About Non-Spherically Symmetric Deformations of an Incompressible Neo-Hookean Sphere*

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Summary

A class of non-spherically symmetric deformations of a neo-Hookean incompressible elastic ball is considered. It is shown that the only possible solutions to the equilibrium equations are the trivial solution, the cavitated radially symmetric solution and the deformation of radial inflation and polar stretching. These are the same solutions as found by Polignone-Warne and Warne [6] for a smaller class of deformations. This fact shows once again that the radial deformations are the only deformations, at least within the class considered, which may support a formation of a cavity in the center of an incompressible, isotropic, elastic sphere.

1 Introduction

The phenomenon of cavitation in nonlinear elastic solid has been the subject of quite extensive theoretical research in recent years mainly due to the celebrated work of Ball [1]. In this pioneering paper Ball studied a class of singular solutions of the equations of nonlinear elasticity. He showed, among other things, that considering the class of radially symmetric deformations of a homogeneous, isotropic, incompressible solid ball subjected to a dead-load traction at the boundary a traction-free spherical cavity bifurcates from the undeformed configuration at the sufficiently large tensile loads.

The effects of material inhomogeneity on formation of cavities were investigated within the realm of the spherically symmetric deformations by Sivaloganathan [2] and Horgan and Pence [3] (see also Horgan and Polignone [4]). In both cases the distribution of inhomogeneities was assumed to be radially symmetric whether varying smoothly or non-smoothly, respectively.

Very little work, however, has been done on non-spherically symmetric deformations. Somewhat relevant for what is being presented in this note is the work of Abeyaratne and Hou [5] where it was shown how the radially symmetric cavitated deformation may become unstable relative to asymmetric non-singular deformations under symmetric dead-load conditions. The authors show that the trivial solution may

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bifurcates not only to a cavitated deformation but also to an asymmetric deformation carring the sphere into an ellipsoid. Moreover, if there is a stable cavitated deformation, then necessarily there exists a stable asymmetric deformation with less energy. The results of [5] may be viewed as partial answer to the question whether the radial solutions with holes are prefered (minimize the elastic strain-energy) when non-radial deformations are allowed to compete. In fact, the results of [5] suggest that the appearance of a spherical cavity is unlikely under dead-load conditions.

In this paper, motivated partially by the work of Abeyaratne and Hou [5], we consider a broad class of not necessarily spherically symmetric smooth deformations of an incompressible, homogeneous neo-Hookean sphere. The class we consider here includes, as a special case, the spherically symmetric deformations but it is different from the class considered in [5]. We try to determine if there are any truly non-spherically symmetric deformations (stable or not) which may support a formation of a cavity. This tedious but relatively simple exercise presented here is an extension of the work done by Polignone-Warne and Warne [6]. The class of deformations we consider here is larger than the class already considered in [6]. Adopting the methodology of [6], in particular, as far as the use of the nominal stress tensor is concerned[1], we show that the results obtained by Polignone-Warne & Warne hold also for this broader class of possibly non-spherically symmetric deformations. That is, the only deformations which can be sustained by the sphere, while no body forces are present, are the radial deformations of Ball [1] (trivial or cavitated) and the deformation of radial inflation coupled with a (linear) polar stretching.

The paper is divided into two sections. In the first part we show the role of the constraint of incompressibility in imposing restrictions on the class of solutions while in the second part we discuss the derivation of the governing equations and the solution of these equations for a neo-Hookean material.

2 The Incompressibility Constraint

Let us consider an incompressible nonlinearly elastic solid sphere B. Our goal is to investigate the existence of non-symmetric (spherically) deformations which can be sustained by the sphere B with no body forces present. We consider, therefore, a class of sufficiently (for our purposes) smooth deformations of the form

$$r = f(R,\Theta)R, \;\; \theta = g(R,\Theta), \;\; \phi = \Phi. \quad (1)$$

where we have employed spherical coordinates in both the undeformed (references) and the deformed configurations and where it is as usually assumed that $0 \leq R < 1$, $0 \leq \Theta < 2\pi$, and $0 \leq \Phi \leq \pi$. Thus, the deformation gradient of the deformation (1) when referred to the spherically coordinates R, Θ, Φ has the following form (cf., Ogden [7])

$$\mathbf{F} = \begin{pmatrix} f + Rf_{;R} & \frac{f_{;\Theta}}{\sin\Phi} & 0 \\ Rfg_{;R}\sin\Phi & fg_{;\Theta} & 0 \\ 0 & 0 & f \end{pmatrix} \quad (2)$$

[1] The results presented in this note were originally obtained using the Cauchy stress tensor. This presentation has been purposely adopted to make use of the nominal stress tensor to conform with the presentation in [6]. Indeed, as remarked therein, the use of the nominal stress tensor seems to be more appropriate in this context; it simplifies derivations significantly.

The semicolon indicates here partial differentiation.

