Clarification and extension of the optical reciprocity theorem

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(Received 18 December 2008; accepted 3 June 2009; published online 10 July 2009)

Clarifications on the optical reciprocity theorem are provided by explicitly proving the equivalence between the Lorentz lemma and the symmetry of the Green dyadic for the electromagnetic wave equation. This is achieved by explicitly including the surface term in the former so that different boundary conditions can be considered as required in the formulation of the latter. In addition, we shall also extend the theorem to include anisotropic magnetic materials with a nonlocal response, leading to a result which will be useful for the study of materials possessing such properties such as certain types of metamaterials. © *2009 American Institute of Physics.* [DOI: 10.1063/1.3162201]

I. INTRODUCTION

Reciprocity in wave propagation is an important concept in many fields of physics such as classical electrodynamics, optics, and quantum mechanics.[1,2] In general, this refers to the symmetry of the propagation under the interchange of the source and the observer. Interesting applications of this symmetry have been found in the theoretical analysis of many problems in optics and spectroscopy.[1,3,4]

Though the reciprocity symmetry in optics or electrodynamics can be formulated mathematically in several different ways,[1] two most common approaches will be in terms of the Lorentz lemma[5] (in obvious notation):

$$\int \mathbf{J}_1 \bullet \mathbf{E}_2 d^3x = \int \mathbf{J}_2 \bullet \mathbf{E}_1 d^3x, \tag{1}$$

and the symmetry for the (electric) Green dyadic associated with the vector wave equation:[6]

$$[\boldsymbol{G}_e(\mathbf{r},\mathbf{r}')]^T = \boldsymbol{G}_e(\mathbf{r}',\mathbf{r}). \tag{2}$$

While the equivalence between (1) and (2) is obvious in the case when infinite free space is considered through the relation between the field and the source (in Gaussian units):

$$\mathbf{E}(\mathbf{r}) = \frac{i\omega}{c}\int \boldsymbol{G}_e(\mathbf{r},\mathbf{r}') \bullet \mathbf{J}(\mathbf{r}')d^3x', \tag{3}$$

it is not as obvious when boundaries of finite extent and real materials are involved where the validity of (2) depends crucially on the types of boundary conditions and the types of materials,[1,7–10] whereas such dependences (the boundary conditions, in particular) are not as

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clearly revealed in (1). To our knowledge, this equivalence in the situation with the most general boundary conditions has not been studied previously in the literature.

It is the purpose of this work to provide an explicit proof of the equivalence of the above two versions for the reciprocity principle, with an account for the specific boundary conditions (e.g., Dirichlet and Neumann conditions) involved. In addition, we shall also generalize our previous work on the conditions for the validity of reciprocity in the presence of nonlocal anisotropic medium to include magnetic materials.

In the literature, the validity of (1) and (2) has been established for various optical media including both electric and magnetic materials characterized by linear, local, and both isotropic and anisotropic responses.[11,12] However, there has been some controversy on its validity in the presence of nonlocal dielectric responses.[1,7] Recently we have clarified this by considering the following general form of linear response:

$$\mathbf{D}(\mathbf{r}) = \int \boldsymbol{\varepsilon}(\mathbf{r},\mathbf{r}') \bullet \mathbf{E}(\mathbf{r}') d^3x', \tag{4}$$

and have shown that reciprocity will break down unless the dielectric tensor satisfies the following symmetry property:[8]

$$\boldsymbol{\varepsilon}_{ij}(\mathbf{r},\mathbf{r}') = \boldsymbol{\varepsilon}_{ji}(\mathbf{r}',\mathbf{r}). \tag{5}$$

In the present work, we are motivated to generalize these results to include the case with a nonlocal anisotropic magnetic permeability into the theory on account of the recent explosion in the research with metamaterials. The so-called double negative left-handed material (with both permittivity and permeability negative) does not have to be anisotropic necessarily; a broader class of artificial (hence "meta-") materials contains in general both electric and magnetic properties and often fabricated with a structure which is highly anisotropic and inhomogeneous, and hence nonlocal effects are significant.[13] Thus besides Eq. (4), we should also consider the following constitutive relation:

$$\mathbf{H}(\mathbf{r}) = \int \boldsymbol{\mu}^{-1}(\mathbf{r},\mathbf{r}') \bullet \mathbf{B}(\mathbf{r}') d^3x', \tag{6}$$

for our study of the validity of (2). In the literature, we have been able to locate only two previous works published recently which studied the reciprocity of chiral[9] and left-handed[12] media incorporating the magnetic permeability, but these have considered only local response of the media. Here we shall establish in the following the additional condition

$$\boldsymbol{\mu}_{ij}(\mathbf{r},\mathbf{r}') = \boldsymbol{\mu}_{ji}(\mathbf{r}',\mathbf{r}) \tag{7}$$

for reciprocity symmetry to hold in a linear anisotropic nonlocal metamaterial of this kind using only classical electrodynamics, in contrast to the quantum mechanical approach adopted in Ref. 12. Although the result in (7) is somewhat expected, we shall see that the mathematical procedures involved are highly nontrivial, and new mathematical identities involving dyadics have to be established along the way.

