An Introduction
to
Applied Partial Differential Equations

Marek Z. Elżanowski

Department of Mathematics and Statistics

Portland State University

Contents

Preface

These notes are written for a one-quarter (pilot) course in elementary partial differential equations. It is assumed that the student has a good background in calculus, vector calculus, and ordinary differential equations. No prior knowledge of any partial differential equations concepts is assumed, nor any required. Some familiarity with the elementary theory of inner vector spaces would be an asset but is not expected. In fact, most of the needed concepts and facts are reviewed in the Appendix.

The main objective of this presentation is to introduce basic analytic techniques useful in solving most fundamental partial differential equations that arise in the physical and engineering sciences. The emphasis are placed on the formulation of a physical problem, deriving explicit analytic results, and on the discussion of properties of solutions. Although the proofs are usually omitted, the underlying mathematical concepts are explained and discussed at length.

The notes are divided into several short chapters and the Appendix. In Chapter 1 we discuss solutions to the equilibrium equations of one-dimensional continuous systems. These are formulated as boundary-value problems for scalar ordinary differential equations. The Green's function technique and the minimum principle are discussed. Chapter 2 deals with the diffusion equation, in particular, the heat propagation equation. In the last section of this chapter we briefly discuss the Burgers' equation. Solutions to a variety of homogeneous and inhomogeneous initial-boundary-value problems are derived using such analytic techniques as the separation of variables method and the concept of the fundamental solution. Laplace's equation and the wave equation are dealt with in Chapter 3 and 4, respectively. Once again, the separation of variables and the Fourier series methods are utilized. The Green's function technique is also researched. d'Alembert's solution of the wave equation is derived. An elementary

discussion on the propagation of one-dimensional non-linear waves is presented. The Appendix consists of two parts. In the first part, the elements of the theory of inner product vector spaces are reviewed. The second part contains the presentation of the theory of Fourier series, and a short section on Fourier integrals.

Although the notes are as self contained as possible, students may find useful to consult some other texts like for example [**Bleecker and Csordas**], [**Boyce and DiPrima**], [**Keane**], [**Knobel**], and [**Davis**], among others.

Portland. Oregon, June 2003

CHAPTER 1

Boundary Value Problems

In this introductory chapter we discuss the equilibrium equations of one-dimensional continuous systems. These are formulated as boundary value problems for scalar ordinary differential equations. We concentrate on deriving the exact analytical formulae for the solutions of these equations giving us the necessary inside into the physical processes we model.

1.1. Elastic Bar

By a *bar* we mean a finite length one-dimensional continuum that can only be stretched or contracted (deformed, in short) in the longitudinal direction, and is not allowed to bend in a transverse direction. Given a point x on a bar we measure its deformation from the reference position by the *displacement* $u(x)$. That is, a material point which was originally at the position x has been moved to the position $x + u(x)$. We adopt the convention that $u(x) > 0$ means that the material is stretched out, while $u(x) < 0$ describes a contraction by the amount $-u(x)$. We also assume that the left (top) end of the bar is fixed, i.e., $u(0) = 0$.

The internal forces experienced by the bar, known as *stress*, do not necessarily depend on how much the bar is stretched as a whole but rather on how much one material point is moved relative to the neighboring points. This relative amount of elongation is measured by the *strain*. Consider two material points (particles) occupying in the reference configuration positions x and $x + \triangle x$, respectively, where $\triangle x$ is the small section of the bar. When the bar experiences the displacement u the section of length $\triangle x$ gets stretched to the new length

$$[x + \triangle x + u(x + \triangle x)] - [x + u(x)] = \triangle x + [u(x + \triangle x) - u(x)]\,. \qquad (1.1.1)$$

The relative elongation of the segment $\triangle x$ is

$$\frac{u(x+\triangle x)-u(x)}{\triangle x}. \tag{1.1.2}$$

Shrinking the segment of the bar to a point, we obtain the dimensionless strain measure at the position x

$$\varepsilon(x) \equiv \lim_{\triangle x \to 0} \frac{u(x+\triangle x)-u(x)}{\triangle x} = \frac{du}{dx}. \tag{1.1.3}$$

The material the bar is made of is defined by the *constitutive relation* for the stress. This constitutive law tells how the stress depends on the strain when the bar undergoes a deformation. Here we shall only consider a linear relation which, in fact, approximates the real case quite adequately as long as the strain is small. If $\mathfrak{s}(x)$ denotes the stress exerted on a material point which was at the reference position x, we postulate

$$\mathfrak{s}(x) = c(x)\varepsilon(x), \tag{1.1.4}$$

where $c(x)$ measures the *stiffness* of the bar at material point x. If the bar is homogeneous $c(x) = c$ is constant.

We also postulate that the internal stresses of the deformed bar balance the external forces imposed. That is, if $f(x)$ denotes the external force applied at x, we assume that

$$\frac{\mathfrak{s}(x+\triangle x)-\mathfrak{s}(x)}{\triangle x} + \frac{1}{\triangle x}\int_x^{x+\triangle x} f(s)ds = 0 \tag{1.1.5}$$

per unit length of a segment of the bar between positions x and $x+\triangle x$, where $f(x) > 0$ if the bar gets stretched. Invoking the Mean Value Theorem and taking the limit of the left hand side of (1.1.5) as $\triangle x \to 0$ we obtain that

$$f = -\frac{d\mathfrak{s}}{dx} \tag{1.1.6}$$

everywhere along the bar. Substituting the constitutive law (1.1.4) into the equation of balance of forces (1.1.6) and using the definition of the strain function (1.1.3) we obtain the equation of equilibrium

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x), \quad 0 < x < l \tag{1.1.7}$$

for the linearly elastic bar of length (in the reference configuration) l. This is a second-order ordinary differential equation for the displacement $u(x)$. Its general solution depends on two arbitrary constants. They can be uniquely determined by the boundary conditions at the ends of the bar. For example, let

$$u(0) = 0, \quad \mathfrak{s}(l) = c(l)\varepsilon(l) = c(l)u'(l) = 0 \tag{1.1.8}$$

as the left(top) end of the bar is fixed and the other end is assumed to be free (stress free).

Example 1.1. Consider a homogeneous bar of unit length subjected to a uniform force, e.g., a bar hanging from a celling and deforming under its own weight. The equilibrium equation (1.1.7) takes the form

$$-c\frac{d^2u}{dx^2} = mg, \tag{1.1.9}$$

where m denotes the mass of the bar and g is the gravitation constant. This is a linear second order equation solution of which is

$$u(x) = -\frac{mg}{2c}x^2 + ax + b. \tag{1.1.10}$$

The arbitrary integration constants a and b can be determined from the boundary conditions (1.1.8). Namely,

$$u(0) = b = 0, \quad u'(1) = -\frac{mg}{2c} + a = 0. \tag{1.1.11}$$

The corresponding unique solution yields the parabolic displacement

$$u(x) = \frac{mg}{c}\left(x - \frac{x^2}{2}\right) \tag{1.1.12}$$

and the linear strain

$$\varepsilon(x) = \frac{mg}{c}(1 - x). \tag{1.1.13}$$

Note that the displacement is maximum at the bottom free end of the bar while strain, and so the stress, are maximum at the fixed end. Note also that as the boundary condition at the free end determines both the strain and the stress at that end the equilibrium equation can be first solved uniquely for stress without calculating the displacement. Such a mechanical configuration is known as *statically determinate*. This is in contrast with the problem in which the displacement is prescribed at both ends of the beam, e.g.,

$$u(0) = 0, \quad u(1) = r. \tag{1.1.14}$$

The general solution to the equilibrium equation (1.1.7) takes the same parabolic form (1.1.10). The unique solution satisfying the boundary conditions (1.1.14) yields

$$u(x) = \frac{mg}{2c}(x - x^2) + rx. \tag{1.1.15}$$

Once the displacement is available we can calculate the stress field

$$\mathfrak{s}(x) = mg(\frac{1}{2} - x) + r. \tag{1.1.16}$$

However, unlike as in case of the bar with the free end, the stress cannot be determined without knowing the displacement. The equilibrium equation (1.1.7) can be re-written in terms of stress

$$-\frac{d\mathfrak{s}}{dx} = mg, \tag{1.1.17}$$

but the integration constant in the general stress solution $\mathfrak{s}(x) = -mgx + a$ cannot be determined as there is no stress boundary condition available. Such a mechanical configuration is called *statically indeterminate*.

Remark 1.2. Our equation (1.1.7) not only describes the mechanical equilibrium of an elastic bar but it models also some other physical systems. For example, this is the thermal equilibrium equation of a bar subjected to an external heat source. Indeed, if $u(x)$ represents the temperature at the position x,

$c(x)$ is the *thermal conductivity* of the material at x, and $f(x)$ denotes the external heat source, then the energy conservation law yields (1.1.7)[1]. A boundary condition $u(l) = r$ corresponds to the situation when an end is kept at a fixed temperature. $u'(l) = 0$, on the other hand, describes a thermally insulated end.

1.2. The Green's Function

The Green's function method is one of the most important approaches to the solution of boundary value problems. It relies on the *superposition principle* for inhomogeneous linear equations. Namely, it builds the general solution out of the solutions to a very particular set of concentrated inhomogeneities.

The superposition principle for a linear homogeneous differential equation states that if $u_1(x)$ and $u_2(x)$ are solutions then every linear combination $\alpha u_1(x) + \beta u_2(x)$ is also a solution, where α and β are arbitrary real numbers. Moreover, if the functions $f_1, \dots, f_n$ represent the inhomogeneities (forcing terms) of the linear differential equation

$$K[u] = f_i, \quad i = 1, \dots, n, \tag{1.2.1}$$

where $K[u]$ denotes the differential operator (the left hand side of an equation, e.g., $K[u] = -cu''$), and if $u_1(x), \dots, u_n(x)$ are the corresponding solutions then the linear *superposition* $\alpha_1 u_1(x) + \alpha_2 u_2(x) \dots + \alpha_n u_n(x)$ is a solution of

$$K[u] = \alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_n f_n \tag{1.2.2}$$

for any choice of the constants $\alpha_1, \dots, \alpha_n$.

Our objective here is to use this superposition principle to solve the boundary value problem for a homogeneous elastic bar. To be able to do this we must solve first the boundary value problem with the unit impulse as a source term. Such a solution is called *the Green's function* and it will be used later to construct a solution to the corresponding boundary value problem with an arbitrary forcing term. First, we shall characterize a unit impulse (a point force) concentrated at a point of the bar by introducing the notion of the *delta function*.

[1]See the derivation of the heat conduction equation (2.1.6) in the next chapter.

The Delta Function.

As the impulse is to be concentrated solely at a single point, say y, the delta function $\delta_y(x)$ should be such that

$$\delta_y(x) = 0, \quad \text{for} \quad x \neq y. \tag{1.2.3}$$

Moreover, as we would like the strength of the impulse to be one, and there is no other external force applied, we require that

$$\int_0^l \delta_y(x)dx = 1, \quad \text{as long as} \quad 0 < y < l. \tag{1.2.4}$$

Looking at both conditions $\delta_y(x)$ must satisfy one realizes quickly that there is no such function.