As our sphere is assumed to be (locally) incompressible the choice of functions f and g is, in general, not completely arbitrary. Namely, they must satisfy the constraint of incompressibility

$$J \equiv \det \mathbf{F} = 1 \qquad (3)$$

which for our class of deformation gradients (2) takes the form

$$f^3 g_{;\Theta} + R f^2 (g_{;\Theta} f_{;R} - g_{;R} f_{;\Theta}) = 1. \qquad (4)$$

This form of the incompressibility constraint is not transparent enough in providing any significant information about our class of deformations (1), and one is forced to resort to the equilibrium equations. However, in some special cases, which we investigate below, equation (4) alone can be quite helpful in producing significant limitations on the class of deformations allowed for it leads to a solvable quasi-linear partial differential equation.

(a) Suppose that $g_{;R} = 0$, and $g_{;\Theta} \neq 0$, that is $\theta = g(\Theta)$. Then, equation (4) reduces to

$$R g_{;\Theta} (f^3)_{;R} + 3 f^3 g_{;\Theta} - 3 = 0 \qquad (5)$$

which can be easily integrated to show that

$$f(R, \Theta) = \left(\frac{1}{g_{;\Theta}} - \frac{A^3(\Theta)}{R^3}\right)^{\frac{1}{3}} \qquad (6)$$

(see also Polignone-Warne and Warne [6]), where $A(\Theta)$, which for obvious reasons may be nonnegative, is still an undetermined function. Notice that should $A(\Theta)$ be allowed to be nonzero this class of deformations would support a cavity formation. Such a cavity might even be non-spherical due to the dependence of A on the polar angle Θ.

(b) Assume that $f_{;\Theta} = 0$. Then, from equation (4) one obtains that $\theta = g(\Theta) \equiv \Theta$. This in turn reduces it to

$$(f^3)_{;R} R + 3 f^3 - 3 = 0 \qquad (7)$$

the solution of which is

$$f(R) = \left(1 + \frac{A^3}{R^3}\right)^{\frac{1}{3}}. \qquad (8)$$

From (1) one then gets that

$$r(R) = (R^3 + A^3)^{\frac{1}{3}} \qquad (9)$$

where A is a nonnegative constant. These are the radially symmetric deformations allowing, for $A > 0$, a spherical cavity to form at some critical dead-load tension applied at the boundary of the sphere B (cf., Ball [1]).

(c) In the case when $g_{;\Theta} = 0$ the incompressibility constraint (4) takes the form

$$R f^2 f_{;\Theta} g_{;R} = -3. \qquad (10)$$

It is obvious that in order to satisfy this constraint the deformation (1) would have to be an eversion. Hence, we postulate that $g_{;\Theta} \neq 0$ to rule out this type of deformations. In fact, we assume that $g_{;\Theta} > 0$.

3 Deformations of a Neo-Hookean Sphere

Consider a homogeneous isotropic nonlinearly elastic solid ball. Thus, as it is well known, there exists a symmetric function $\phi : I\!R^3_{++} \rightarrow I\!R$; $I\!R^3_{++} \equiv \{(c_1, c_2, c_3) \in I\!R^3; \quad c_i > 0\}$ such that the density (per unit undeformed volume) of the strain-energy function

$$\mathrm{W}(\mathbf{F}) = \phi(\nu_1, \nu_2, \nu_3) \tag{11}$$

where ν_1, ν_2, ν_3 are the eigenvalues of the root of the right Cauchy-Green tensor $\mathbf{B} \equiv \mathbf{F}\mathbf{F}^T$. If we restrict our choice even more and limit ourselves to the neo-Hookean materials only then

$$\mathrm{W}(\mathbf{F}) = \frac{\mu}{2}(\nu_1^2 + \nu_2^2 + \nu_3^2 - 3) \tag{12}$$

where the shear modulus $\mu > 0$. The corresponding nominal stress tensor $\mathbf{N} \equiv J\mathbf{F}^{-1}\mathbf{T}$ (cf., Ogden [7]), where $\mathbf{T}$ denotes the usual Cauchy stress tensor, is given by

$$\mathbf{N} = \mu\mathbf{F}^T - p\mathbf{F}^{-1} \tag{13}$$

where $p \equiv p(R, \Theta, \Phi)$ is the unknown pressure associated with the constraint of incompressibility. Note that the nominal stress tensor $\mathbf{N}$ is the transpose of the first Piola-Kirchhoff stress tensor $\mathbf{S} \equiv J\mathbf{T}\mathbf{F}^{-T}$, and that given the orthonormal bases $\{\mathbf{E}_\alpha\}$ and $\{\mathbf{e}_i\}$ in the undeformed and the deformed configurations respectively its component $N_{\alpha i}$ is the $\mathbf{e}_i$-component of force on a surface element in the current configuration whose normal was in the $\mathbf{E}_\alpha$-direction in the (reference) undeformed configuration. To obtain the components $N_{\alpha j}$ ($\alpha = R, \Theta, \Phi; \quad j = r, \theta, \phi$) of the nominal stress tensor $\mathbf{N}$ corresponding to the deformation (1) we first need to identify $\mathbf{F}^{-1}$. This can easily be obtained from (1) and the incompressibility constraint (4). Namely, $\mathbf{F}^{-1}$ equals to

$$f \begin{pmatrix} f g_{;\Theta} & \frac{-f_{;\Theta}}{\sin\Phi} & 0 \\ -Rfg_{;R}\sin\Phi & f + Rf_{;R} & 0 \\ 0 & 0 & \frac{1}{f^2} \end{pmatrix}. \tag{14}$$