II. EQUIVALENCE BETWEEN LORENTZ LEMMA AND GREEN DYADIC SYMMETRY

To demonstrate the equivalence between Lorentz lemma and the symmetry properties of the Green dyadic, let us start with a slightly more general form of Eq. (1) by retaining the surface terms:[5]

$$\frac{4\pi}{c}\int(\mathbf{J}_1\bullet\mathbf{E}_2-\mathbf{J}_2\bullet\mathbf{E}_1)d^3x=\oint_S\mathbf{n}\bullet(\mathbf{E}_1\times\mathbf{H}_2-\mathbf{E}_2\times\mathbf{H}_1)da. \tag{8}$$

Note that Eq. (8) is a direct consequence from Maxwell's equations, and the surface terms are kept to allow for the explicit discussion of various types of boundary conditions in the presence of finite boundaries and nontrivial dielectric materials. Although these surface terms are often discarded,[1,4,5,7] they have also been considered in some studies in the literature,[14,15] and here we *must* keep them to demonstrate the exact equivalence between the two versions of reciprocity symmetry.

To demonstrate the equivalence between (2) and (8), let us consider two point current sources due to electric dipole (with moment p) as follows:

$$\mathbf{J}_1=-i\omega p\,\delta(\mathbf{r}-\mathbf{r}'')\mathbf{e}_i,\quad \mathbf{J}_2=-i\omega p\,\delta(\mathbf{r}-\mathbf{r}')\mathbf{e}_j, \tag{9}$$

and the electric fields at each of their locations are then given in terms of the column component of the dyadic as follows:

$$\mathbf{E}_1=\frac{\omega^2p}{c}\boldsymbol{G}_{ei}(\mathbf{r},\mathbf{r}''),\quad \mathbf{E}_2=\frac{\omega^2p}{c}\boldsymbol{G}_{ej}(\mathbf{r},\mathbf{r}'). \tag{10}$$

Substituting (9) and (10) into (8) leads to the following result:

$$-\frac{4\pi i\omega p}{c}[\mathbf{e}_i\bullet\boldsymbol{G}_{ej}(\mathbf{r}'',\mathbf{r}')-\mathbf{e}_j\bullet\boldsymbol{G}_{ei}(\mathbf{r}',\mathbf{r}'')]=\oint_S\mathbf{n}\bullet[\boldsymbol{G}_{ei}(\mathbf{r},\mathbf{r}'')\times\mathbf{H}_2(\mathbf{r})-\boldsymbol{G}_{ej}(\mathbf{r},\mathbf{r}')\times\mathbf{H}_1(\mathbf{r})]da. \tag{11}$$

Hence using Maxwell's equation and the vector triple product, we obtain

$$\begin{aligned}
&-\frac{4\pi i\omega p}{c}\{[\boldsymbol{G}_e(\mathbf{r}'',\mathbf{r}')]_{ij}-[\boldsymbol{G}_e(\mathbf{r}',\mathbf{r}'')]_{ji}\}\\
&=\oint_S\{\mathbf{H}_2(\mathbf{r})\bullet[\mathbf{n}\times\boldsymbol{G}_{ei}(\mathbf{r},\mathbf{r}'')]-\mathbf{H}_1(\mathbf{r})\bullet[\mathbf{n}\times\boldsymbol{G}_{ej}(\mathbf{r},\mathbf{r}')]\}da\\
&=\frac{\omega}{i}\oint_S\{(\boldsymbol{\mu}^{-1}\bullet\nabla\times\mathbf{E}_2(\mathbf{r}))\bullet[\mathbf{n}\times\boldsymbol{G}_{ei}(\mathbf{r},\mathbf{r}'')]-(\boldsymbol{\mu}^{-1}\bullet\nabla\times\mathbf{E}_1(\mathbf{r}))\bullet[\mathbf{n}\times\boldsymbol{G}_{ej}(\mathbf{r},\mathbf{r}')]\}da\\
&=\frac{\omega p}{i}\oint_S\{(\boldsymbol{\mu}^{-1}\bullet\nabla\times\boldsymbol{G}_{ej}(\mathbf{r},\mathbf{r}'))\bullet[\mathbf{n}\times\boldsymbol{G}_{ei}(\mathbf{r},\mathbf{r}'')]-(\boldsymbol{\mu}^{-1}\bullet\nabla\times\boldsymbol{G}_{ei}(\mathbf{r},\mathbf{r}''))\bullet[\mathbf{n}\times\boldsymbol{G}_{ej}(\mathbf{r},\mathbf{r}')]\}da.
\end{aligned} \tag{12}$$