The mathematically correct definition of such a *generalized function*, which can for example be found in [**Ziemer**] (see also [**Lang**]), is well beyond the scope of these notes. It relies on the assumption that for any bounded continuous function $u(x)$

$$\int_0^l \delta_y(x)u(x)dx = u(y), \quad \text{if} \quad 0 < y < l. \tag{1.2.5}$$

Here, we will present the "approximate" definition of the delta function which regards $\delta_y(x)$, considered over the infinite domain $(-\infty, \infty)$, as the limit of a sequence of continuous functions. To this end, let

$$f_n(x; y) \equiv \frac{n}{\pi(1 + n^2(x - y)^2)}. \tag{1.2.6}$$

These functions are such that

$$\int_{-\infty}^{\infty} f_n(x; y)dx = \frac{1}{\pi} \arctan(nx)\Big|_{-\infty}^{\infty} = 1, \tag{1.2.7}$$

and

$$\lim_{n\to\infty} f_n(x; y) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases} \tag{1.2.8}$$

pointwise, but not uniformly[2]. Hence, we identify $\delta_y(x)$ with the limit

$$\lim_{n\to\infty} f_n(x;y) = \delta_y(x). \tag{1.2.9}$$

Note, however, that this construction of the delta function should only be viewed as a visualization of such a generalized function, and not as its correct mathematical definition. Indeed, within the context of the Riemann's integration theory

$$1 = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x;y)dx \neq \int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x;y)dx = 0 \tag{1.2.10}$$

as the limit of the integral is not necessarily the integral of the limit. On the other hand, this can be made to work if we adopt a somewhat different definition of the limit. This can allow us to justify the formula (1.2.5) as the limit of the approximating integrals. In these notes we will use both definitions of the delta function interchangeably.

Let us consider now the calculus of the delta function, that is, its integration and differentiation. Firstly, assuming that $a < y$ and using the definition (1.2.5) we obtain that

$$\int_a^x \delta_y(s)ds = \sigma_y(x) \equiv \begin{cases} 0, & a < x < y, \\ 1, & x > y > a, \end{cases} \tag{1.2.11}$$

is the *step function*. This is a function which is continuous everywhere except at $x = y$, where its is not defined and experiences a jump discontinuity (1.2.15). The value of the step function at $x = y$ is often left undefined. Motivated by Fourier theory[3] we set $\sigma_y(y) = \frac{1}{2}$. Interestingly enough we obtain the same result using the characterization of the delta function as the limit of the sequence $f_n(x;y)$. Indeed, if

$$g_n(x) \equiv \int_{-\infty}^{x} f_n(t,0)dt = \frac{1}{\pi}\arctan(nx) + \frac{1}{2}, \tag{1.2.12}$$

then

[2]See Definition B.18.

[3]See Theorem B.4.

$$\lim_{n\to\infty} g_n(x) = \sigma(x) \equiv \sigma_0(x) \tag{1.2.13}$$

pointwise. In tern, the Fundamental Theorem of Calculus allows us to identify the derivative of the step function with the delta function;

$$\frac{d\sigma_y(x)}{dx} = \delta_y(x). \tag{1.2.14}$$

In fact, this enables us to differentiate any discontinuous function having finite jump discontinuities at isolated points. Suppose the function $f(x)$ is differentiable everywhere except at a single point y at which it has a *jump discontinuity*

$$[f(y)] = f_+(y) - f_-(y) \equiv \lim_{x\to y^+} f(x) - \lim_{x\to y^-} f(x). \tag{1.2.15}$$

We can write

$$f(x) = g(x) + [f]\sigma_y(x), \tag{1.2.16}$$

where $g(x) = f(x) - [f]\sigma_y(x)$ is a continuous function, but is not necessarily differentiable at y. Therefore,

$$f'(x) = \begin{cases} g'(x), & x \neq y \\ [f]\delta_y(x), & x = y. \end{cases} \tag{1.2.17}$$

In short, we write

$$f'(x) = g'(x) + [f]\delta_y(x). \tag{1.2.18}$$

Example 1.3. Consider the function

$$f(x) = \begin{cases} -x+1, & x < 0, \\ 0, & 0 < x < 1, \\ x^2, & x > 1. \end{cases} \tag{1.2.19}$$

It has two jump discontinuities: $[f] = -1$ at $x = 0$, and $[f] = 1$ at $x = 1$. Utilizing the construction (1.2.16) one gets that

$$g(x) = f(x) + \sigma(x) - \sigma_1(x) = \begin{cases} -x+1, & x < 0, \\ 1, & 0 < x < 1, \\ x^2, & x > 1 \end{cases}$$

and

$$f'(x) = g'(x) - \delta(x) + \delta_1(x) = -\delta(x) + \delta_1(x) + \begin{cases} -1, & x < 0, \\ 0, & 0 < x < 1, \\ 2x, & x > 1. \end{cases}$$

To find the derivative $\delta_y'(x)$ of the delta function let us determine its effect on a function $u(x)$ by looking at the limiting integral

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{df_n(x;0)}{dx} u(x)dx = \lim_{n\to\infty} f_n(x;0)u(x)\Big|_{-\infty}^{\infty} - \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x;0)u'(x)dx \tag{1.2.20}$$

$$= -\int_{-\infty}^{\infty} \delta_0(x)u'(x)dx = -u'(0),$$

where the integration by parts was used and where the function $u(x)$ is assume continuously differentiable and bounded to guarantee that

$$\lim_{n\to\infty} f_n(x;0)u(x)|_{-\infty}^{\infty} = 0. \tag{1.2.21}$$

Hence, we postulate that $\delta_y'(x)$ is a generalized function such that

$$\int_0^l \delta_y'(x)u(x)dx = -u'(y). \tag{1.2.22}$$

Note that this definition of the derivative of the delta function is compatible with the formal integration by parts procedure

$$\int_0^l \delta_y'(x)u(x)dx = \delta_y(x)u(x)\Big|_0^l - \int_0^l \delta_y(x)u'(x)dx = -u'(y). \tag{1.2.23}$$

Note also that one may view the derivative $\delta_y'(x)$ as the limit of the sequence of derivatives of the "approximating" functions $f_n(x;0)$. That is,

$$\delta_0'(x) = \lim_{n\to\infty} \frac{df_n(x;0)}{dx} = \lim_{n\to\infty} \frac{-2n^3\pi}{\pi(1+n^2x^2)^2}. \tag{1.2.24}$$

These are interesting rational functions. First of all, it easy to see that the sequence converges pointwise, but not uniformly, to 0. Also, elementary calculations revile, that the graphs of the functions consist of two increasingly concentrated symmetrically positioned at $x = \pm\frac{1}{n\sqrt{3}}$ spikes, and that the amplitudes of these spikes approach $\mp\infty$, respectively, as $n \to \infty$.

The Green's Function.

Once we have familiarized ourselves with the delta function we may try now to solve the boundary value problem for a homogeneous elastic bar with the delta function (unit impulse) as its source term. As we have explained earlier, the main idea behind this approach is to use the superposition principle to obtain the solution for a general external force by putting together the solutions to the impulse problems.

Consider a linearly elastic bar of the reference length l subjected to a unit point force $\delta_y(x)$ applied at position $0 < y < l$. The equation governing such a system (1.1.7) takes the form

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = \delta_y(x), \quad 0 < x < l. \tag{1.2.25}$$

The solution $G(x;y)$ to the boundary value problem associated with (1.2.25) is called the *Green's function* of the problem. To illustrate how such a solution comes about let us consider the following homogeneous boundary value problem

$$-u'' = \delta_y(x), \quad u(0) = u(1) = 0, \tag{1.2.26}$$

for a bar of length $l = 1$ and the stiffness $c = 1$, where $0 < y < l$. Integrating the equation twice we obtain that

$$u(x) = \begin{cases} ax + b, & x \le y, \\ -(x-y) + ax + b, & x \ge y. \end{cases} \tag{1.2.27}$$

Taking into consideration the boundary conditions we have that

$$u(0) = b = 0, \quad \text{and} \quad u(1) = -(1-y) + a + b = 0.$$

This implies that $b = 0$, $a = 1 - y$, and the Green's function of this boundary value problem is

$$G(x;y) = \begin{cases} x(1-y), & x \le y, \\ y(1-x), & x \ge y. \end{cases} \tag{1.2.28}$$

This is a continuous, piecewise differentiable function. Its first derivative experiences a jump of magnitude -1 at $x = y$. In fact, this is a piecewise affine function as its graph consists of straight line segments only. Note also that the Green's function, viewed as a function of two variables, is symmetric in x and y. This symmetry has an interesting physical interpretation that the deformation of the bar measured at position x due to the point force applied at position y is exactly the same as the deformation of the bar at position y due to the concentrated force being applied at position x.

Once we have the Green's function available we can solve the general inhomogeneous problem

$$-u'' = f(x), \quad u(0) = u(1) = 0, \tag{1.2.29}$$

by the linear superposition method. To be able to do this we need first to express the forcing term $f(x)$ as a superposition of point forces (impulses) distributed throughout the bar. The delta function comes handy again as, according to (1.2.5), it enables us to write the external forcing term as

$$f(x) = \int_0^1 f(y)\delta_x(y)dy. \tag{1.2.30}$$

One may interpret the external source f as the superposition of an infinitely many point sources $f(y)\delta_x(y)$ of the amplitude $f(y)$ applied throughout the bar at $0 < x < l$. Re-writing the differential equation (1.2.29) as

$$-u'' = \int_0^1 f(y)\delta_x(y)dy \tag{1.2.31}$$

renders the solution $u(x)$ as a linear combination

$$u(x) = \int_0^1 f(y)G(x;y)dy \tag{1.2.32}$$

of solutions to the unit impulse problems. We can verify by direct computation that the formula (1.2.32) gives us the correct answer to the boundary value problem (1.2.29). Indeed, using the formula for the Green's function (1.2.28) we may write the solution of (1.2.29) as

$$u(x) = \int_0^x (1-x)yf(y)dy + \int_x^1 x(1-y)f(y)dy. \tag{1.2.33}$$

Differentiating it once gives us

$$u'(x) = -\int_0^1 yf(y)dy + \int_x^1 f(y)dy.$$

Differentiating it once again shows that

$$u''(x) = -f(x).$$

For a particular forcing term $f(x)$ it may be easier to solve the problem directly rather than by using the corresponding Green's function. However, the advantage of the Green's function method is that it provides the general framework for any and all inhomogeneous equations with the homogeneous boundary conditions. The case of the inhomogeneous boundary value problem will be discussed in the next chapter.

EXAMPLE 1.4. Consider now a different boundary value problem for a uniform bar of length l. Namely,

$$-cu''(x) = \delta_y(x), \quad u(0) = 0, \quad u'(l) = 0, \tag{1.2.34}$$

where c denotes the elastic constant. This problem models the deformation of the bar with one end fixed and the other end free. Integrating this equation twice, we find the general solution

$$u(x) = -\frac{1}{c}\rho(x-y) + ax + b,$$

where

$$\rho(x-y) \equiv \begin{cases} x-y, & x > y, \\ 0, & x < y, \end{cases} \tag{1.2.35}$$

is called the (first order) *ramp function*. Utilizing the given boundary conditions we find the Green's function for this problem as

$$G(x;y) = \begin{cases} x/c, & x \le y, \\ y/c, & x \ge y. \end{cases} \tag{1.2.36}$$

Again, this function is symmetric and affine. Also, it is continuous but not differentiable at the point of application of the external force $x = y$, where its derivative experiences a $-1/c$ magnitude jump. The formula for the solution of the corresponding boundary value problem for the inhomogeneous equation

$$-cu''(x) = f(x), \quad u(0) = 0, \quad u'(l) = 0, \tag{1.2.37}$$

takes the form

$$u(x) = \int_0^l G(x;y) f(y) dy = \frac{1}{c}\left[\int_0^x x f(y) dy + \int_x^l y f(y) dy\right].$$

1.3. Minimum Principle

In this section we shall discuss how the solution to a boundary value problem is a unique minimizer of the corresponding "energy" functional. This minimization property proves to be particularly significant for the design of numerical techniques such as the finite elements method.