Substituting this into equation (13) one gets the following relations for the components of the stress tensor $\mathbf{N}$.

$$\begin{aligned} N_{Rr} &= f + Rf_{;R} - \frac{p}{\mu} f^2 g_{;\Theta}, \\ \sin\Phi N_{R\theta} &= Rfg_{;R}\sin^2\Phi + \frac{p}{\mu} f f_{;\Theta}, \\ \sin\Phi N_{\Theta r} &= f_{;\Theta} + \frac{p}{\mu} Rf^2 g_{;R}\sin^2\Phi, \\ N_{\Theta\theta} &= f g_{;\Theta} - \frac{p}{\mu} f(f + Rf_{;R}), \\ f N_{\Phi\phi} &= f^2 - \frac{p}{\mu}, \\ N_{\Phi r} &= N_{\Phi\theta} = 0, \\ N_{R\phi} &= N_{\Theta\phi} = 0. \end{aligned} \tag{15}$$

We are now in the position to consider the equations of equilibrium

$$\mathrm{Div}\mathbf{N} = 0 \tag{16}$$

where divergence must be taken with respect to the coordinates of the undeformed (reference) state, i.e., R, Θ and Φ. The spherical form of the equations of equilibrium for the nominal stress $\mathbf{N}$ was originally derived by Polignone-Warne and Warne [6]. These equations when formulated for our particular class of deformations (1) for the neo-Hookean material (12) reduce to the following set of three equations

$$g_{;\Theta} R \sin\Phi N_{Rr} + R N_{R\theta;R} + 2N_{R\theta} + \\ + g_{;\Theta} N_{\Theta r} + \frac{1}{\sin\Phi} N_{\Theta\theta;\Theta} = 0, \qquad (17)$$

$$g_{;R} R \sin\Phi N_{R\theta} - R N_{Rr;R} - 2N_{Rr} + \\ + g_{;\Theta} N_{\Theta\theta} + N_{\Phi\phi} - \frac{1}{\sin\Phi} N_{\Theta\theta;\Theta} = 0, \qquad (18)$$

$$(g_{;\Theta} N_{\Theta\theta} - N_{\Phi\phi}) \cot\Phi = \\ = N_{\Phi\phi;\Phi} - g_{;R} R \cos\Phi N_{R\theta}. \qquad (19)$$

After substituting the formulae for the components of the nominal stress (15) the last equilibrium equation (19) enables one to determine the dependence of the unknown hydrostatic pressure p on the spherical variable Φ. Namely, making use of the incompressibility constraint (4) one obtains that

$$p(R,\Theta,\Phi) = \mu f^2 (1 - g_{;\Theta}^2) \ln(\sin\Phi) - \\ -\frac{\mu}{2}(R f g_{;R} \sin\Phi)^2 + p_0(R,\Theta) \qquad (20)$$

where $p_0(R,\Theta)$ must still be determined from the equilibrium equations (17 & 18).

After rather lengthy and tedious manipulations[2] one can finally conclude that there are only two different types of deformations of the investigated class (1) which satisfy the equilibrium equations (17,18 & 19) subject to the constraint of incompressibility (3). These are the trivial solution, the radially symmetric cavitated deformations

$$r(R) = (1 + \frac{A^3}{R^3})^{\frac{1}{3}} R = (R^3 + A^3)^{\frac{1}{3}},$$

$$\theta = \Theta, \quad \phi = \Phi, \qquad (21)$$

and the deformation of inflation and (polar) stretching

$$r(R) = \frac{R}{\alpha}, \ \theta = \alpha^3 \Theta, \ \phi = \Phi, \qquad (22)$$

where α is a positive constant. The pressures corresponding to these deformations are respectively

$$p(R) = \mu \frac{R(3R^3 + 4A^3)}{2(R^3 + A^3)^{\frac{4}{3}}} + p_0, \qquad (23)$$

and

$$p(R,\Phi) = \frac{\mu}{\alpha}(1 - \alpha^{\frac{2}{3}}) \ln(R \sin\Phi) + p_0. \quad (24)$$

where p_0 is a constant. When $A = 0$ or $\alpha = 1$ we recover the trivial (undeformed) solution maintained by a constant hydrostatic pressure.

The first solution is the well known and extensively studied radially symmetric solution which allows a cavity to form at some critical load. The second class is completely non-singular; a cavity cannot form. It is specially remarkable as well as surprising that Polignone-Warne and Warne [6] when considering a more restricted class of non-spherically symmetric solutions, namely $g_{;R} = 0$, obtained exactly

[2] Some of these manipulations were done with the aid of MAPLE.

the same set of admissible deformations. Thus, if there are still non-spherically symmetric solutions allowing a cavity to form they may only come from a class of deformations explicitly dependent on the spherical angle Φ. We are currently investigating such a class of deformations as well as the effects of material inhomogeneity on cavity formation.

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