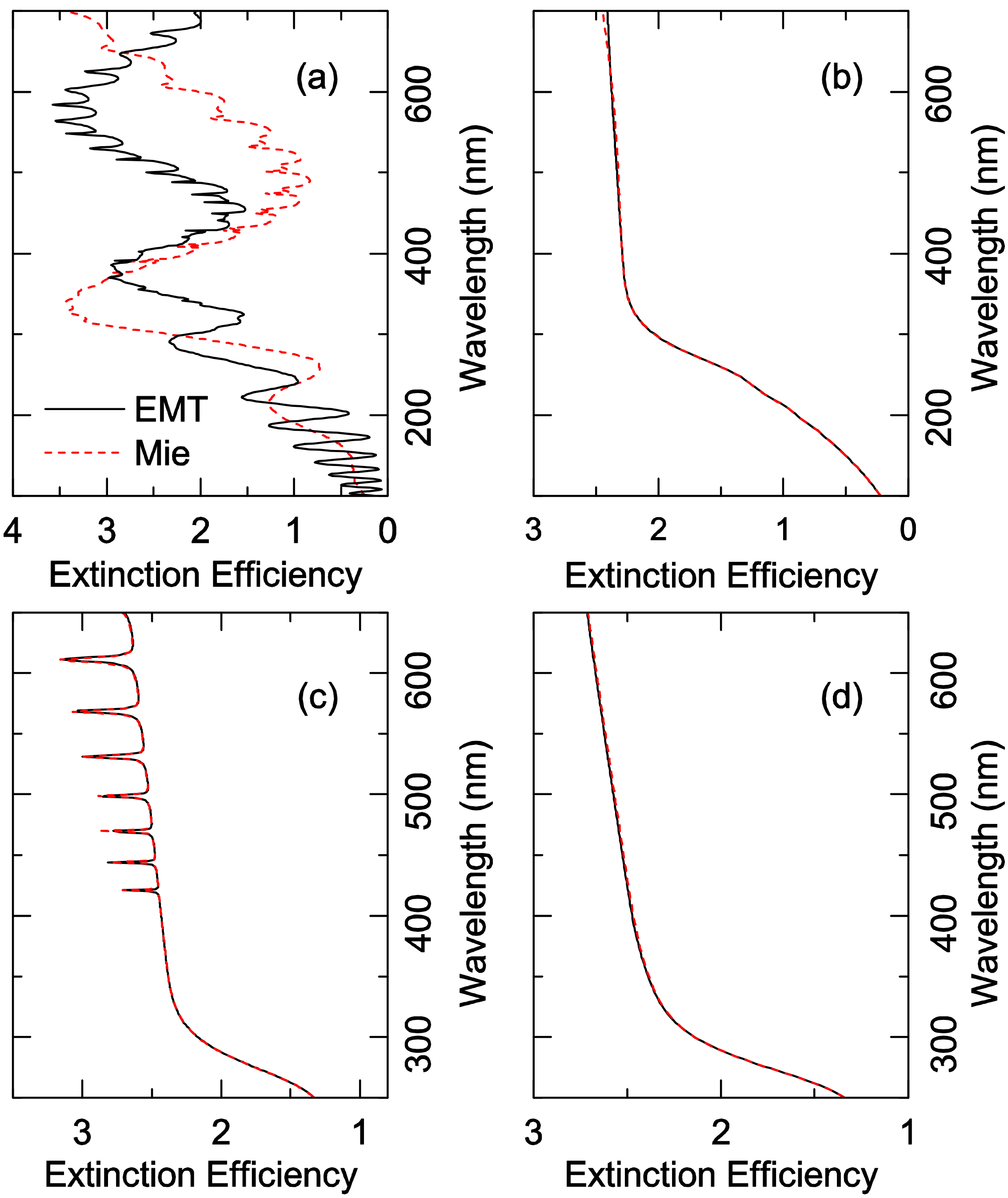


Fig. 1