We start by taking a short detour to discuss the concept of an adjoint of a linear operator on an inner product vector space [4] . Let $L : U \to W$ denote a linear operator from the inner product vector space U into another (not necessarily different) inner product vector space W. An *adjoint* of the linear operator L is the operator $L^\star : W \to U$ such that

$$\langle L[u]; w\rangle_W = \langle u; L^\star[u]\rangle_U \quad \text{for all} \quad u \in U, \quad w \in W, \tag{1.3.1}$$

[4]The fundamentals of Inner Product Vector Spaces are reviewed in Appendix A.

where the inner products are evaluated on the respective spaces as signified by the corresponding subscripts. Note that if $U = W = \mathbb{R}^n$, with the standard dot product, and the operator L is represented by a $n \times n$ matrix A, then the adjoint $L^\star$ can be identified with the transpose A^T.

In the context of an equilibrium equation for a one-dimensional continuum the main linear operator is the derivative $D[u] = du/dx$. It operates on the space of all possible displacements U into the space of possible strains W. To evaluate its adjoint we impose on both vector spaces the same standard L^2- inner product, i.e.,

$$\langle u; \tilde{u} \rangle_U \equiv \int_a^b u(x)\tilde{u}(x)dx, \quad \langle w; \tilde{w} \rangle_W \equiv \int_a^b w(x)\tilde{w}(x)dx. \tag{1.3.2}$$

According to (1.3.1) the adjoint $D^\star$ of the operator D must satisfy

$$\langle D[u]; w \rangle_W = \langle \frac{du}{dx}; w \rangle_W = \int_a^b \frac{du}{dx} w(x)dx = \langle u; D^\star[w] \rangle_U = \int_a^b u(x) D^\star[w](x)dx, \tag{1.3.3}$$

for all $u \in U$ and $w \in W$. Note, however, that the integration by part yields

$$\begin{aligned} \langle D[u]; w \rangle_W &= \int_a^b \frac{du}{dx} w(x)dx = [u(b)w(b) - u(a)w(a)] - \int_a^b u(x)\frac{dw}{dx}dx \\ &= [u(b)w(b) - u(a)w(a)] + \langle u; \frac{dw}{dx} \rangle_U. \end{aligned} \tag{1.3.4}$$

This suggests that

$$\left(\frac{d}{dx}\right)^\star = -\frac{d}{dx}, \tag{1.3.5}$$

provided the functions $u \in U$ and $w \in W$ are such that

$$[u(b)w(b) - u(a)w(a)] = 0. \tag{1.3.6}$$

This will be possible if we impose suitable boundary conditions. In other words, if we define the vector space

$$U = \{u(x) \in C^1[a, b] : \quad u(a) = u(b) = 0\} \tag{1.3.7}$$

as containing all continuously differentiable functions (displacements) vanishing at the boundary, and restrict the operator D to U, the definition of the adjoint (1.3.5) will hold. Obviously, these are not the only boundary conditions guaranteeing (1.3.5). The other possible choice is the space of displacements

$$U = \{u(x) \in C^1[a,b] : \quad u'(a) = u'(b) = 0\}$$

Yet another possibility is the strain space

$$W = \{w(x) \in C^1[a,b] : \quad w(a) = w(b) = 0\}.$$

This is the case when due to the lack of support at the ends the displacement is undetermined and the stresses vanish.

Consider now a homogeneous bar[5] of length l, with material stiffness $c = 1$, and suitable boundary conditions (guaranteeing (1.3.5)). Then, the equilibrium equation (1.1.7) takes the form

$$K[u] = f, \quad \text{where} \quad K = D^\star \circ D = -D^2 \tag{1.3.8}$$

is *self-adjoint*, that is, $K^\star = K$. Indeed,

$$K^\star = (D^\star \circ D)^\star = D^\star \circ D^{\star\star} = D^\star \circ D = K. \tag{1.3.9}$$

Moreover, K is *positive definite*, where a linear operator $K : U \to U$ is said to be positive definite if it is self-adjoint and

$$\langle K[u]; u\rangle_U > 0, \quad \text{for all} \quad 0 \neq u \in U. \tag{1.3.10}$$

To verify the positivity condition for $K = D^\star \circ D$ note that

$$\langle K[u]; u\rangle_U = \langle D^\star[D[u]]; u\rangle_U = \langle D[u]; D[u]\rangle_U = ||D[u]||^2 \geq 0, \tag{1.3.11}$$

and its is positive if and only if $D[u] = 0$ only for $u \equiv 0$. This is not true in general. However, if we impose the homogeneous boundary conditions $u(0) = u(l) = 0$

[5]To use the same framework for the analysis of a bar, or a beam, made of the inhomogeneous material one needs to modify the inner product on the space of strains by introducing, as shown in Section 1.4, a weighted L^2-inner product.

the vanishing of the first derivative of u implies its vanishing everywhere, proving that K is positive-definite. In fact, the same is true for the mixed boundary value problem $u(0) = u'(l) = 0$. However, this is not the case when $u'(0) = u'(l) = 0$ as $D[u] = 0$ for a constant, non-vanishing displacement. The corresponding linear operator K is self-adjoint but not positive-definite.

The minimum principle for the boundary value problem

$$-u'' = f, \quad u(0) = u(l) = 0 \tag{1.3.12}$$

can be formulated as the minimization of the functional

$$\mathfrak{P}[u] = \frac{1}{2}\,||D[u]||^2 - \langle u; f\rangle_U = \int_0^l \left[\frac{1}{2}(u'(x))^2 - f(x)u(x)\right] dx \tag{1.3.13}$$

over the space of functions $U = \{u(x) \in C^2[0,l] : \quad u(0) = u(l) = 0\}$.

Remark 1.5. Formally, we seek a function, say $\overline{u}(x)$, from among the functions belonging to the space U, such that

$$\mathfrak{P}[\overline{u}] = \min_{u\in U} \mathfrak{P}[u].$$

Suppose now that $\overline{u}$ is such a minimum and assume that

$$\mathfrak{P}[u] = \int_0^l f(x, u, u')dx.$$

Also, let $w_\epsilon(x) = \overline{u}(x) + \epsilon\eta(x)$ represent a curve of functions in the space U. Note that this implies that the function $\eta(x)$ vanishes at both ends of the interval $[0, l]$. Restricting the functional $\mathfrak{P}$ to the curve w_ϵ consider the real-valued function

$$\mathfrak{i}(\epsilon) = \mathfrak{P}[w_\epsilon].$$

Since $\overline{u}$ is a minimizer of $\mathfrak{P}[\cdot]$ we observe that $\mathfrak{i}[\cdot]$ has a minimum at $\epsilon = 0$. Therefore

$$\mathfrak{i}'(0) = 0.$$

Computing explicitly the derivative we obtain that

$$\mathfrak{i}'(\epsilon) = \int_0^l \left[\frac{\partial f}{\partial w}\eta(x) + \frac{\partial f}{\partial w'}\eta'(x)\right] dx.$$

Integrating the second term by parts and taking into account the boundary conditions for the variation $\eta(x)$, we get that

$$\mathfrak{i}'(\epsilon) = \int_0^l \eta(x) \left[\frac{\partial f}{\partial w} - \frac{d}{dx} \left(\frac{\partial f}{\partial w'} \right) \right] dx.$$

As it is valid for all variations $\eta(x)$, it vanishes when

$$\frac{\partial f}{\partial w} - \frac{d}{dx} \left(\frac{\partial f}{\partial w'} \right) = 0. \tag{1.3.14}$$

This partial differential equation equation is known as the *Euler-Lagrange equation*. Its solution is the minimizer $\overline{u}$. Although any minimizer of $\mathfrak{P}[\cdot]$ is a solution of the corresponding Euler-Lagrange equation, the converse is not necessarily true.

The functional $\mathfrak{P}[u]$ represents the total potential energy of the bar due to the deformation $u(x)$. The first term measures the internal energy due to the strain $u'(x)$ (the *strain energy*) while the second part is the energy due to the external source $f(x)$. The solution to (1.3.12) is the minimizer of $\mathfrak{P}[u]$ over all functions satisfying the given boundary conditions.

Example 1.6. To illustrate the importance of the positive-definiteness of the given boundary value problem let us consider the following boundary value problem in strains

$$-u'' = f, \quad u'(0) = u'(l) = 0. \tag{1.3.15}$$

Integrating the equation twice, we find

$$u(x) = ax + b - \int_0^x \left(\int_0^y f(s) ds \right) dy. \tag{1.3.16}$$

Since

$$u'(x) = a - \int_0^x f(s) ds,$$

the boundary condition at $x = 0$ yields $a = 0$. The second boundary condition at $x = l$ implies that

$$u'(l) = \int_0^l f(s)ds = 0. \tag{1.3.17}$$

This is not true in general, unless the source term has the zero mean. But even if the distribution of external forces is such that the mean is zero, the solution

$$u(x) = b - \int_0^x \left(\int_0^y f(s)ds \right) dy \tag{1.3.18}$$

is not unique as the constant b remains unspecified. Physically, this corresponds to an unstable situation. Indeed, if the ends of the bar are left free there exists translation instability in the longitudinal direction.

1.4. Elastic Beam

In this short section we briefly discuss the use (possibly with some necessary adaptation) of the methods developed in the previous sections to analyze the deformation of an elastic (planar) beam. Here by a *beam* we understand a one-dimensional continuum which in addition to being able to stretch in the longitudinal direction is also allowed to bend in a plane, say (x, y). However, to simplify matters, we will only consider that it can bend, neglecting its longitudinal deformations. Consider therefore a beam of a reference length l, and let $y = u(x)$ denote the displacement in the transversal direction. As the beam bends we postulate that its bending moment $\omega(x)$ is proportional to the curvature of the beam

$$\kappa(x) \equiv \frac{u''}{(1 + u'^2)^{3/2}}. \tag{1.4.1}$$

Hence,

$$\omega(x) = c(x)\kappa(x) = \frac{c(x)u''}{(1 + u'^2)^{3/2}}. \tag{1.4.2}$$

If we assume that $u'(x)$ is small, i.e., the beam does not bend too far from its natural straight position, then the curvature is approximately equal to $u''(x)$, and the linearized constitutive relation for the beam assumes the form

$$\omega(x) = c(x)u''(x). \tag{1.4.3}$$

The linearized curvature $\kappa(x) = u''(x)$ plays the role of the (bending) *strain* [**Ogden**].

Relying on the law of balance of moments of forces and using (1.4.3) we obtain the equilibrium equation for the beam as the forth order ordinary differential equation

$$\frac{d^2}{dx^2}\left(c(x)\frac{d^2u}{dx^2}\right) = f(x). \tag{1.4.4}$$

To be able to determine any particular equilibrium configuration, equation (1.4.4) must be supplemented by a set of boundary conditions. As the equation is of order four we need four boundary conditions; two at each end of the beam. For example, we may assume that $u(0) = \omega(0) = \omega(l) = \omega'(l) = 0$ which describes the situation in which one end of the beam is simply supported while the other is free.

Note that the balance law (1.4.4) can be viewed, with the proper choice of the inner product and the boundary conditions, as written in the adjoint form. To this end, let us consider the differential operator $L \equiv D^2 = D \circ D$, where as before $D \equiv \frac{d}{dx}$. The equilibrium equation (1.4.4) takes the form

$$L[cu''] = f. \tag{1.4.5}$$

Let us also introduce the weighted inner product

$$\langle \upsilon; \tilde{\upsilon} \rangle \equiv \int_0^l \upsilon(x)\tilde{\upsilon}(x)c(x)dx \tag{1.4.6}$$

on the space of strains $\upsilon(x) \equiv \kappa(x) = u''(x)$, where $c(x) > 0$. One can easily check that this is indeed an inner product. To compute the adjoint operator $L^\star$ we need to evaluate

$$\langle L[u]; \upsilon \rangle = \int_0^l cL[u]\upsilon dx = \int_0^l c\frac{d^2u}{dx^2}\upsilon dx \tag{1.4.7}$$

where differentiating by parts twice

$$\int_0^l c\frac{d^2u}{dx^2}\upsilon dx = \left[c\frac{du}{dx}\upsilon - u\frac{d(c\upsilon)}{dx}\right]\Bigg|_0^l + \int_0^l u\frac{d^2(c\upsilon)}{dx^2}dx. \tag{1.4.8}$$

Consequently, if the functions u and v are such that

$$\left[c\frac{du}{dx}v - u\frac{d(cv)}{dx}\right]\Big|_0^l = [u'(l)\omega(l) - u(l)\omega'(l)] - [u'(0)\omega(0) - u(0)\omega'(0)] = 0 \tag{1.4.9}$$

then

$$L^\star[v] = \frac{d^2}{dx^2}(cv). \tag{1.4.10}$$

The equilibrium equation (1.4.4) can be written as

$$L^\star[v] = f, \tag{1.4.11}$$

and there is a quite a variety of possible *self-adjoint boundary conditions* as determined by (1.4.9).

