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Singular behaviour of the electrodynamic fields of an oscillating dipole

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Abstract

The singularity of the exact electromagnetic fields is derived to include the 'source terms' for harmonically oscillating electric (and magnetic) dipoles, so that the fields will be consistent with the full Maxwell equations with a source. It is shown explicitly, as somewhat expected, that the same δ-function terms for the case of static dipole fields will have to be included in the dynamic case for this purpose, while the associated magnetic (electric) field will remain unaffected by these singular source terms.

Introduction

For the electrostatic and magnetostatic dipole fields, it is well known that a term with the Dirac δ-function has to be included explicitly to make them consistent with Gauss's law with (and without) a source, so that the correct field expressions (in Gaussian units) take the following form [1]:

$$\vec{E}^{(2)} = \frac{3(\vec{p}\cdot\vec{r})\vec{r} - r^2\vec{p}}{r^5} - \frac{4\pi}{3}\vec{p}\delta(\vec{r}), \tag{1}$$

$$\vec{B}^{(2)} = \frac{3(\vec{m}\cdot\vec{r})\vec{r} - r^2\vec{m}}{r^5} + \frac{8\pi}{3}\vec{m}\delta(\vec{r}). \tag{2}$$

In a recent publication [2], we have clarified that the δ-function term in (1) guarantees $\vec{E}^{(2)}$ to satisfy the correct Gauss's law with a source in the form

$$\vec{\nabla}\cdot\vec{E}^{(2)} = 4\pi\rho, \tag{3}$$

with a singular effective charge distribution given by the following expression [1]:

$$\rho(\vec{r}) = -\vec{p}\cdot\vec{\nabla}\delta(\vec{r}), \tag{4}$$

while that in (2) guarantees $\vec{\nabla}\cdot\vec{B}^{(2)} = 0$.

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In this paper, we further show that the same singular terms have to be included in the exact dynamical fields for an oscillating dipole, so that consistency with Maxwell's equations and various field integrals [1] are preserved. In addition, we shall see that the associated components of the fields—i.e. the magnetic field for a radiating electric dipole and the electric field radiated by a magnetic dipole—will remain unaffected by these additional source terms.

While the subject of singular electromagnetic fields and sources has received much attention recently [3], such singular behaviour for an oscillating electric dipole has been obtained previously in the literature based on the singularity of Green's function for the Helmholtz equation [4]. Here we derived the singularities for the fields from *both* electric and magnetic oscillating dipoles, following perhaps a slightly more pedagogical approach without resorting to Green's function of the dynamic equations. In addition, a general pattern will emerge from our results, leading to an implication for the singular behaviours of all the higher multipole radiation fields.

Fields from an oscillating electric dipole

Let us begin by recapitulating the well-known results for a harmonically radiating electric dipole located in vacuum at the origin. In the Lorentz gauge, exact results can be obtained for the vector potential and fields as follows [1]:

$$\vec{A} = -\mathrm{i}k\vec{p}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}, \tag{5}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = k^2(\hat{n} \times \vec{p})\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\left(1 - \frac{1}{\mathrm{i}kr}\right), \tag{6}$$

$$\begin{aligned}\vec{E} &= \frac{i}{k}\vec{\nabla} \times \vec{B} - \frac{4\pi\mathrm{i}}{\omega}\vec{J} \\ &= \frac{i}{k}\vec{\nabla} \times \vec{B} \\ &= [3\hat{n}(\hat{n} \cdot \vec{p}) - \vec{p}]\left(\frac{1}{r^3} - \frac{\mathrm{i}k}{r^2}\right)\mathrm{e}^{\mathrm{i}kr} + k^2(\hat{n} \times \vec{p}) \times \hat{n}\frac{\mathrm{e}^{\mathrm{i}kr}}{r},\end{aligned} \tag{7}$$

where $\hat{n} \equiv \vec{r}/r$, and we have ignored the source current in the last two lines in (7) assuming that the origin is excluded from the observation location. This will make the electric field necessarily divergenceless and hence cannot satisfy the full Maxwell equations with a source if the origin were included. Because of this, the conflict with the electric field integral will arise just like in the case of the electrostatic dipole [1]. To remedy this deficiency, one has to (i) include the source term and (ii) employ various differentiation identities for the potential function $(1/r)$ involving the δ-function as established in the literature [5].

To begin, we first note that the various results established in [5] will not lead to any modification of the result in (6) since with (5) into (6), one will end up with only a first-order derivative of the potential function [5]. In other words, the result in (6) automatically satisfies the divergenceless requirement of the magnetic field, without further modifications needed as is in the case of the magnetostatic dipole field [1, 2]. Note that this is the magnetic field associated with the electric dipole radiation and *not* the field from the magnetic dipole radiation which would otherwise require modifications using the δ-function (see below).