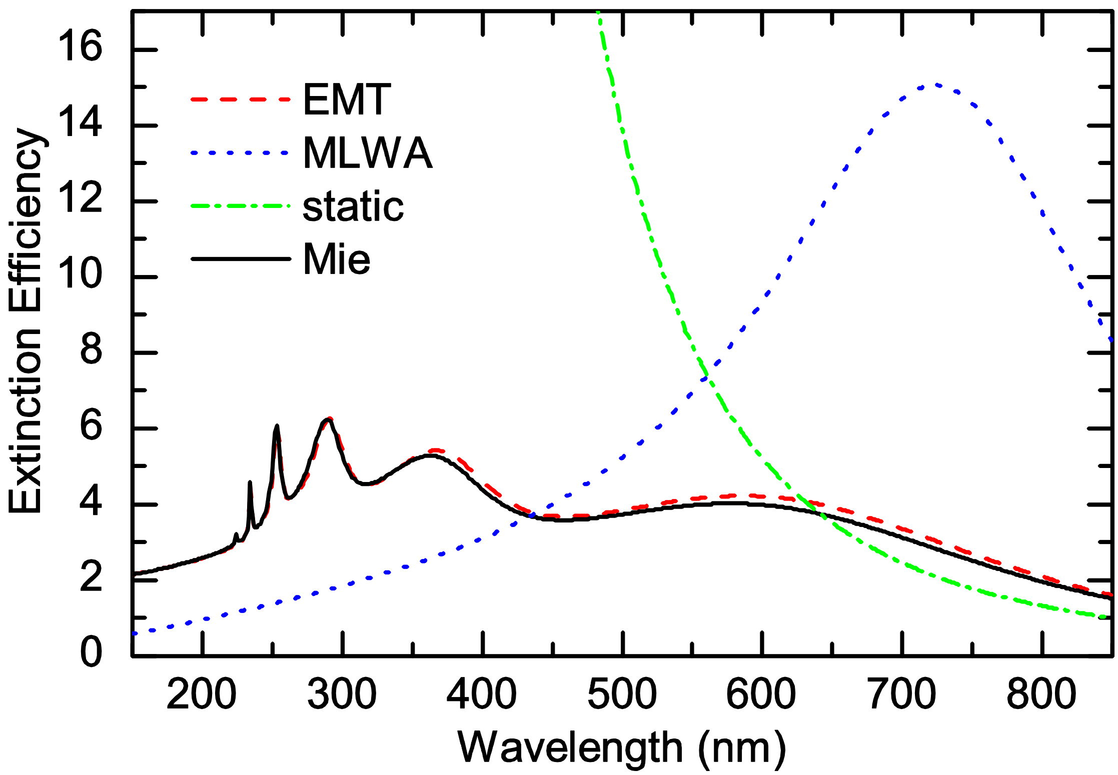
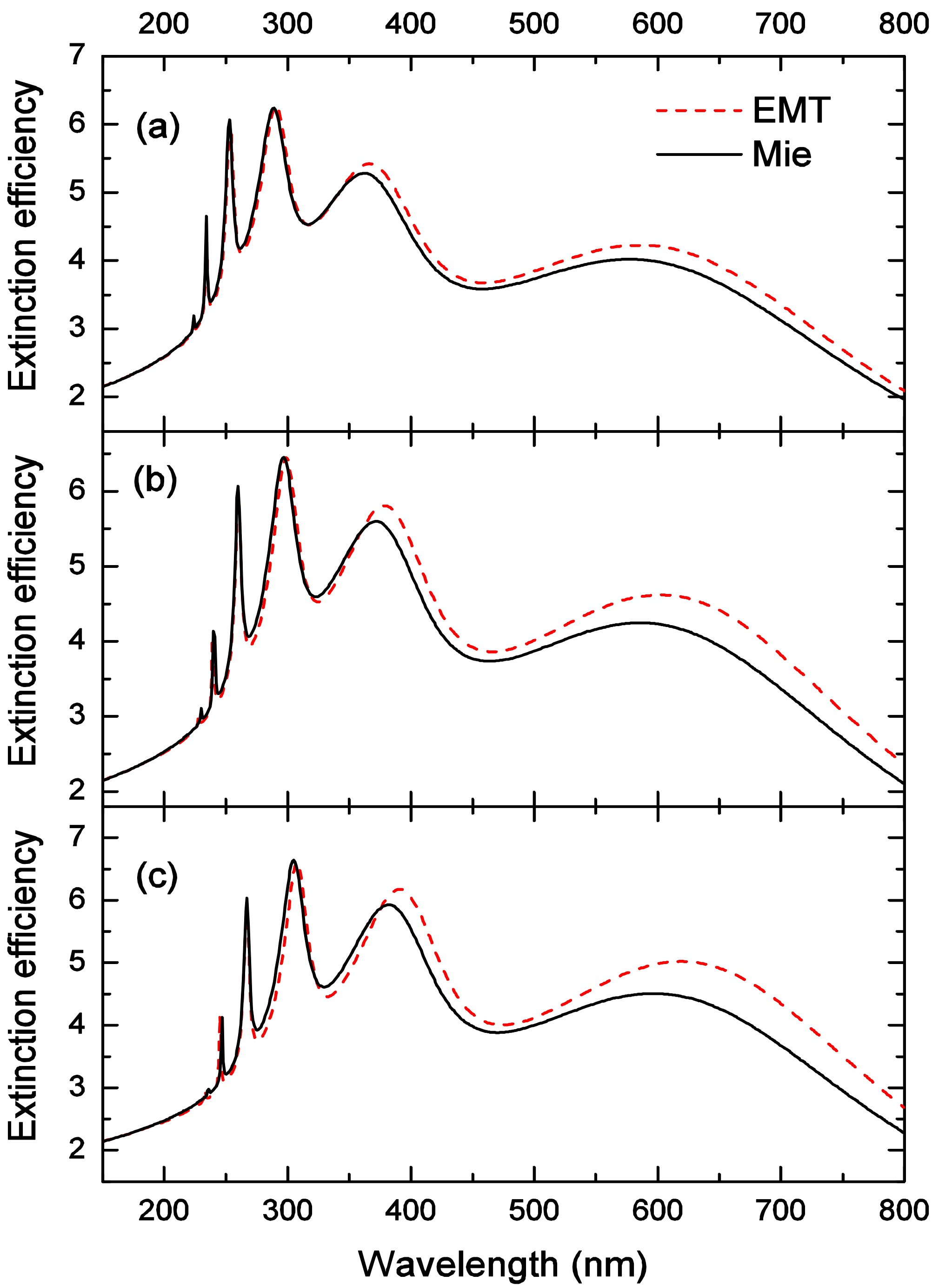