Hence we have

$$\begin{aligned}
&\frac{4\pi}{c}\{[\boldsymbol{G}_e(\mathbf{r}'',\mathbf{r}')]_{ij}\mathbf{e}_i-[\boldsymbol{G}_e(\mathbf{r}',\mathbf{r}'')]_{ji}\mathbf{e}_i\}\\
&=\oint_S\{[\mathbf{n}\times\boldsymbol{G}_e(\mathbf{r},\mathbf{r}'')]^T\bullet(\boldsymbol{\mu}^{-1}\bullet\nabla\times\boldsymbol{G}_{ej}(\mathbf{r},\mathbf{r}'))-[\boldsymbol{\mu}^{-1}\bullet\nabla\times\boldsymbol{G}_e(\mathbf{r},\mathbf{r}'')]^T\bullet[\mathbf{n}\times\boldsymbol{G}_{ej}(\mathbf{r},\mathbf{r}')]\}da,
\end{aligned} \tag{13}$$

and we therefore obtain in component form:

$$\frac{4\pi}{c}\{[\boldsymbol{G}_e(\mathbf{r}'',\mathbf{r}')]_{ij}\mathbf{e}_i\mathbf{e}_j-[\boldsymbol{G}_e(\mathbf{r}',\mathbf{r}'')]_{ji}\mathbf{e}_i\mathbf{e}_j\}$$
$$=\oint_S\{[\mathbf{n}\times\boldsymbol{G}_e(\mathbf{r},\mathbf{r}'')]^T\bullet(\boldsymbol{\mu}^{-1}\bullet\nabla\times\boldsymbol{G}_e(\mathbf{r},\mathbf{r}'))-[\boldsymbol{\mu}^{-1}\bullet\nabla\times\boldsymbol{G}_e(\mathbf{r},\mathbf{r}'')]^T\bullet[\mathbf{n}\times\boldsymbol{G}_e(\mathbf{r},\mathbf{r}')]\}da. \tag{14}$$

Equation (14) in dyadic form will yield the following result:

$$\frac{4\pi}{c}\{\boldsymbol{G}_e(\mathbf{r}'',\mathbf{r}')-[\boldsymbol{G}_e(\mathbf{r}',\mathbf{r}'')]^T\}$$
$$=\oint_S\{[\mathbf{n}\times\boldsymbol{G}_e(\mathbf{r},\mathbf{r}'')]^T\bullet(\boldsymbol{\mu}^{-1}\bullet\nabla\times\boldsymbol{G}_e(\mathbf{r},\mathbf{r}'))-[\boldsymbol{\mu}^{-1}\bullet\nabla\times\boldsymbol{G}_e(\mathbf{r},\mathbf{r}'')]^T\bullet[\mathbf{n}\times\boldsymbol{G}_e(\mathbf{r},\mathbf{r}')]\}da. \tag{15}$$

By imposing on S either the dyadic Dirichlet condition:[6]

$$\mathbf{n}\times\boldsymbol{G}_e(\mathbf{r},\mathbf{r}')=0, \tag{16}$$

or the dyadic Neumann condition:[6,16]

$$\mathbf{n}\times[\boldsymbol{\mu}^{-1}\bullet\nabla\times\boldsymbol{G}_e(\mathbf{r},\mathbf{r}')]=0, \tag{17}$$

the surface integral in Eq. (15) can be made to vanish by applying the dyadic triple product in the Neumann case.[8] Hence under either one of these boundary conditions, Eq. (15) will lead to the symmetry property of the Green dyadic in (2).

III. EXTENSION TO ANISOTROPIC METAMATERIALS

Having established the necessary equivalence, we shall next focus on the dyadic symmetry equation (2) and examine its validity in the presence of an anisotropic metamaterial with nonlocal electric and magnetic properties. We shall present our results in two steps to establish the conditions leading to the symmetry of the Green dyadic of the problem. As explained above, we shall limit to linear anisotropic and nonlocal responses[17] of the materials and to the case with either Dirichlet or Neumann boundary conditions.

A. Anisotropic local response

First we consider only local response which is simpler and sets the framework for the treatment of the more complicated nonlocal case. Thus we assume the following constitutive relations: $\mathbf{D}(\mathbf{r})=\boldsymbol{\varepsilon}(\mathbf{r})\bullet\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})=\boldsymbol{\mu}(\mathbf{r})\bullet\mathbf{H}(\mathbf{r})$. For fields with harmonic time dependence ($\sim e^{-i\omega t}$), we have

$$\nabla\times\boldsymbol{\mu}^{-1}(\mathbf{r})\bullet\nabla\times\mathbf{E}(\mathbf{r})-\frac{\omega^2}{c^2}\boldsymbol{\varepsilon}(\mathbf{r})\bullet\mathbf{E}(\mathbf{r})=i\omega\frac{4\pi}{c^2}\mathbf{J}(\mathbf{r}), \tag{18}$$

which implies the following differential equation for the electric dyadic of the problem:

$$\nabla\times\boldsymbol{\mu}^{-1}(\mathbf{r})\bullet\nabla\times\boldsymbol{G}_e(\mathbf{r},\mathbf{r}')-\frac{\omega^2}{c^2}\boldsymbol{\varepsilon}(\mathbf{r})\bullet\boldsymbol{G}_e(\mathbf{r},\mathbf{r}')=\frac{4\pi}{c}\boldsymbol{I}\delta(\mathbf{r}-\mathbf{r}'), \tag{19}$$

where $\boldsymbol{I}$ and $\delta(\mathbf{r}-\mathbf{r}')$ denote the unit dyadic and Dirac delta function, respectively.