Although the beam operator $L^\star \circ L$ is not self-adjoint it is positive definite for the appropriate boundary conditions, as evident from its form. As usually, the key condition is that D^2 vanishes only on the constant zero function of the appropriate space of functions. Since $D^2[u]$ vanishes if and only if u is affine, the boundary conditions must be such that they force all affine functions to vanish everywhere. For example, having one fixed end ($u(0) = u'(0) = 0$) will be sufficient, while having one simply supported end ($u(0) = \omega(0) = 0$) and one free end ($\omega(l) = \omega'(l) = 0$) will not do.

If the homogeneous boundary conditions are chosen so that the beam operator L is positive definite, it can be shown that the solution to the boundary value problem (1.4.11) is the unique minimizer of the corresponding energy functional

$$\mathfrak{P}[u] = \frac{1}{2}\left|\left|D^2[u]\right|\right|^2 - \langle u; f\rangle = \int_0^l \left[\frac{1}{2}c(x)u''(x) - f(x)u(x)\right] dx. \tag{1.4.12}$$

Example 1.7. Consider a uniform beam, with $c(x) \equiv 1$, of the reference length $l = 1$ and such that one end is fixed and the other end is free. In case there are no external forces the equilibrium equation (1.4.4) takes a very simple form

$$\frac{d^4u}{dx^4} = 0. \tag{1.4.13}$$

Its general solution

$$u = ax^3 + bx^2 + cx + d \tag{1.4.14}$$

must satisfy the following boundary conditions:

$$u(0) = u'(0) = \omega(1) = \omega'(1) = 0. \tag{1.4.15}$$

This yields the solution

$$u = \frac{1}{6}x^2(x-3). \tag{1.4.16}$$

To solve the forced beam problem we start by finding the appropriate Green's function. This means that we need to solve first the equation

$$\frac{d^4u}{dx^4} = \delta_y(x) \tag{1.4.17}$$

for the (1.4.15) boundary conditions. Integrating the equation four times and using the fact that the integral of the delta impulse is the ramp function (1.2.35) we obtain the general solution

$$u(x) = ax^3 + bx^2 + cx + d + \begin{cases} \frac{1}{6}(x-y)^3, & x > y, \\ 0, & x < y, \end{cases}. \tag{1.4.18}$$

The boundary conditions imply that

$$u(0) = d = 0, \quad u'(0) = c = 0, \quad \omega(1) = 6a+2b+1-y = 0, \quad \omega'(1) = 6a+1 = 0. \tag{1.4.19}$$

Therefore, the Green's function has the form

$$G(x,y) = \begin{cases} \frac{1}{2}x^2(y - \frac{x}{3}), & x < y, \\ \frac{1}{2}y^2(x - \frac{y}{3}), & x > y. \end{cases} \tag{1.4.20}$$

The Green's function is again symmetric in x and y as the boundary value problem we are dealing with is self-adjoint.

The general solution of the corresponding forced boundary value problem

$$\frac{d^4u}{dx^4} = f(x), \quad u(0) = u'(0) = \omega(1) = \omega'(1) = 0 \tag{1.4.21}$$

is given by the superposition formula

$$u(x) = \int_0^1 G(x,y)f(y)dy = \frac{1}{2}\int_0^x y^2(x - \frac{y}{3})f(y)dy + \frac{1}{2}\int_x^1 x^2(y - \frac{x}{3})f(y)dy. \tag{1.4.22}$$

CHAPTER 2

The Diffusion Equation

In this chapter we study the one-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2} + p(x,t),$$

which describes such physical situations as the heat conduction in a one-dimensional solid body, spread of a die in a stationary fluid, population dispersion, and other similar processes. In the last section we will also discuss the quasilinear version of the diffusion equation, known as, the Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \gamma \frac{\partial^2 u}{\partial x^2} = p(x,t)$$

which arises in the context of modelling the motion of a viscous fluid as well as traffic flow.

We begin with a derivation of the heat equation from the principle of the energy conservation.

2.1. Heat Conduction

Consider a thin, rigid, heat-conducting body (we shall call it a bar) of length l. Let $\theta(x,t)$ indicate the temperature of this bar at position x and time t, where $0 \le x \le l$ and $t \ge 0$. In other words, we postulate that the temperature of the bar does not vary with the thickness. We assume that at each point of the bar the energy density per unit volume ε is proportional to the temperature, that is

$$\varepsilon(x,t) = c(x)\theta(x,t), \tag{2.1.1}$$

where $c(x)$ is called *heat capacity* and where we also assumed that the mass density is constant throughout the body and normalized to equal one. Although the body has been assumed rigid, and with constant mass density, its material properties, including the heat capacity, may vary from one point to another.

To derive the "homogeneous" heat-conduction equation we assume that there are no internal sources of heat along the bar, and that the heat can only enter the bar through its ends. In other words, we assume that the lateral surface of the bar is perfectly insulated so no heat can be gained or lost through it. The fundamental physical law which we employ here is the law of conservation of energy . It says that the rate of change of energy in any finite part of the bar is equal to the total amount of heat flowing into this part of the bar. Let $q(x,t)$ denote the *heat flux* that is the rate at which heat flows through the body at position x and time t, and let us consider the portion of the bar from x to $x+\triangle x$. The rate of change of the total energy of this part of the bar equals the total amount of heat that flows into this part through its ends, namely

$$\frac{\partial}{\partial t}\int_x^{x+\triangle x} c(z)\theta(z,t)dz = -q(x+\triangle x,t)+q(x,t). \tag{2.1.2}$$

We use here commonly acceptable convention that the heat flux $q(x,t)>0$ if the flow is to the right.

In order to obtain the equation describing the heat conduction at an arbitrary point x we shall consider the limit of (2.1.2) as $\triangle x\to 0$. First, assuming that the integrand $c(z)\theta(z,t)$ is sufficiently regular, we are able to differentiate inside the integral. Second, dividing both sides of the equation by $\triangle x$, invoking the Mean-Value Theorem for Integrals, and taking $\triangle x\to 0$ we obtain the equation

$$c(x)\frac{\partial\theta}{\partial t} = -\frac{\partial q}{\partial x} \tag{2.1.3}$$

relating the rate of change of temperature with the gradient of the heat flux. We are ready now to make yet another assumption; a constitutive assumption which relates the heat flux to the temperature. Namely, we postulate what is known as *Fourier's Law of Cooling*, that the heat flows at the rate directly proportional to the (spatial) rate of change of the temperature. If in addition we accept that the heat flows, as commonly observed, from hot to cold we get that

$$q(x,t) = -\kappa(x)\frac{\partial\theta}{\partial x}. \tag{2.1.4}$$

where the proportionality factor $\kappa(x)>0$ is called the *thermal conductivity*. Notice the choice of the sign in the definition of the heat flux guarantees that if

the temperature is increasing with x the heat flux is negative and the heat flows from right to left, i.e., from hot to cold.

Combining (2.1.3) and (2.1.4) produces the partial differential equation

$$c(x)\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}(\kappa(x)\frac{\partial\theta}{\partial x}), \qquad 0 < x < l, \tag{2.1.5}$$

governing the heat flow in a inhomogeneous (κ is in general point dependent) one-dimensional body. However, if the bar is made of the same material throughout, whereby the heat capacity $c(x)$ and the thermal conductivity $\kappa(x)$ are point independent, (2.1.5) reduces to

$$\frac{\partial\theta}{\partial t} = \gamma\frac{\partial^2\theta}{\partial x^2}, \qquad 0 < x < l, \tag{2.1.6}$$

where

$$\gamma = \frac{\kappa}{c}. \tag{2.1.7}$$

This equation is known as the *heat equation*, and it describes the evolution of temperature within a finite, one-dimensional, homogeneous continuum, with no internal sources of heat, subject to some initial and boundary conditions. Indeed, in order to determine uniquely the temperature $\theta(x,t)$, we must specify the temperature distribution along the bar at the initial moment, say $\theta(x,0) = g(x)$ for $0 \le x \le l$. In addition, we must tell how the heat is to be transmitted through the boundaries. We already know that no heat may be transmitted through the lateral surface but we need to impose boundary conditions at the ends of the bar. There are two particularly relevant physical types of such conditions. We may for example assume that

$$\theta(l,t) = \alpha(t) \tag{2.1.8}$$

which means that the right hand end of the bar is kept at a prescribed temperature $\alpha(t)$. Such a condition is called the *Dirichlet boundary condition*. On the other hand, the *Neumann boundary condition* requires specifying how the heat flows out of the bar. This means prescribing the flux

$$q(l,t) = \kappa(l)\frac{\partial\theta}{\partial x}(l,t) = \beta(t). \tag{2.1.9}$$

at the right hand end. In particular, $\beta(t) \equiv 0$ corresponds to insulating the right hand end of the bar. If both ends are insulated we deal with the *homogeneous Neumann boundary conditions*.

REMARK 2.1. Other boundary conditions like the periodic one are also possible.

2.2. Separation of Variables

The most basic solutions to the heat equation (2.1.6) are obtained by using the *separation of variables* technique, that is, by seeking a solution in which the time variable t is separated from the space variable x. In other words, assume that

$$\theta(x,t) = T(t)u(x), \tag{2.2.1}$$

where $T(t)$ is a x-independent function while $u(x)$ is a time-independent function. Substituting the separable solution into (2.1.6) and gathering the time-dependent terms on one side and the x-dependent terms on the other side we find that the functions $T(t)$ and $u(x)$ must solve an equation

$$\frac{T'}{T} = \gamma\frac{u''}{u}. \tag{2.2.2}$$

The left hand side of equation (2.2.2) is a function of time t only. The right hand side, on the other hand, is time independent while it depends on x only. Thus, both sides of equation (2.2.2) must be equal to the same constant. If we denote the constant as $-\lambda$ and specify the initial condition

$$\theta(x,0) = u(x), \qquad 0 \le x \le l, \tag{2.2.3}$$

we obtain that

$$\theta(x,t) = e^{-\lambda t}u(x) \tag{2.2.4}$$

solves the heat equation (2.1.6) provided we are able to find $u(x)$ and λ such that

$$-\gamma u'' = \lambda u \tag{2.2.5}$$

along the bar. This is an *eigenvalue problem* for the second order differential operator $K \equiv -\gamma \frac{d^2}{dt^2}$ with the *eigenvalue* λ and the *eigenfunction* $u(x)$. The particular eigenvalues and the corresponding eigenfunctions will be determined by the boundary conditions that u inherits from θ. Once we find all eigenvalues and eigenfunctions we will be able to write the general solution as a linear combinations of basic solutions (2.2.4).

Homogeneous Boundary Conditions.