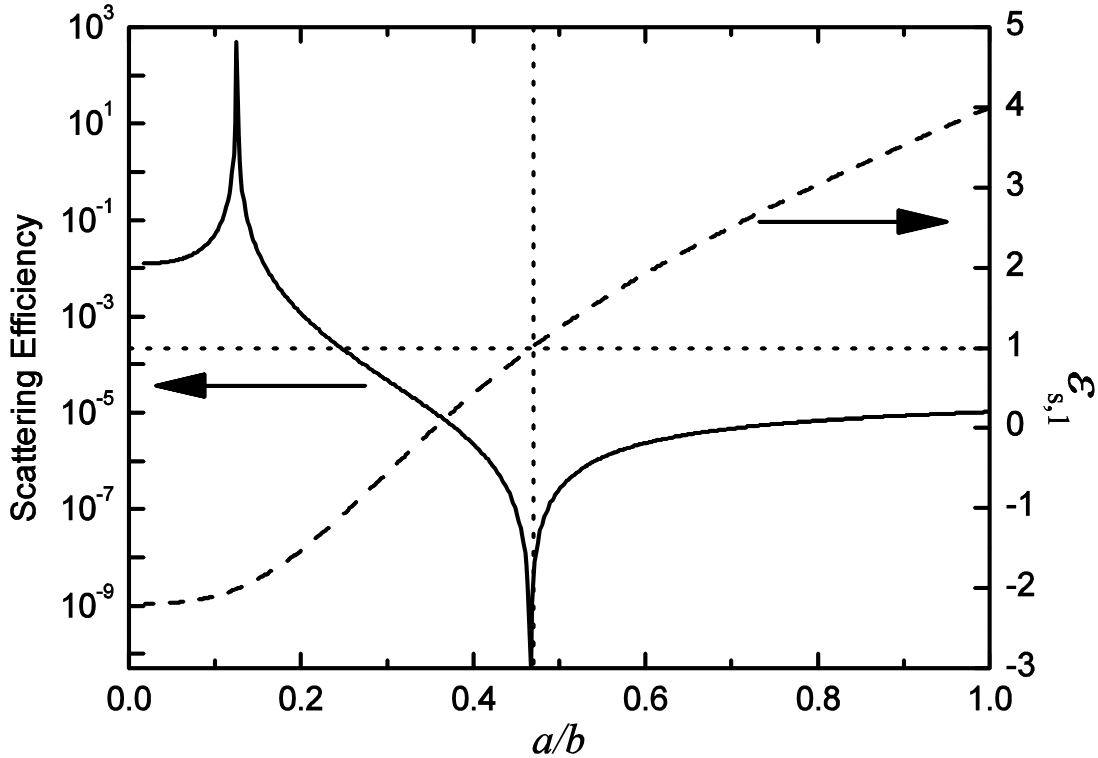