Next, we come to the modifications of the result in (7) by including the source and singular terms carefully. To this end, let us consider a component of the electric field and obtain the following from (7):

$$E_\ell = \frac{i}{k}(\vec{\nabla} \times \vec{\nabla} \times \vec{A})_\ell - \frac{4\pi \mathrm{i}}{\omega} J_\ell. \tag{8}$$

Using the result in (5), we obtain

$$\begin{aligned}\frac{i}{k}(\vec{\nabla} \times \vec{\nabla} \times \vec{A})_\ell &= \varepsilon_{\ell mn}\partial_m \left[\varepsilon_{nqs}\partial_q \left(p_s \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right)\right] \\ &= (\delta_{\ell q}\delta_{ms} - \delta_{\ell s}\delta_{mq}) p_s \partial_m \partial_q \frac{\mathrm{e}^{\mathrm{i}kr}}{r} \\ &= p_m \partial_m \partial_\ell \frac{\mathrm{e}^{\mathrm{i}kr}}{r} - p_\ell \nabla^2 \frac{\mathrm{e}^{\mathrm{i}kr}}{r}.\end{aligned} \tag{9}$$

By carrying out the differentiation in (9) and using the appropriate identities in [5], one obtains the following two δ-function terms:

$$\partial_m \partial_\ell \frac{1}{r} \sim -\frac{4\pi}{3}\delta_{m\ell}\delta(\vec{r}) + \cdots, \tag{10}$$

$$\nabla^2 \frac{1}{r} = -4\pi\delta(\vec{r}). \tag{11}$$

Before moving forward, we remark here that it is straightforward (though a little tedious) to show that the rest of the results in the differentiation of the term $\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right)$ in (9), excluding the two δ-function terms in (10) and (11), simply reproduces the well-known result in the last row of (7). To continue, we shall now show that the singular term in (11), when being plugged into (9) and then into (8), will turn out to be exactly cancelled by the term from the source current in (8). To demonstrate explicitly this cancellation, we first recall the following result for harmonic sources [1]:

$$\int \vec{J} d^3x = -\mathrm{i}\omega\vec{p}, \tag{12}$$

which implies

$$\vec{J}(\vec{r}) = -\mathrm{i}\omega\vec{p}\delta(\vec{r}). \tag{13}$$

Since the particular contribution to the electric field obtained from substituting (11) into (9) will appear in the form $4\pi\,\vec{p}\delta(\vec{r})\,\mathrm{e}^{\mathrm{i}kr} \sim 4\pi\,\vec{p}\delta(\vec{r})$,[2] it will be exactly cancelled by the source term when being plugged into (8) with the current given as in (13). Hence, by collecting all the results, we obtain the following exact dipole radiating field [4] with the same explicit singular term as in the case of a static field:

$$\vec{E} = -\frac{4\pi}{3}\vec{p}\delta(\vec{r}) + [3\hat{n}(\hat{n}\cdot\vec{p}) - \vec{p}]\left(\frac{1}{r^3} - \frac{\mathrm{i}k}{r^2}\right)\mathrm{e}^{\mathrm{i}kr} + k^2(\hat{n}\times\vec{p})\times\hat{n}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}. \tag{14}$$

Note that it is straightforward to show that in the near zone, (14) simply reduces back to (1). In addition, (14) will now consistently satisfy Maxwell's equations with the radiating dipole as the source and hence the corresponding field integral. In particular, one can easily show that (14) does satisfy Gauss's law as displayed in (3) and (4). This is not surprising since from the equation of continuity [1] and the result in (13), one can easily show that the 'equivalent scalar charge density' for a harmonically radiating dipole is still given by the same expression as that for the case of a static dipole field, i.e. the result in (4).

[2] Note that $\delta(\vec{r})\,\mathrm{e}^{\mathrm{i}kr} \sim \delta(r)\,\mathrm{e}^{\mathrm{i}kr} \sim \delta(r)\,\mathrm{e}^{\mathrm{i}k\cdot 0} \sim \delta(r) \sim \delta(\vec{r})$.

Fields from an oscillating magnetic dipole

For a harmonically radiating magnetic dipole, the vector potential (applicable for all field zones) can be obtained in the following form [1]:

$$\vec{A}_m = \mathrm{i}k\frac{\mathrm{e}^{\mathrm{i}kr}}{r}(\hat{n}\times\vec{m})\left(1-\frac{1}{\mathrm{i}kr}\right), \tag{15}$$

which is seen to have the same functional form in spatial coordinates as the magnetic field in (6) has. Hence, the calculation of the magnetic field from (15) is almost identical to that of (7) from (6), *except that no source current is ever involved in the present step*. Thus by analogy to the steps as in (8)–(11) with the current term in (8) removed, one obtains the following result:

$$\begin{aligned}\vec{B}_m &= \vec{\nabla}\times\vec{A}_m \\ &= \frac{8\pi}{3}\vec{m}\delta(\vec{r}) + [3\hat{n}(\hat{n}\cdot\vec{m})-\vec{m}]\left(\frac{1}{r^3}-\frac{\mathrm{i}k}{r^2}\right)\mathrm{e}^{\mathrm{i}kr} + k^2(\hat{n}\times\vec{m})\times\hat{n}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}.\end{aligned} \tag{16}$$