Using the following dyadic-dyadic formula which is valid under the condition $\boldsymbol{\lambda}_{ij}=\boldsymbol{\lambda}_{ji}$ (see the Appendix):

$$\int \{[\nabla \times \boldsymbol{\lambda} \bullet \nabla \times \mathbf{B}]^T \bullet \mathbf{A} - \mathbf{B}^T \bullet \nabla \times \boldsymbol{\lambda} \bullet \nabla \times \mathbf{A}\} d^3x$$
$$= \oint_S \{[\mathbf{n} \times \mathbf{B}]^T \bullet (\boldsymbol{\lambda} \bullet \nabla \times \mathbf{A}) - [\boldsymbol{\lambda} \bullet \nabla \times \mathbf{B}]^T \bullet \mathbf{n} \times \mathbf{A}\} da, \tag{20}$$

we obtain the following by setting $\mathbf{A}=\boldsymbol{G}_e(\mathbf{r},\mathbf{r}')$, $\mathbf{B}=\boldsymbol{G}_e(\mathbf{r},\mathbf{r}'')$ and $\boldsymbol{\lambda}=\boldsymbol{\mu}^{-1}$:

$$\int \{[\nabla \times \boldsymbol{\mu}^{-1} \bullet \nabla \times \boldsymbol{G}_e(\mathbf{r},\mathbf{r}'')]^T \bullet \boldsymbol{G}_e(\mathbf{r},\mathbf{r}') - [\boldsymbol{G}_e(\mathbf{r},\mathbf{r}'')]^T \bullet \nabla \times \boldsymbol{\mu}^{-1} \bullet \nabla \times \boldsymbol{G}_e(\mathbf{r},\mathbf{r}')\} d^3x$$
$$= \oint_S \{[\mathbf{n} \times \boldsymbol{G}_e(\mathbf{r},\mathbf{r}'')]^T \bullet (\boldsymbol{\mu}^{-1} \bullet \nabla \times \boldsymbol{G}_e(\mathbf{r},\mathbf{r}')) - [\boldsymbol{\mu}^{-1} \bullet \nabla \times \boldsymbol{G}_e(\mathbf{r},\mathbf{r}'')]^T \bullet \mathbf{n} \times \boldsymbol{G}_e(\mathbf{r},\mathbf{r}')\} da. \tag{21}$$

Hence from either the dyadic Dirichlet condition Eq. (16) or the dyadic Neumann condition Eq. (17), Eq. (21) leads to

$$\int \{[\nabla \times \boldsymbol{\mu}^{-1} \bullet \nabla \times \boldsymbol{G}_e(\mathbf{r},\mathbf{r}'')]^T \bullet \boldsymbol{G}_e(\mathbf{r},\mathbf{r}') - [\boldsymbol{G}_e(\mathbf{r},\mathbf{r}'')]^T \bullet \nabla \times \boldsymbol{\mu}^{-1} \bullet \nabla \times \boldsymbol{G}_e(\mathbf{r},\mathbf{r}')\} d^3x = 0. \tag{22}$$

Substituting (19) into (22), we have

$$\int \left\{ \frac{\omega^2}{c^2} [\boldsymbol{\varepsilon}(\mathbf{r}) \bullet \boldsymbol{G}_e(\mathbf{r},\mathbf{r}'')]^T + \frac{4\pi}{c} \boldsymbol{I} \delta(\mathbf{r}-\mathbf{r}'') \right\} \bullet \boldsymbol{G}_e(\mathbf{r},\mathbf{r}') d^3x$$
$$- \int [\boldsymbol{G}_e(\mathbf{r},\mathbf{r}'')]^T \bullet \left\{ \frac{\omega^2}{c^2} \boldsymbol{\varepsilon}(\mathbf{r}) \bullet \boldsymbol{G}_e(\mathbf{r},\mathbf{r}') + \frac{4\pi}{c} \boldsymbol{I} \delta(\mathbf{r}-\mathbf{r}') \right\} d^3x = 0, \tag{23}$$

which implies the symmetry of the Green dyadic:

$$[\boldsymbol{G}_e(\mathbf{r}',\mathbf{r}'')]^T = \boldsymbol{G}_e(\mathbf{r}'',\mathbf{r}'), \tag{24}$$

provided that the dielectric tensor is symmetric: $\boldsymbol{\varepsilon}_{ij}=\boldsymbol{\varepsilon}_{ji}$. We remark that the validity of Eq. (20) has already required $\boldsymbol{\mu}_{ij}=\boldsymbol{\mu}_{ji}$ as explained above. It is worth to note that these symmetry conditions so derived are in general not satisfied in the presence of dissipation in the materials. For a dissipative anisotropic medium, the dielectric and permeability tensors are Hermitian and complex which therefore are in general asymmetric. In such case, the symmetry of the Green dyadics are not guaranteed.[18]