Let us consider a simple Dirichlet boundary value problem for the heat conduction in a (uniform) bar held at zero temperature at both ends, i.e.,

$$\theta(0,t) = \theta(l,t) = 0, \qquad t \geq 0, \tag{2.2.6}$$

where initially

$$\theta(x,0) = g(x), \qquad 0 < x < l. \tag{2.2.7}$$

This amounts, as we have explained earlier, to finding the eigenvalues and the eigenfunctions of (2.2.5) subject to the boundary conditions

$$u(0) = u(l) = 0. \tag{2.2.8}$$

Notice first that as evident from the form of the equation (2.2.5) the eigenvalues λ must be real. Also, it can be easily checked using the theory of second order ordinary linear differential equations with constant coefficients that if $\lambda \leq 0$, then the boundary conditions (2.2.8) yield only the trivial solution $u(x) \equiv 0$. Hence, the general solution of the differential equation (2.2.5) is a combination of trigonometric functions

$$u(x) = a \cos \omega x + b \sin \omega x \tag{2.2.9}$$

where we let $\lambda = \gamma \omega^2$ with $\omega > 0$. The boundary condition $u(0) = 0$ implies that $a = 0$. Because of the second boundary condition

$$u(l) = b \sin \omega l = 0 \tag{2.2.10}$$

ωl must be an integer multiple of π. Thus, the eigenvalues and the eigenfunctions of the eigenvalue problem (2.2.5) with boundary conditions (2.2.8) are

$$\lambda_i = \gamma \left(\frac{i\pi}{l} \right)^2, \qquad u_i(x) = \sin \frac{i\pi}{l} x, \qquad i = 1, 2, 3, \ldots. \tag{2.2.11}$$

The corresponding basic solutions (2.2.4) to the heat equation are

$$\theta_i(x,t) = \exp\left(-\frac{\gamma i^2\pi^2}{l^2}t\right)\sin\frac{i\pi}{l}x, \qquad i = 1,2,3,\ldots. \tag{2.2.12}$$

By linear superposition of these basic solutions we get a formal series

$$\theta(x,t) = \sum_{i=1}^{\infty} a_i u_i(x,t) = \sum_{i=1}^{\infty} \exp\left(-\frac{\gamma i^2\pi^2}{l^2}t\right)\sin\frac{i\pi}{l}x. \tag{2.2.13}$$

Assuming that the series converges we have a general series solution of the heat equation with the initial temperature distribution

$$\theta(x,0) = g(x) = \sum_{i=1}^{\infty} a_i \sin\frac{i\pi}{l}x. \tag{2.2.14}$$

This is a Fourier sine series on the interval $[0,l]$ of the initial condition $g(x)$[1]. Its coefficients a_i can be evaluated explicitly thanks to the remarkable *orthogonality* property of the eigenfunctions. Indeed, it is a matter of a simple exercise on integration by parts to show that

$$\int_0^l \sin\frac{k\pi}{l}x \sin\frac{n\pi}{l}x dx \neq 0 \tag{2.2.15}$$

only if $n = k$, and that

$$\int_0^l \sin^2\frac{k\pi}{l}x = \frac{l}{2}. \tag{2.2.16}$$

Multiplying the Fourier series of $g(x)$ by the k-th eigenfunction and integrating over the interval $[0,l]$ one gets that

$$a_k = \frac{2}{l}\int_0^l g(x)\sin\frac{k\pi}{l}x dx, \qquad k = 1,2,3,\ldots. \tag{2.2.17}$$

EXAMPLE 2.2. Consider the initial-boundary value problem

$$\theta(0,t) = \theta(2,t) = 0, \qquad \theta(x,0) = g(x) = \begin{cases} x, & 0 \le x \le 1, \\ -x+2, & 1 \le x \le 2, \end{cases} \tag{2.2.18}$$

for the heat equation for a homogeneous bar of length 2. The Fourier coefficients of $g(x)$ are

$$a_{2k+2} \equiv 0, \quad a_{2k+1} = (-1)^k \frac{8}{(2k+1)^2\pi^2}, \quad k = 0,1,2,\ldots. \tag{2.2.19}$$

[1]Fourier series are introduced and treated extensively in Appendix B

The resulting series solution is

$$\theta(x,t) = 8\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^2\pi^2} \exp\left(-\frac{(2i+1)^2\pi^2 t}{4}\right) \sin(i+\frac{\pi}{2})x. \tag{2.2.20}$$

Notice first that although the initial data is piecewise differentiable the solution is smooth for any $t > 0$. Also, as long as the initial profile is integrable (e.g., piecewise continuous) on $[0, 2]$ its Fourier coefficients are uniformly bounded, namely:

$$|a_k| \leq \int_0^2 |g(x) \sin k\pi x|\, dx \leq \int_0^2 |g(x)|\, dx \equiv M. \tag{2.2.21}$$

Consequently, the series solution (2.2.20) is bounded by an exponentially decaying time series

$$|\theta(x,t)| \leq M\sum_{i=0}^{\infty} \exp\left(-\frac{(2i+1)^2\pi^2 t}{4}\right). \tag{2.2.22}$$

This means that solution decays to the zero temperature profile, a direct consequence of the fact that both ends are hold at zero temperature.

This simple example shows that in the case of homogeneous boundary conditions any initial heat distributed throughout the bar will eventually dissipate away. Moreover, as the Fourier coefficients in (2.2.20) decay exponentially as $t \to \infty$, the solution gets very smooth despite the fact that the initial data was not. In fact, this is an illustration of the general *smoothing property* of the heat equation.

THEOREM 2.3. *If $u(t,x)$ is a solution to the heat equation with the initial condition such that its Fourier coefficients are uniformly bounded, then for all $t > 0$ the solution is an infinitely differentiable function of x. Also, $u(t,x) \to 0$ as $t \to \infty$, in such a way that there exists $K > 0$ such that $|u(t,x)| < Ke^{-\gamma\pi^2 t/l^2}$ for all $t \geq t_0 > 0$.*

The smoothing effect of the heat equation means that it can be effectively used to de-noise signals by damping the high frequency modes. This, however, means also that it is impossible to reconstruct the initial temperature by measuring the temperature distribution at some later time. The heat equation cannot be run backwards in time. There is no temperature distribution at $t < 0$ which would

produce a non-smooth temperature distribution at $t = 0$. Had we tried to run it backwards, we would only get noise due to the fact that the Fourier coefficients grow exponentially as $t < 0$. The backwards heat equation is *ill posed*.

Inhomogeneous Boundary Conditions.

There is a simple *homogenization transformations* that converts a homogeneous heat equation with inhomogeneous Dirichlet boundary conditions

$$\theta(0,t) = \alpha(t), \quad \theta(l,t) = \beta(t), \quad t \geq 0, \tag{2.2.23}$$

into an inhomogeneous heat equation with homogeneous Dirichlet boundary conditions. Suppose

$$\omega(x,t) = \theta(x,t) - \alpha(t) + \frac{\alpha(t) - \beta(t)}{l} x \tag{2.2.24}$$

where $\theta(x,0) = g(x)$. $\theta(x,t)$ is a solution of a homogeneous heat equation if and only if $\omega(x,t)$ satisfies the inhomogeneous equation

$$\frac{\partial \omega}{\partial t} - \frac{\partial^2 \omega}{\partial t^2} = \frac{\alpha' - \beta'}{l} x - \alpha' \tag{2.2.25}$$

subject to the initial condition

$$\omega(x,0) = g(x) - \alpha(0) + \frac{\alpha(0) - \beta(0)}{l} x, \tag{2.2.26}$$

where

$$\omega(0,t) = \omega(0,l) = 0. \tag{2.2.27}$$

Note that $\omega(x,t)$ is a solution to the homogeneous heat equation if and only if the Dirichlet boundary conditions are constant. As the homogeneous boundary conditions are essential in being able to superpose basic solutions (eigensolutions) the Fourier series method can be used now in conjunction with the separation of variables to obtain solutions of (2.2.25).

Example 2.4. Consider

$$\frac{\partial \omega}{\partial t} - \frac{\partial^2 \omega}{\partial t^2} = x \cos t, \quad 0 < x < 1, \quad t > 0, \tag{2.2.28}$$

subject to the initial condition

$$\omega(x,0) = x, \tag{2.2.29}$$

and the following homogeneous boundary conditions:

$$\omega(0,t) = \frac{\partial \omega}{\partial x}(1,t) = 0, \quad t > 0. \tag{2.2.30}$$

First, let us look for a solution of the homogeneous version of (2.2.28) with the given boundary conditions (2.2.30) using the separation of variables method. To this end the reader can easily show that the eigenfunctions are:

$$u_i(x) = \sin \lambda_i x, \quad \lambda_i = \frac{(2i+1)\pi}{2}, \quad i = 0, 1, 2 \ldots. \tag{2.2.31}$$

By the analogy with the form of the solution to the homogeneous heat equation let us suppose a solution of (2.2.28) as a series of eigenfunctions

$$\omega(x,t) = \sum_{i=0}^{\infty} \alpha_i(t) \sin \lambda_i x. \tag{2.2.32}$$

Also, represent the right-hand side of (2.2.28) as a series of eigenfunctions. Namely, write

$$x \cos t = \left(\sum_{i=0}^{\infty} b_i \sin \lambda_i x \right) \cos t, \tag{2.2.33}$$

where

$$b_i = \int_0^1 x \sin \lambda_i x dx = \frac{(-1)^i}{\lambda_i^2}. \tag{2.2.34}$$

Substituting the solution (2.2.32) with (2.2.33) for its right hand side we are able to show that the unknown functions $\alpha_i(t)$ satisfy an inhomogeneous ordinary differential equation

$$\frac{d\alpha_i}{dt} + \lambda_i^2 \alpha_i = \frac{(-1)^i}{\lambda_i^2} \cos t. \tag{2.2.35}$$

Using the method of undetermined coefficients it is easy to obtain its solution

$$\alpha_i(t) = Ae^{-\lambda_i^2 t} + (-1)^i [\frac{\cos t}{1+\lambda_i^4} + \frac{\sin t}{\lambda_i^2(1+\lambda_i^4)}]. \tag{2.2.36}$$

From the initial condition (2.2.29) and using (2.2.32) one can calculate that

$$A = \frac{(-1)^{i+1}(\lambda_i^4 - \lambda_i^2 + 1)}{\lambda_i^2(\lambda_i^4 + 1)}. \tag{2.2.37}$$

This enables us to construct the solution (2.2.32).

Periodic Boundary Conditions.

Heat flow in a circular ring is governed by the same homogeneous heat equation as is heat conduction in a rod (2.1.6), however, this time subject to *periodic boundary conditions*

$$\theta(\pi,t)=\theta(-\pi,t),\quad \frac{\partial\theta}{\partial x}(\pi,t)=\frac{\partial\theta}{\partial x}(-\pi,t),\quad t\geq 0, \tag{2.2.38}$$

where $-\pi<x<\pi$ is the angular variable, and where we assume that the heat can only flow along the ring as no radiation of heat from one side of the ring to another is permitted[2].