This result surely meets all expectations such as $\vec{\nabla}\cdot\vec{B}_m = 0$ (and hence the field integral requirement [1]) as well as the correct near-field limit given in (2). To calculate the corresponding electric field with possible additional singular terms, it will be very complicated to start with Ampere's law as done in (8). Instead, we resort to Faraday's law which yields

$$\vec{B}_m = \frac{1}{\mathrm{i}k}\vec{\nabla}\times\vec{E}_m. \tag{17}$$

Hence by comparing the results in (16) and (17), one obtains the following result for the electric field:

$$\vec{E}_m = \mathrm{i}k(\vec{A}_m + \vec{\nabla}\Lambda) \equiv \mathrm{i}k\vec{A}_m + \vec{\varepsilon}, \tag{18}$$

where $-\mathrm{i}k\Lambda$ can be regarded as a kind of a scalar potential term for the problem. In order to determine the field $\vec{\varepsilon}$ associated with Λ, we first recall the equivalent current for the magnetic dipole source and then apply Gauss's law. Thus, from

$$\vec{m} = \int \vec{M}d^3x = \frac{1}{2c}\int \vec{r}\times\vec{J}d^3x, \tag{19}$$

one obtains, with $\vec{M} = \vec{m}\delta(\vec{r})$ [1][3],

$$\vec{J} = c\vec{\nabla}\times\vec{M} = c\vec{m}\times\vec{\nabla}\delta(\vec{r}). \tag{20}$$

Hence from the equation of continuity for a harmonic source, we obtain

$$\rho = \frac{1}{\mathrm{i}\omega}\vec{\nabla}\cdot\vec{J} = 0. \tag{21}$$

As a consequence, we deduce from Gauss's law that $\vec{\nabla}\cdot\vec{E}_m = 0$. Since as pointed out above that $\vec{A}_m$ has the same form as the magnetic field in (6) and therefore it must be divergenceless, we conclude that there is no need to add any additional source term $\vec{\varepsilon}$ in (18), and by using the result in (15), the corresponding electric field for an oscillating magnetic dipole will simply be given by the usual expression [1]

$$\vec{E}_m = \mathrm{i}k\vec{A}_m = -k^2\frac{\mathrm{e}^{\mathrm{i}kr}}{r}(\hat{n}\times\vec{m})\left(1-\frac{1}{\mathrm{i}kr}\right). \tag{22}$$

Although it is clear that (22) is the correct associated electric field from our consideration above using Faraday's law, it will be of interest to show explicitly that the results in (16), (20) and (22) do satisfy the appropriate Maxwell–Ampere law as required.

[3] We thank Professor G J Ni for a clarification of this point.

Discussion and conclusion

We believe that the results obtained in (14) and (16) are of interest for a course in electrodynamics which clarifies the singular behaviour of the exact E1 and M1 radiation fields. We have not seen this being discussed previously in standard textbooks of electromagnetism. As a conjecture, we speculate that the singular behaviour of any exact dynamical multipole field will be determined solely by its near-field component. Moreover, the other associated component field (i.e. the $\vec{B}$ field of the electric multipoles and the $\vec{E}$ field of the magnetic multipoles) will in general remain unaffected since the specific type of source associated with a particular type of multipole will not lead to source terms for the other type of (associated) field. Although this is not too surprising, an explicit demonstration of the result for higher order multipole fields (both electric and magnetic) will be of interest, as is done here for the dipole case.

Aside from being consistent with Maxwell's equations, it is also of interest to explore the possible applications of these various singular terms in the field expressions. For example, it will be intriguing to look for a significant application of the results for the electric dipole field in (1) and in (14) to real physical problems, such as that for the singular magnetostatic field which finds prominent significance in the understanding of the hyperfine interaction of the ground state of the H atom [1]. A possibility will be in the study of the quantum mechanics of the charge–dipole interaction (i.e. via the inversed square potential), a problem which has been intensively studied in recent literature [6].

In addition, the explicit inclusion of the singular source terms in the field expressions may also find relevant applications in certain numerical methods for electromagnetic field computations[4]. For example, in the 'method of moments', a certain arbitrary size (diameter) for the wire elements has to be introduced into the algorithm to avoid the 'blow-up' of the self-fields of these elements [7]. With the explicit inclusion of the singular terms derived in the present work[5], it may become unnecessary for this arbitrary parameter (wire diameter) to be introduced and the numerical algorithm may be more rigorously implemented. A future demonstration of this possibility will be of interest in the area of numerical electromagnetism.

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[4] We thank an anonymous referee for suggesting this possibility.
[5] We assume that the wire elements are small enough to be approximated by dipolar currents. Otherwise it will be necessary to derive the corresponding singular terms for higher multipole fields.