B. Anisotropic nonlocal response

We now consider both electric and magnetic nonlocal responses as follows:

$$\mathbf{D}(\mathbf{r}) = \int \boldsymbol{\varepsilon}(\mathbf{r},\mathbf{r}') \bullet \mathbf{E}(\mathbf{r}') d^3x', \tag{25}$$

$$\mathbf{H}(\mathbf{r}) = \int \boldsymbol{\mu}^{-1}(\mathbf{r},\mathbf{r}') \bullet \mathbf{B}(\mathbf{r}') d^3x', \tag{26}$$

and then generalize the dyadic differential equation in (19) to

$$\nabla\times\int \boldsymbol{\mu}^{-1}(\mathbf{r},\mathbf{r}_1)\cdot\nabla_1\times \boldsymbol{G}_e(\mathbf{r}_1,\mathbf{r}')d^3x_1-\frac{\omega^2}{c^2}\int \boldsymbol{\varepsilon}(\mathbf{r},\mathbf{r}_1)\cdot \boldsymbol{G}_e(\mathbf{r}_1,\mathbf{r}')d^3x_1=\frac{4\pi}{c}\boldsymbol{I}\delta(\mathbf{r}-\mathbf{r}'). \tag{27}$$

Using the corresponding generalization of (20) also under the condition $\boldsymbol{\lambda}_{ij}(\mathbf{r},\mathbf{r}')=\boldsymbol{\lambda}_{ji}(\mathbf{r}',\mathbf{r})$ (see the Appendix):

$$\begin{aligned}&\int d^3x\int d^3x_1\{[\nabla\times\boldsymbol{\lambda}(\mathbf{r},\mathbf{r}_1)\cdot\nabla_1\times\mathbf{B}(\mathbf{r}_1)]^T\cdot\mathbf{A}(\mathbf{r})-[\mathbf{B}(\mathbf{r})]^T\cdot\nabla\times\boldsymbol{\lambda}(\mathbf{r},\mathbf{r}_1)\cdot\nabla_1\times\mathbf{A}(\mathbf{r}_1)\}\\&=\oint_S da\int d^3x_1\{[\mathbf{n}\times\mathbf{B}(\mathbf{r})]^T\cdot[\boldsymbol{\lambda}(\mathbf{r},\mathbf{r}_1)\cdot\nabla_1\times\mathbf{A}(\mathbf{r}_1)]-[\boldsymbol{\lambda}(\mathbf{r},\mathbf{r}_1)\cdot\nabla_1\times\mathbf{B}(\mathbf{r}_1)]^T\cdot\mathbf{n}\times\mathbf{A}(\mathbf{r})\},\end{aligned} \tag{28}$$

we obtain by setting

$$\begin{cases}\mathbf{A}(\mathbf{r})=\boldsymbol{G}_e(\mathbf{r},\mathbf{r}')\\ \mathbf{A}(\mathbf{r}_1)=\boldsymbol{G}_e(\mathbf{r}_1,\mathbf{r}')\end{cases},\quad \begin{cases}\mathbf{B}(\mathbf{r})=\boldsymbol{G}_e(\mathbf{r},\mathbf{r}'')\\ \mathbf{B}(\mathbf{r}_1)=\boldsymbol{G}_e(\mathbf{r}_1,\mathbf{r}'')\end{cases}\quad \text{and}\quad \boldsymbol{\lambda}=\boldsymbol{\mu}^{-1} \tag{29}$$

the following:

$$\begin{aligned}&\int d^3x\int d^3x_1\{[\nabla\times\boldsymbol{\mu}^{-1}(\mathbf{r},\mathbf{r}_1)\cdot\nabla_1\times\boldsymbol{G}_e(\mathbf{r}_1,\mathbf{r}'')]^T\cdot\boldsymbol{G}_e(\mathbf{r},\mathbf{r}')\}\\&-\int d^3x\int d^3x_1\{[\boldsymbol{G}_e(\mathbf{r},\mathbf{r}'')]^T\cdot\nabla\times\boldsymbol{\mu}^{-1}(\mathbf{r},\mathbf{r}_1)\cdot\nabla_1\times\boldsymbol{G}_e(\mathbf{r}_1,\mathbf{r}')\}\\&=\oint_S da\int d^3x_1\{[\mathbf{n}\times\boldsymbol{G}_e(\mathbf{r},\mathbf{r}'')]^T\cdot[\boldsymbol{\mu}^{-1}(\mathbf{r},\mathbf{r}_1)\cdot\nabla_1\times\boldsymbol{G}_e(\mathbf{r}_1,\mathbf{r}')]\}\\&-\oint_S da\int d^3x_1\{[\boldsymbol{\mu}^{-1}(\mathbf{r},\mathbf{r}_1)\cdot\nabla_1\times\boldsymbol{G}_e(\mathbf{r}_1,\mathbf{r}'')]^T\cdot\mathbf{n}\times\boldsymbol{G}_e(\mathbf{r},\mathbf{r}')\}.\end{aligned} \tag{30}$$