EMT
MLWA
static
Mie
Extinction Efficiency
Wavelength (nm)
0
2
4
6
8
10
12
14
16
200
300
400
500
600
700
800

Fig. 2



200
300
400
500
600
700
800
Extinction efficiency
(a)
EMT
Mie
(b)
(c)
Wavelength (nm)

Fig. 3



Scattering Efficiency
10^3
10^1
10^{-1}
10^{-3}
10^{-5}
10^{-7}
10^{-9}
0.0
0.2
0.4
0.6
0.8
1.0
a/b
5
4
3
2
1
0
-1
-2
-3
$\varepsilon_{s,1}$

Fig. 4

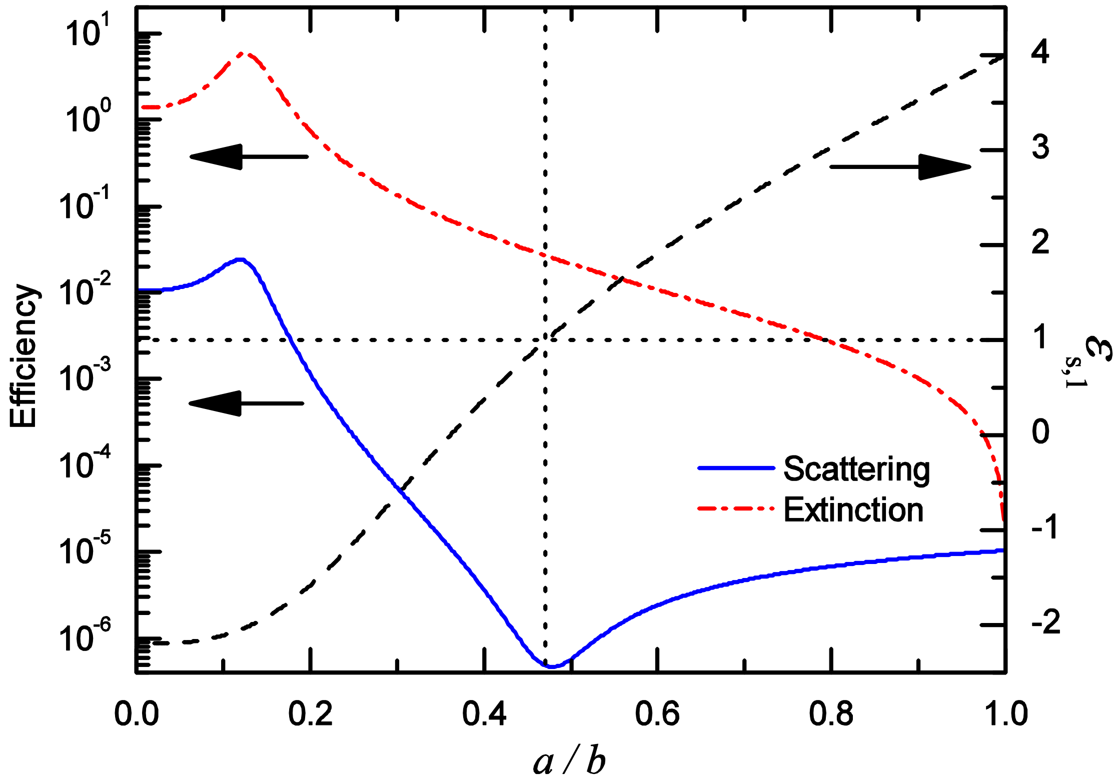


Fig. 5