Benefiting from the separation of variables technique we are seeking a solution in the form $\theta(x,t)=e^{-\lambda t}u(x)$. Assuming for simplicity that $\gamma=1$, we arrive, as before, at the associated eigenvalue problem

$$\frac{d^2u}{dx^2}+\lambda u=0,\quad u(\pi)=u(-\pi),\quad u'(\pi)=u'(-\pi). \tag{2.2.41}$$

Its solutions are combinations of trigonometric sine and cosine functions

$$u_i(x)=a_i\cos ix+b_i\sin ix,\qquad i=0,1,2,\ldots, \tag{2.2.42}$$

with the eigenvalues

$$\lambda_i=i^2,\qquad i=0,1,2,\ldots. \tag{2.2.43}$$

The resulting infinite series solution is

$$\theta(x,t)=\frac{1}{2}a_0+\sum_{i=1}^{\infty}e^{-i^2t}\left[a_i\cos ix+b_i\sin ix\right]. \tag{2.2.44}$$

If we postulate the initial condition $\theta(x,0)=g(x)$ the coefficients a_i and b_i must be such that

$$g(x)=\frac{1}{2}a_0+\sum_{i=1}^{\infty}\left[a_i\cos ix+b_i\sin ix\right], \tag{2.2.45}$$

[2]The heat conduction equation for a heated ring can easily be derived from the two-dimensional heat equation

$$\frac{\partial\theta}{\partial t}=\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2} \tag{2.2.39}$$

by rewriting its right hand side in polar coordinates (r,α)

$$\frac{\partial\theta}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\theta}{\partial\alpha^2}, \tag{2.2.40}$$

and assuming that the solution is r independent.

which is precisely the Fourier series of the initial condition $g(x)$ provided

$$a_i = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos ix dx, \quad b_i = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin ix dx, \quad i = 0, 1, 2, \ldots. \quad (2.2.46)$$

2.3. Uniqueness of Solutions

In this section we investigate the uniqueness of solutions to the initial-boundary value problem for the heat equation. To this end let us consider solutions of the homogeneous heat equation

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2}, \qquad 0 < x < l, \quad 0 < t < \infty, \quad (2.3.1)$$

with the initial condition

$$\theta(x, 0) = g(x), \quad (2.3.2)$$

and the boundary conditions

$$\theta(0, t) = \alpha(t), \quad \theta(l, t) = \beta(t). \quad (2.3.3)$$

Suppose that θ_1 and θ_2 are two solutions of (2.3.1) both satisfying the initial condition (2.3.2) and boundary conditions (2.3.3). As the equation (2.3.1) is linear the function $\omega(x, t) \equiv \theta_1 - \theta_2$ is also a solution but with the zero initial profile and the homogeneous boundary conditions.

Let us multiply (2.3.1) by $\omega(x, t)$ and integrate the resulting equation with respect x on the interval $[0, l]$ to obtain

$$\int_0^l \omega\frac{\partial\omega}{\partial t}dx = \int_0^l \frac{\partial^2\omega}{\partial x^2}\omega dx. \quad (2.3.4)$$

Assuming that $\omega(x, t)$ is regular enough, and integrating the right-hand side by parts we reduce the relation (2.3.4) to

$$\frac{1}{2}\frac{d}{dt}\int_0^l \omega^2 dx = \omega\frac{\partial\omega}{\partial x}\Big|_0^l - \int_0^l \left(\frac{\partial\omega}{\partial x}\right)^2 dx = -\int_0^l \left(\frac{\partial\omega}{\partial x}\right)^2 dx \le 0. \quad (2.3.5)$$

Let

$$I(t) \equiv \frac{1}{2}\int_0^l \omega^2 dx \ge 0. \quad (2.3.6)$$

Then,

$$I(t) - I(0) = -\int_0^t\int_0^l \left(\frac{\partial\omega}{\partial x}\right)^2 dxdt \le 0. \quad (2.3.7)$$

However, $I(0) = 0$ implying that $I(t) \leq 0$. On the other hand according to its definition $I(t) \geq 0$. Hence, $I(t) \equiv 0$. This is possibly only if $\omega(x,t) \equiv 0$ proving that $\theta_1(x,t) = \theta_2(x,t)$ everywhere. Note that the same technique can be used to prove uniqueness of solutions to other boundary value problems as long as $\omega\frac{\partial\omega}{\partial x} = 0$ at $x = 0$ and $x = l$.

2.4. Fundamental Solutions

The idea of the *fundamental solution* of a partial differential equation is an extension of the Green's function method for solving boundary value problems of ordinary differential equations. To set the stage for further considerations let us briefly review the main points of the that method[3].

Consider a homogeneous boundary value problem for the linear ordinary differential equation

$$L(u) = \delta(x-\xi), \qquad u(0) = u(l) = 0, \qquad 0 < x < l, \tag{2.4.1}$$

where $L(u)$ denotes a linear second-order differential operator actig on the function $u(x)$ defined on $[0,l]$ interval, while $\delta(x-\xi) \equiv \delta_\xi(x)$ is the (Dirac) delta function at ξ. Note that if the boundary conditions are inhomogeneous we can use the homogenization transformation (2.2.24) to transform the problem into one with the homogeneous boundary conditions and a different inhomogeneous right hand side.

Let $u(x,\xi) = G(x,\xi)$ denote the solution to (2.4.1). This is the Green's function of this particular boundary value problem. Once we found this solution we can use linearity to obtain the general solution of

$$L(u) = f(x), \qquad u(0) = u(l) = 0, \qquad 0 < x < l, \tag{2.4.2}$$

in the form of the *superposition integral*. Indeed, let

$$u(x) \equiv \int_0^l G(x,\xi) f(\xi) d\xi. \tag{2.4.3}$$

[3]Details can be found in Section 1.2

It is easy to see that $u(x)$ solves the boundary value problem (2.4.2) as

$$L(u) = \int_0^l L_x(G)(x,\xi) f(\xi) d\xi = \int_0^l \delta(x-\xi) f(\xi) d\xi = f(x) \tag{2.4.4}$$

and the boundary conditions are satisfied. $L_x(u)$ denotes here the partial differential operator induced by L. We will try to use the same idea in the context of the heat equation.

Consider first the initial value problem for the heat conduction in an infinite homogeneous bar subjected initially to a concentrated unit heat source applied at a point y. We assume for simplicity that the thermal diffusivity $\gamma = 1$. This requires solving the heat equation with the initial condition

$$u(x,0) = \delta(x-y), \qquad -\infty < x < \infty. \tag{2.4.5}$$

To avoid any specific boundary conditions but to guarantee the uniqueness of solutions (see Section 2.3) we require the solution to be square integrable at all times, that is

$$\int_{-\infty}^{\infty} |u(x,t)|^2 \, dx < \infty \qquad \text{for all} \qquad t \geq 0. \tag{2.4.6}$$

This, in fact, implies that the solution vanishes at infinity.

Let us now take the complex separable solution to the heat equation

$$u(x,t) = e^{-k^2 t} e^{\mathrm{i}kx}, \tag{2.4.7}$$

where, as there are no boundary conditions, there are no restrictions on the choice of frequencies k. Mimicking the Fourier series superposition solution when there are infinitely many frequencies allowed we may combine these solutions into a Fourier integral (see Appendix B.5)

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k^2 t} e^{\mathrm{i}kx} \widehat{\delta}_y(k) dk \tag{2.4.8}$$

to realize, provided we can differentiate under the integral, that it solves the heat equation. Moreover, the initial condition is also satisfied as

$$u(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{i}kx} \widehat{\delta}_y(k) dk = \delta(x-y), \tag{2.4.9}$$

where $\widehat{\delta}_y(k)$ denotes the Fourier transform of the delta function $\delta(x-y)$, that is

$$\widehat{\delta}_y(k) = \frac{1}{\sqrt{2\pi}} e^{-\mathrm{i}ky}. \tag{2.4.10}$$

Combining (2.4.9) with (2.4.10) we find that the *fundamental solution* of the heat equation is

$$F(x-y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k^2 t} e^{\mathrm{i}k(x-y)} dk = \frac{1}{2\sqrt{\pi t}} e^{\frac{-(x-y)^2}{4t}}. \tag{2.4.11}$$

It is worth pointing out here that although the individual component of the Fourier series (2.4.8) are not square integrable the resulting fundamental solution (2.4.11) is. Another interesting derivation of the fundamental solution based on the concept of the similarity transformation can be found in [**Kevorkian**].

REMARK 2.5. It is important to point out here that one of the drawbacks of the heat equation model is - as evident from the form of the fundamental solution - that the heat propagates at infinite speed. Indeed, a very localized heat source at y is felt immediately at the entire infinite bar because the fundamental solution is at all times nonzero everywhere.

With the fundamental solution $F(x-y,t)$ at hand we can now adopt the superposition integral formula (2.4.3) to construct the solution to the heat conduction problem of an infinite homogeneous bar with the an arbitrary initial temperature distribution $u(x,0) = g(x)$ as

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4t}} g(y) dy. \tag{2.4.12}$$

That is, the general solution is obtained by a convolution of the initial data with the fundamental solution. In other words, the solution with the initial temperature profile $g(x)$ is an infinite superposition over the entire bar of the point source solutions of the initial strength

$$g(y) = \int_{-\infty}^{\infty} \delta(x-y) g(x) dx. \tag{2.4.13}$$

Inhomogeneous Heat Equation for the Infinite Bar.

The Green's function method can also be used to solve the inhomogeneous heat conduction problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = p(x,t), \quad -\infty < x < \infty, \quad t > 0, \tag{2.4.14a}$$

where the bar is subjected to a heat source $p(x,t)$ which may vary in time and along its length. We impose the zero initial condition

$$u(x,0) = 0, \tag{2.4.14b}$$

and some homogeneous boundary conditions. The main idea behind this method is to solve first the heat equation with the concentrated source applied instantaneously at a single moment, and to use the method of superposition to obtain the general solution with an arbitrary source term. We therefore begin by solving the heat equation (2.4.14a) with the source term

$$p(x,t) = \delta(x-y)\delta(t-s). \tag{2.4.15}$$

It represents a unit heat input applied instantaneously at time s and position y. We postulate the same homogeneous initial and boundary conditions as in the general case. Let

$$u(x,t) = G(x-y, t-s) \tag{2.4.16}$$

denote the solution to this problem. We will refer to it as the *general fundamental solution* or a *Green's function*. Thanks to the linearity of the heat equation the solution of the general problem is given by the superposition integral

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-y, t-s)p(y,s)dyds, \tag{2.4.17}$$

where the forcing term may be also rewritten by the superposition formula as

$$p(x,t) = \int_0^{\infty} \int_{-\infty}^{\infty} p(y,s)\delta(t-s)\delta(x-y)dyds. \tag{2.4.18}$$

If we replace the zero initial condition by $u(x,0) = f(x)$, then once again due to the linearity of the differential equation we may write the solution as a combination of a solution to the homogeneous equation with inhomogeneous initial data

and the solution with the homogeneous initial condition but a nonzero forcing term

$$u(x,t) = \int_{-\infty}^{\infty} F(x-y,t)f(y)dy + \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s)p(y,s)dyds. \quad (2.4.19)$$

To find the general fundamental solution in an explicit form let us take the Fourier transform with respect to variable x of both sides of the differential equation (2.4.14a) with the forcing term (2.4.15). Using (2.4.10) we find that

$$\frac{d\widehat{u}}{dt} + k^2\widehat{u} = \frac{1}{\sqrt{2\pi}} e^{-\mathrm{i}ky}\delta(t-s), \quad (2.4.20)$$

where $\widehat{u}(k,t)$ denotes the Fourier transform of $u(x,t)$, and where k is viewed as a parameter. This is an inhomogeneous first order linear ordinary differential equation for the Fourier transform of $u(x,t)$ with the initial condition

$$\widehat{u}(k,0) = 0 \qquad \text{for} \qquad s > 0. \quad (2.4.21)$$

Using the integrating factor method with the integrating factor $e^{k^2 t}$ we obtain that

$$\widehat{u}(k,t) = \frac{1}{\sqrt{2\pi}} e^{k^2(t-s)-\mathrm{i}ky}\sigma(t-s), \quad (2.4.22)$$

where $\sigma(t-s)$ is the usual step function. The Green's function is than obtained by the inverse Fourier transform

$$G(x-y,t-s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{i}kx}\widehat{u}(k,t)dk. \quad (2.4.23)$$

Using the formula (2.4.11) of the fundamental solution we deduce that

$$\begin{aligned} G(x-y,t-s) &= \frac{\sigma(t-s)}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{i}k(x-y)+k^2(t-s)}dk \quad (2.4.24)\\ &= \frac{\sigma(t-s)}{2\sqrt{\pi(t-s)}} \exp\left\{-\frac{(x-y)^2}{4(t-s)}\right\}. \end{aligned}$$

The general fundamental solution (Green's function) is just a shift of the fundamental solution for the initial value problem at $t = 0$ to the starting time $t = s$. More importantly, its form shows that the effect of a concentrated heat source applied at the initial moment is the same as that of a concentrated initial temperature.