Again, with either the dyadic Dirichlet or the dyadic Neumann condition,[17] we obtain from (30):

$$\begin{aligned}&\int d^3x\int d^3x_1\{[\nabla\times\boldsymbol{\mu}^{-1}(\mathbf{r},\mathbf{r}_1)\cdot\nabla_1\times\boldsymbol{G}_e(\mathbf{r}_1,\mathbf{r}'')]^T\cdot\boldsymbol{G}_e(\mathbf{r},\mathbf{r}')\}\\&-\int d^3x\int d^3x_1\{[\boldsymbol{G}_e(\mathbf{r},\mathbf{r}'')]^T\cdot\nabla\times\boldsymbol{\mu}^{-1}(\mathbf{r},\mathbf{r}_1)\cdot\nabla_1\times\boldsymbol{G}_e(\mathbf{r}_1,\mathbf{r}')\}=0.\end{aligned} \tag{31}$$

Substituting (27) into (31), we have

$$\begin{aligned}&\int\left[\frac{\omega^2}{c^2}\int\boldsymbol{\varepsilon}(\mathbf{r},\mathbf{r}_1)\cdot\boldsymbol{G}_e(\mathbf{r}_1,\mathbf{r}'')d^3x_1+\frac{4\pi}{c}\boldsymbol{I}\delta(\mathbf{r}-\mathbf{r}'')\right]^T\cdot\boldsymbol{G}_e(\mathbf{r},\mathbf{r}')d^3x\\&=\int[\boldsymbol{G}_e(\mathbf{r},\mathbf{r}'')]^T\cdot\left\{\frac{\omega^2}{c^2}\int\boldsymbol{\varepsilon}(\mathbf{r},\mathbf{r}_1)\cdot\boldsymbol{G}_e(\mathbf{r}_1,\mathbf{r}')d^3x_1+\frac{4\pi}{c}\boldsymbol{I}\delta(\mathbf{r}-\mathbf{r}')\right\}d^3x,\end{aligned} \tag{32}$$

which again implies the symmetry of the Green dyadic:

$$[\boldsymbol{G}_e(\mathbf{r}',\mathbf{r}'')]^T=\boldsymbol{G}_e(\mathbf{r}'',\mathbf{r}'), \tag{33}$$

in a way similar to the above case for local response.[8]

IV. CONCLUSION

In this work, we have proven explicitly the equivalence between two popular statements for the optical reciprocity principle, namely, the Lorentz lemma and the symmetry property of the Green dyadic for the vector wave equation. We have further generalized this principle to include both anisotropic nonlocal electric and magnetic responses with arbitrary finite boundaries. We found explicitly that the dielectric tensors must satisfy the following symmetry properties:

$$\boldsymbol{\varepsilon}_{ij}(\mathbf{r},\mathbf{r}') = \boldsymbol{\varepsilon}_{ji}(\mathbf{r}',\mathbf{r}),$$

$$\boldsymbol{\mu}_{ij}(\mathbf{r},\mathbf{r}') = \boldsymbol{\mu}_{ji}(\mathbf{r}',\mathbf{r}), \tag{34}$$

in order for reciprocity to hold. These results reduce to the well-known conditions in the case of local responses. Note that while the symmetry in $\mathbf{r}$ and $\mathbf{r}'$ will be valid for most materials on a macroscopic scale,[19] that in the tensorial indices will not be valid in general for complex materials such as bianisotropic or chiral materials.[18,20] Hence it will be of interest to design some optical experiments to observe the breakdown of reciprocity symmetry with these systems in the study of anisotropic metamaterials. One possible way is to observe transmission asymmetry in the light propagating through these materials. As is well known, interesting applications can be developed based on both the validity and the breakdown of the reciprocity principle.[1] Our results established in this paper will thus provide some guidelines for the application of this principle to the exciting field in the optical studies of anisotropic metamaterials.

ACKNOWLEDGMENTS

The authors are grateful for the research support from the National Science Council of Taiwan, Republic of China, under Project Nos. NSC-97-2120-M-002-013, NSC-96-2923-M-002-002-MY3, and NSC 97-2811-M-002-018, respectively, and to the National Taiwan University and National Center for Theoretical Sciences, Taipei Office. P. T. Leung would like to thank the additional support from a Fulbright Scholarship.

APPENDIX: PROOF OF TWO NEW DYADIC INTEGRAL FORMULA

Here we provide a rigorous derivation of both Eqs. (20) and (28). For the case with anisotropic local response [Eq. (20)], let us first establish the following simpler vector identity:

$$\nabla \cdot [\mathbf{B} \times \boldsymbol{\lambda} \cdot \nabla \times \mathbf{A} - \mathbf{A} \times \boldsymbol{\lambda} \cdot \nabla \times \mathbf{B}] = \mathbf{A} \cdot \nabla \times \boldsymbol{\lambda} \cdot \nabla \times \mathbf{B} - \mathbf{B} \cdot \nabla \times \boldsymbol{\lambda} \cdot \nabla \times \mathbf{A}, \tag{A1}$$

under the condition $\boldsymbol{\lambda}_{ij}=\boldsymbol{\lambda}_{ji}$. In explicit Einstein summation convention, we have for the left-hand side of (A1):

$$\begin{aligned}\nabla \cdot [\mathbf{B} \times \boldsymbol{\lambda} \cdot \nabla \times \mathbf{A} - \mathbf{A} \times \boldsymbol{\lambda} \cdot \nabla \times \mathbf{B}]\\ = \varepsilon_{ijk}\varepsilon_{lmn}(B_j\partial_i\boldsymbol{\lambda}_{kl}\partial_m A_n - A_j\partial_i\boldsymbol{\lambda}_{kl}\partial_m B_n) + \varepsilon_{ijk}\varepsilon_{lmn}\boldsymbol{\lambda}_{kl}[(\partial_i B_j)(\partial_m A_n) - (\partial_i A_j)(\partial_m B_n)],\end{aligned} \tag{A2}$$

and the right-hand side of (A1):

$$\mathbf{A} \cdot \nabla \times \boldsymbol{\lambda} \cdot \nabla \times \mathbf{B} - \mathbf{B} \cdot \nabla \times \boldsymbol{\lambda} \cdot \nabla \times \mathbf{A} = \varepsilon_{ijk}\varepsilon_{lmn}(A_i\partial_j\boldsymbol{\lambda}_{kl}\partial_m B_n - B_i\partial_j\boldsymbol{\lambda}_{kl}\partial_m A_n). \tag{A3}$$

Thus (A2) and (A3) are equal under the condition $\boldsymbol{\lambda}_{ij}=\boldsymbol{\lambda}_{ji}$ and thus (A1) is established. From here and by following the method in Ref. 21, it is straightforward to show that (A1) with the application of the divergence theorem will lead to the result given in Eq. (20) with the ranks of **A** and **B** raised by 1 to assume dyadic forms.

Next for the case with anisotropic nonlocal response, the proof is similar with the establishment of the following slightly more complicated identity:

$$\int d^3x \int d^3x_1 \nabla \cdot [\mathbf{B}(\mathbf{r}) \times \boldsymbol{\lambda}(\mathbf{r},\mathbf{r}_1) \cdot \nabla_1 \times \mathbf{A}(\mathbf{r}_1) - \mathbf{A}(\mathbf{r}) \times \boldsymbol{\lambda}(\mathbf{r},\mathbf{r}_1) \cdot \nabla_1 \times \mathbf{B}(\mathbf{r}_1)]$$
$$= \int d^3x \int d^3x_1 [\mathbf{A}(\mathbf{r}) \cdot \nabla \times \boldsymbol{\lambda}(\mathbf{r},\mathbf{r}_1) \cdot \nabla_1 \times \mathbf{B}(\mathbf{r}_1) - \mathbf{B}(\mathbf{r}) \cdot \nabla \times \boldsymbol{\lambda}(\mathbf{r},\mathbf{r}_1) \cdot \nabla_1 \times \mathbf{A}(\mathbf{r}_1)], \tag{A4}$$

under the condition $\boldsymbol{\lambda}_{ij}(\mathbf{r},\mathbf{r}')=\boldsymbol{\lambda}_{ji}(\mathbf{r}',\mathbf{r})$. Again we express the left side as