Finally, the superposition integral (2.4.17) gives us the solution

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{p(y,s)}{2\sqrt{\pi(t-s)}} \exp\left\{-\frac{(x-y)^2}{4(t-s)}\right\} dsdy \tag{2.4.25}$$

of the heat conduction problem for the infinite homogeneous bar with a heat source.

Heat Equation for the Semi-infinite Bar.

To illustrate how the Green's function method can be applied in the case of the semi-infinite domain we consider the heat equation with the concentrated forcing term

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \delta(x-y)\delta(t), \qquad 0 \le x < \infty, \qquad t > 0 \tag{2.4.26a}$$

and impose the zero initial condition, i.e., $u(x,0) = 0$, and the homogeneous boundary conditions

$$u(0,t) = 0, \quad \lim_{x\to\infty} u(x,t) = 0. \tag{2.4.26b}$$

As we have remarked earlier the effect of such a concentrated instantaneous heat source is the same as that of the concentrated initial distribution. Thus, the only difference between this case and the case of the fundamental solution (2.4.5) is the imposition of the boundary condition at $x = 0$. One possible way to tackle this difficulty is to consider in place of this semi-infinite problem such an infinite domain problem in which the homogeneous "boundary" condition at $x = 0$ is permanently satisfied.

Hence, consider the heat conduction problem for an infinite homogeneous bar with $\delta(t)[\delta(x-y) - \delta(x+y)]$ as the forcing term, homogeneous initial condition, and the homogeneous boundary conditions at infinities. In other words, in the infinite domain we apply a unit strength source at $x = y$, and simultaneously a negative source of unit strength at $x = -y$. This approach is known as the *method of images*. Once again, due to the linearity of the heat equation and that of the forcing term, the temperature profile at $t > 0$ will be the sum of two fundamental solutions $F(x-y,t)$ and $F(x+y,t)$ each corresponding to one of the source terms. In particular, due to the skew-symmetry of these solutions the combined solution will always be vanishing at $x = 0$. Moreover since all the

boundary conditions of the original problem (2.4.26a) are satisfied, and since the second source term $\delta(t)\delta(x+y)$ is outside of the original semi-infinite domain, the Green's function

$$G(x-y,t) \equiv F(x-y,t) - F(x+y,t) \tag{2.4.27}$$

is the solution of (2.4.26a), where the fundamental solution F is defined by (2.4.11).

In conclusion, the solution of the inhomogeneous heat conduction problem for a semi-infinite bar

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = p(x,t), \quad 0 \le x < \infty, \quad t > 0, \tag{2.4.28}$$

with the initial condition $u(x,0) = 0$ and the homogeneous boundary conditions (2.4.26b) has the form

$$u(x,t) = \int_0^t \int_0^\infty \frac{p(y,s)}{2\sqrt{\pi(t-s)}} \left\{ \exp\left[-\frac{(x-y)^2}{4(t-s)} \right] - \exp\left[-\frac{(x+y)^2}{4(t-s)} \right] \right\} dyds. \tag{2.4.29}$$

Example 2.6. Suppose that a semi-infinite homogeneous bar is initially heated to a unit temperature along a finite interval $[a,b]$, where $a > 0$. Assume also that at $x = 0$ the temperature is held at zero (by attaching an infinite rod of this temperature) and vanishes at infinity. This corresponds to the following initial value problem for the heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = \sigma(x-a) - \sigma(x-b) = \begin{cases} 0, & \text{if } 0 < x < a, \\ 1, & \text{if } a < x < b, \\ 0, & \text{if } x > b, \end{cases} \tag{2.4.30a}$$

with the homogeneous boundary conditions

$$u(0,t) = 0, \quad \lim_{x\to\infty} u(x,t) = 0, \quad t > 0.$$

The method of images and the superposition formula (2.4.12) yield the solution

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi t}} \left(\int_a^b e^{-\frac{(x-y)^2}{4t}} dy + \int_{-a}^{-b} e^{-\frac{(x-y)^2}{4t}} \right) dy \\ &= \frac{1}{2} \left\{ \operatorname{erf}\left(\frac{x-a}{2\sqrt{t}} \right) + \operatorname{erf}\left(\frac{x+a}{2\sqrt{t}} \right) \right\} - \frac{1}{2} \left\{ \operatorname{erf}\left(\frac{x-b}{2\sqrt{t}} \right) + \operatorname{erf}\left(\frac{x+b}{2\sqrt{t}} \right) \right\}, \end{aligned} \tag{2.4.31}$$

where the *error function*

$$\text{erf}z \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-\eta^2} d\eta. \tag{2.4.32}$$

Note that the error function is odd and that its asymptotic value at infinity is 1.

2.5. Burgers' Equation

In this last section we will study the quasilinear version of the diffusion equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \epsilon \frac{\partial^2 u}{\partial x^2} = 0, \qquad \epsilon > 0, \tag{2.5.1}$$

to show how the solution methods developed in previous sections for the heat equation may be used to obtain solutions for other equations. Also, Burgers' equation is a fundamental example of an *evolution equation* modelling situations in which viscous and nonlinear effects are equally important. Moreover, it plays somewhat important role in discussing discontinuous solutions (*shocks*) of the one-dimensional conservation law

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{2.5.2}$$

a topic which will not be discussed here (see for example [**Knobel**], [**Smoller**]).

We start by looking at ways at which the methods for solving the initial-boundary value problems of heat equations can be used to solve (2.5.1).

The Cole-Hopf Transformation.

This is a change of dependent variable $w = W(u)$ which enables us to transform Burgers' equation into the linear diffusion equation studied already in this chapter. Let

$$u \equiv -2\epsilon \frac{w_x}{w}, \tag{2.5.3}$$

where w_x denotes partial differentiation. Calculating all derivatives and substituting them into (2.5.1) yields

$$w_x(\epsilon w_{xx} - w_t) - w(\epsilon w_{xx} - w_t)_x = 0. \tag{2.5.4}$$

In particular, if $w(x,t)$ solves the diffusion equation

$$\epsilon w_{xx} - w_t = 0, \tag{2.5.5}$$

the function $u(x,t)$ given by (2.5.3) satisfies Burgers' equation (2.5.1).

Initial Value Problem on the Infinite Domain.

Let us consider the following initial value problem:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \epsilon\frac{\partial^2 u}{\partial x^2} = 0, \quad u(x,0) = g(x), \quad -\infty < x < \infty, \tag{2.5.6}$$

and suppose that we are looking for the solutions which satisfy the corresponding diffusion equation (2.5.5). According to (2.5.3), the initial condition for the new variable $w(x,0)$ must be such that

$$g(x)w(x,0) = -2\epsilon w_x(x,0). \tag{2.5.7}$$

The general solution of this linear ordinary differential equation for $w(x,0)$ is

$$w(x,0) = A\exp\left[-\frac{1}{2\epsilon}\int_0^x g(s)ds\right], \tag{2.5.8}$$

where A is a constant, and where we assume that the integral exists. Hence, we essentially need to solve the following initial value problem for the homogeneous diffusion equation with the inhomogeneous initial condition:

$$\epsilon w_{xx} - w_t = 0, \quad w(x,0) = h(x), \quad -\infty < x < \infty. \tag{2.5.9}$$

Its solution has the form of (2.4.12) :

$$w(x,t) = \frac{1}{2\sqrt{\pi\epsilon t}}\int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4\epsilon t}} h(y)dy \tag{2.5.10}$$

where we replaced t by ϵt Note that the parameter ϵ may be eliminated from the equation, and so from the solution, by an appropriate scaling of variables. We retain it, however, so we one can later study the asymptotic behavior of solutions when $\epsilon \to 0$. Differentiating with respect to x and using the Cole-Hopf formula (2.5.3) we compute that

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \frac{(x-y)}{t}\exp\left[\frac{-(x-y)^2}{4\epsilon t}\right] h(y)dy}{\int_{-\infty}^{\infty}\exp\left[\frac{-(x-y)^2}{4\epsilon t}\right] h(y)dy}, \tag{2.5.11}$$

where

$$h(y) \equiv \exp\left[-\frac{1}{2\epsilon}\int_0^x g(s)ds\right], \tag{2.5.12}$$

as the constant A cancels out.

Boundary Value Problem on a Finite Interval.

Using separation of variables method, we solve here the following initial-boundary value problem:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \epsilon\frac{\partial^2 u}{\partial x^2} = 0, \quad u(x,0) = g(x), \quad 0 < x < a, \tag{2.5.13a}$$

$$u(0,t) = u(a,t) = 0, \qquad t > 0. \tag{2.5.13b}$$

After Cole-Hope transformation we obtain the corresponding initial-boundary value problem for the diffusion equation:

$$w_t - \epsilon w_{xx} = 0, \quad w(x,0) = Ah(x), \quad 0 < x < a, \tag{2.5.14a}$$

$$w_x(0,t) = w_x(a,t) = 0, \qquad t > 0. \tag{2.5.14b}$$

As the boundary condition are homogeneous the solution $w(x,t)$ can easily be derived (see page 27) as

$$w(x,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k e^{-(\frac{k\pi}{a})^2 t} \cos\frac{k\pi}{a}x, \tag{2.5.15}$$

where

$$a_k = \frac{2A}{a\pi}\int_0^a h(x)\cos\frac{k\pi}{a}x dx. \tag{2.5.16}$$

From the Cole-Hope transformation formula (2.5.3) one now gets the solution to the initial vale problem (2.5.13)

$$u(x,t) = 2\epsilon\frac{\frac{\pi}{a}\sum_{k=1}^{\infty} k a_k \exp\left[-(\frac{k\pi}{a})^2 t\right]\sin\frac{k\pi}{a}x}{\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \exp\left[-(\frac{k\pi}{a})^2 t\right]\cos\frac{k\pi}{a}x} \tag{2.5.17}$$

Example 2.7. Consider Burgers' equation

$$u_t + uu_x - \epsilon u_{xx} = 0, \qquad -\infty < x, \infty, \tag{2.5.18}$$

with the the piecewise initial condition

$$u(x,0) = 2\sigma(x) - 1 = \begin{cases} 1, & \text{if } x < 0 \\ -1, & \text{if } x > 0. \end{cases} \tag{2.5.19}$$

The initial condition of the associated diffusion equation (2.5.5) may now be obtained from (2.5.7):

$$w(x,0) = Ae^{\frac{1}{2\epsilon}|x|}. \tag{2.5.20}$$

The solution $w(x,t)$ takes the form of (2.4.12) with t replaced by ϵt. Namely,

$$w(x,t) = \frac{1}{2\sqrt{\pi\epsilon t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4\epsilon t}} e^{\frac{|y|}{2\epsilon}} dy. \tag{2.5.21}$$

Therefore, the solution of the original initial value problem is given by (2.5.11):

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \frac{(x-y)}{t} \exp\left[\frac{-(x-y)^2}{4\epsilon t}\right]) e^{\frac{|y|}{2\epsilon}} dy}{\int_{-\infty}^{\infty} \exp\left[\frac{-(x-y)^2}{4\epsilon t}\right] e^{\frac{|y|}{2\epsilon}} dy}. \tag{2.5.22}$$

Integrating independently from $-\infty$ to 0 and from 0 to ∞, and using the substitution

$$\eta = \frac{(x - y \pm t)}{2\sqrt{\epsilon t}},$$

respectively, we finally obtain

$$u(x,t) = \frac{e^{-\frac{x}{\epsilon}} \operatorname{erfc}\left(\frac{x-t}{2\sqrt{\epsilon t}}\right) - \operatorname{erfc}\left(-\frac{x+t}{2\sqrt{\epsilon t}}\right)}{e^{-\frac{x}{\epsilon}} \operatorname{erfc}\left(\frac{x-t}{2\sqrt{\epsilon t}}\right) + \operatorname{erfc}\left(-\frac{x+t}{2\sqrt{\epsilon t}}\right)}, \tag{2.5.23}$$

where the *complimentary error function*

$$\operatorname{erfc}(z) \equiv 1 - \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-\eta^2} d\eta. \tag{2.5.24}$$

CHAPTER 3

Laplace's Equation

In this chapter we discuss the solution techniques for the Laplace's and Poisson equations in two dimensions. The *Laplace's equation*

$$\triangle u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and its inhomogeneous counterpart, the *Poisson equation*

$$-\triangle u = f(x, y),$$

are one of the most fundamental partial differential equations. They arise in a variety of physical and mathematical situations, ranging from heat conduction, solid and fluid mechanics, and electromagnetism to geometry, probability, and number theory.