$$\int d^3x \int d^3x_1 \, \nabla \cdot [\mathbf{B}(\mathbf{r}) \times \boldsymbol{\lambda}(\mathbf{r},\mathbf{r}_1) \cdot \nabla_1 \times \mathbf{A}(\mathbf{r}_1) - \mathbf{A}(\mathbf{r}) \times \boldsymbol{\lambda}(\mathbf{r},\mathbf{r}_1) \cdot \nabla_1 \times \mathbf{B}(\mathbf{r}_1)]$$
$$= \varepsilon_{ijk}\varepsilon_{lmn} \int d^3x \int d^3x_1 \{[B_j(\mathbf{r}) \partial_i^{\mathbf{r}} \boldsymbol{\lambda}_{kl}(\mathbf{r},\mathbf{r}_1) \partial_m^{\mathbf{r}_1} A_n(\mathbf{r}_1) - A_j(\mathbf{r}) \partial_i^{\mathbf{r}} \boldsymbol{\lambda}_{kl}(\mathbf{r},\mathbf{r}_1) \partial_m^{\mathbf{r}_1} B_n(\mathbf{r}_1)]\}$$
$$+ \varepsilon_{ijk}\varepsilon_{lmn} \int d^3x \int d^3x_1 \boldsymbol{\lambda}_{kl}(\mathbf{r},\mathbf{r}_1) \{[\partial_i^{\mathbf{r}} B_j(\mathbf{r})][\partial_m^{\mathbf{r}_1} A_n(\mathbf{r}_1)] - [\partial_i^{\mathbf{r}} A_j(\mathbf{r})][\partial_m^{\mathbf{r}_1} B_n(\mathbf{r}_1)]\}, \tag{A5}$$

and the right side as

$$\int d^3x \int d^3x_1 [\mathbf{A}(\mathbf{r}) \cdot \nabla \times \boldsymbol{\lambda}(\mathbf{r},\mathbf{r}_1) \cdot \nabla_1 \times \mathbf{B}(\mathbf{r}_1) - \mathbf{B}(\mathbf{r}) \cdot \nabla \times \boldsymbol{\lambda}(\mathbf{r},\mathbf{r}_1) \cdot \nabla_1 \times \mathbf{A}(\mathbf{r}_1)]$$
$$= \varepsilon_{ijk}\varepsilon_{lmn} \int d^3x \int d^3x_1 [A_i(\mathbf{r}) \partial_j^{\mathbf{r}} \boldsymbol{\lambda}_{kl}(\mathbf{r},\mathbf{r}_1) \partial_m^{\mathbf{r}_1} B_n(\mathbf{r}_1) - B_i(\mathbf{r}) \partial_j^{\mathbf{r}} \boldsymbol{\lambda}_{kl}(\mathbf{r},\mathbf{r}_1) \partial_m^{\mathbf{r}_1} A_n(\mathbf{r}_1)]. \tag{A6}$$

Hence (A5) is equal to (A6) by imposing $\boldsymbol{\lambda}_{ij}(\mathbf{r},\mathbf{r}')=\boldsymbol{\lambda}_{ji}(\mathbf{r}',\mathbf{r})$ and the result in Eq. (28) can again be obtained by the same method as in the local case by following Ref. 21.

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$$H_{ij} = \left\{ \partial_i^r \kappa(\mathbf{r},\omega) \partial_j^r - \left[\partial_l^r \kappa(\mathbf{r},\omega) \partial_l^r + \frac{\omega^2}{c^2} \varepsilon(\mathbf{r},\omega) \right] \delta_{ij} \right\} \delta(\mathbf{r}-\mathbf{r}').$$

Thus we believe the proof in this paper is questionable.
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[16] Note that the condition introduced in Eq. (17) is a generalization of that used in Ref. 6 with the explicit inclusion of the magnetic permeability tensor.
[17] Note that the proof of the equivalence between the two versions of the reciprocity principle in Sec. II remains valid for the case with nonlocal response, with Eq. (12) generalized to the following form:

$$-\frac{4\pi i\omega p}{c}\{[\boldsymbol{G}_e(\mathbf{r}'',\mathbf{r}')]_{ij}-[\boldsymbol{G}_e(\mathbf{r}',\mathbf{r}'')]_{ji}\}=\frac{\omega p}{i}\oint_S da\int d^3x_1(\boldsymbol{\mu}^{-1}(\mathbf{r},\mathbf{r}_1)\bullet\boldsymbol{\nabla}_1\times\boldsymbol{G}_{ej}(\mathbf{r}_1,\mathbf{r}'))\bullet[\mathbf{n}\times\boldsymbol{G}_{ei}(\mathbf{r},\mathbf{r}'')]$$
$$-\frac{\omega p}{i}\oint_S da\int d^3x_1(\boldsymbol{\mu}^{-1}(\mathbf{r},\mathbf{r}_1)\bullet\boldsymbol{\nabla}_1\times\boldsymbol{G}_{ei}(\mathbf{r}_1,\mathbf{r}''))\bullet[\mathbf{n}\times\boldsymbol{G}_{ej}(\mathbf{r},\mathbf{r}')],$$

and the Neumann condition in Eq. (17) in the nonlocal case has also to be generalized to the following form: $\int\mathbf{n}\times[\boldsymbol{\mu}^{-1}(\mathbf{r},\mathbf{r}_1)\bullet\boldsymbol{\nabla}_1\times\boldsymbol{G}_e(\mathbf{r}_1,\mathbf{r}')]d^3x_1=0$.

[18] We thank an anonymous referee for the suggestion to include this clarification for the symmetry condition of the dielectric and permeability tensors.

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