3.1. Physical Interpretation

In a typical interpretation u denotes the density per unit volume of some physical quantity in equilibrium (e.g. a thermal energy of a homogeneous material, like in (2.1.1), or a chemical concentration). If Ω is a smooth region in $\mathbb{R}^2$, the net flux of u through the boundary $\partial\Omega$ vanishes:

$$\int_{\partial\Omega} \mathbf{q}(x, y) \cdot \mathbf{n}\, dS = 0. \tag{3.1.1}$$

$\mathbf{q}(x, y)$ denotes here the flux density per unit surface area and $\mathbf{n}$ is the unit outer normal. Applying the Gauss-Green Theorem, we have

$$\int_{\partial\Omega} \mathbf{q}(x, y) \cdot \mathbf{n}\, dS = \int_{\Omega} \operatorname{div} \mathbf{q}(x, y)\, dV = 0,$$

and so

$$\operatorname{div} \mathbf{q}(x, y) = 0 \quad \text{in} \quad \Omega, \tag{3.1.2}$$

since the region Ω is arbitrary. Assuming, what is often physically reasonable, that the flux $\mathbf{q}$ is proportional to the gradient ∇u, but points in the opposite direction (the flow is from the region of higher to lower concentration)[1], we obtain Laplace's equation

$$\operatorname{div}(\nabla u) = 0. \tag{3.1.3}$$

3.2. Separation of Variables

In order to construct an explicit solution to the Laplace's equation, we resort first to the method of *separation of variables*.

We start by writing a solution ansatz in the form

$$u(x, y) = w(x)v(y). \tag{3.2.1}$$

Let us substitute this expression into the Laplace's equation

$$\triangle u = w''(x)v(y) + w(x)v''(y) = 0$$

and re-write it in the form

$$w''(x)v(y) = -w(x)v''(y).$$

Dividing both sides by the product $w(x)v(y)$, while assuming that it is not identically zero, yields

$$\frac{w''(x)}{w(x)} = -\frac{v''(y)}{v(y)} \tag{3.2.2}$$

effectively separating the x and y variables on the opposite sites of the equation. Assuming that the domain on which the equation is considered, say Ω, is connected leads to the obvious conclusion that for the separation to work both sides must be constant functions. This reduces our problem to a pair of second order linear ordinary differential equations

$$w'' - \lambda w = 0, \quad v'' + \lambda v = 0, \tag{3.2.3}$$

for the individual components of the separable solution $u(x, y) = w(x)v(y)$ of the Laplace's equation $\triangle u = 0$. The undetermined constant λ is called the *separation constant*. These are relatively simple linear ordinary differential equations. We know how to solve them, and what form of a solution to expect depending on

[1]Some such examples are Fick's Law of Diffusion and Fourier's Law of Cooling.

the sign of the separation constant λ. Indeed, if $\lambda \neq 0$ the solution will be the combinations of products of exponential functions with sine and cosine functions. On the other hand, when $\lambda = 0$ the solutionis a second degree polynomial in x and y.

Applying the separation of variables method to the boundary value problems forces us to consider rather restricted geometry of the domain of solutions. Thus, let us consider the Laplace's equation on a rectangle, e.g.,

$$\triangle u = 0 \quad \text{on} \quad D = \{0 < x < a, 0 < y < b\}, \tag{3.2.4a}$$

with Dirichlet boundary conditions

$$u(x,0) = f(x), \quad u(x,b) = u(0,y) = u(a,y) = 0. \tag{3.2.4b}$$

These boundary conditions are not the most general boundary conditions one might like to consider. However, observe that an arbitrary inhomogeneous Dirichlet boundary condition can always be decomposed into four different boundary conditions like (3.2.4b), and use the superposition of solutions - all due to the linearity of the Laplace's equation. Proceeding to solve our boundary value problem (3.2.4b) let us look first at all the homogeneous conditions. To this end, we immediately obtain that

$$w(0) = w(a) = 0 \quad \text{and} \quad v(b) = 0. \tag{3.2.5}$$

Standard analysis of the associated eigenvalue problems (3.2.3) shows that the only possible solutions are the products

$$u_n(x,y) = \sin\frac{n\pi x}{a}\sinh\frac{n\pi(b-y)}{a}, \quad n = 1,2,3,\ldots, \tag{3.2.6}$$

where the separation constants

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2, \quad n = 1,2,3,\ldots. \tag{3.2.7}$$

It remains now to analyze the role of the inhomogeneous boundary condition $u(x,0) = f(x)$. As none of the $u_n(x,y)$ functions could, in general, satisfy such an arbitrary condition we try a linear superposition of solutions in the form of

an infinite series

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi(b-y)}{a}, \tag{3.2.8}$$

where the coefficients c_n are to be determined from the boundary condition at $y = 0$. Indeed, we have that

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi b}{a} = f(x). \tag{3.2.9}$$

This has the form of the Fourier sine series for the function $f(x)$. Let us therefore take

$$c_n = \frac{b_n}{\sinh\frac{n\pi b}{a}} \tag{3.2.10}$$

where b_n are the Fourier sine coefficients on the interval $(0,a)$ of $f(x)$ as given by the formula (B.1.2). This yields that the solution to the boundary value problem (3.2.4) is the infinite series

$$u(x,y) = \sum_{n=1}^{\infty} b_n \frac{\sinh\frac{n\pi(b-y)}{a}}{\sinh\frac{n\pi b}{a}} \sin\frac{n\pi x}{a}. \tag{3.2.11}$$

In can be shown that if the function $f(x)$ is integrable on $(0,a)$, i.e.,

$$\int_0^a |f(x)|dx < \infty, \tag{3.2.12}$$

the series converges on the rectangle D.

The Unit Disk.

As the illustration of how the method of separation of variables can be used in some other geometric situations, let us consider the boundary value problem for the Laplace's equation on the unit disk. Namely, let

$$\triangle u = 0 \quad \text{on} \quad x^2 + y^2 < 1, \tag{3.2.13a}$$

and

$$u(x,y) = f(x,y) \quad \text{whenever} \quad x^2 + y^2 = 1. \tag{3.2.13b}$$

One possible physical interpretation of what the solution $u(x,y)$ might represent is the displacement of a circular drum of a given hight; the (circumferential) hight of the drum cannot change but the membrane of the drum can deform.

When dealing with a circular geometry it seems natural to adopt polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{3.2.14}$$

Rewriting the boundary conditions (3.2.13b) we obtain that

$$u(\cos\theta, \sin\theta) = f(\cos\theta, \sin\theta) = f(\theta). \tag{3.2.15}$$

Elementary calculations yield the Laplacian in *polar coordinates*

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}. \tag{3.2.16}$$

This enables us to reformulate our boundary value problem (3.2.13) for the function $u = u(r, \theta)$ in the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad u(1, \theta) = f(\theta). \tag{3.2.17}$$

Observe that we must also require that the solution $u(r, \theta)$, as well as the boundary condition $f(\theta)$, are 2π periodic, i.e.,

$$u(r, \theta + 2\pi) = u(r, \theta), \quad f(\theta + 2\pi) = f(\theta). \tag{3.2.18}$$

The method of separation of variables is based on the postulate that

$$u(r, \theta) = v(r)\mu(\theta). \tag{3.2.19}$$

Substituting (3.2.19) into the polar version of the Laplace's equation (3.2.17), we obtain that

$$v''(r)\mu(\theta) + \frac{1}{r}v'(r)\mu(\theta) + \frac{1}{r^2}v(r)\mu''(\theta) = 0. \tag{3.2.20}$$

Multiplying both sides of the above equation by r^2 and dividing by the product $v\mu$ we are able to move all terms involving r to one side of the equation and all the terms involving θ to the other side. Hence, the equivalent form of the equation (3.2.20) is

$$\frac{r^2 v''(r) + r v'(r)}{v(r)} = -\frac{\mu''(\theta)}{\mu(\theta)}. \tag{3.2.21}$$

This, as we know from our previous similar considerations, reduces to a pair of ordinary differential equations

$$r^2 v'' + r v' - \lambda r = 0, \quad \mu'' + \lambda\mu = 0, \tag{3.2.22}$$

for the respective components of the separable solution $v\mu$. The separation constant λ plays here the role of the eigenvalue. The latter equation, with the added periodicity requirement (3.2.18), was solved in Section 2.2, where we showed that the eigenvalues are

$$\lambda = n^2, \quad n = 0, 1, 2, \dots, \tag{3.2.23}$$

with the corresponding eigenfunctions

$$1, \quad \sin n\theta, \quad \cos n\theta. \tag{3.2.24}$$

The first differential equation, on the other hand, has the form of a second order *Euler equation* for the function $v(r)$, where $\lambda = n^2$. Its linearly independent solutions are usually obtained by seeking the power solution $v(r) = r^k$. The resulting *characteristic equation* requires

$$k^2 - n^2 = 0.$$

Hence,

$$k = \pm n.$$

If $n \neq 0$ there are two linearly independent solutions,

$$v_1(r) = r^n, \quad v_2(r) = r^{-n}, \quad n = 1, 2, \dots. \tag{3.2.25}$$

If, on the other hand, $n = 0$ we get two other solutions[2]

$$v_1(r) = 1, \quad v_2(r) = \log r. \tag{3.2.26}$$

Since we require our solution $u(r, \theta)$ to be bounded and continuous - even at the center of the disk - we are forced to discard these elementary solutions which are singular when $r \to 0$. Thus, the series solution takes the form

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left[a_n \cos n\theta + b_n \sin n\theta \right]. \tag{3.2.27}$$

Taking into account the boundary condition $u(1, r) = f(\theta)$ we conclude that

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta, \tag{3.2.28}$$

[2]These solutions can easily be obtained by the reduction of order method.

are precisely the Fourier coefficients (B.1.2) of the boundary value $f(\theta)$. If $f(\theta)$ is integrable on $(-\pi, \pi)$ then, similarly to the rectangle problem, it can be showed that the series solution (3.2.27) converges uniformly to the solution of our boundary value problem. Moreover, such a solution is analytic inside the disk.

EXAMPLE 3.1. Consider the boundary value problem (3.2.17) where

$$u(1, \theta) = \theta, \quad -\pi < \theta < \pi.$$

Observe that the boundary condition experiences a jump at $(x, y) = (-1, 0)$. The Fourier series of $f(\theta) = \theta$ is

$$2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin n\theta}{n}.$$

Therefore, according to (3.2.27) the solution to our boundary value problem is

$$u(r, \theta) = 2\sum_{n=1}^{\infty}(-1)^{n+1}r^n\frac{\sin n\theta}{n}. \tag{3.2.29}$$

This series can be explicitly summed. Indeed, write

$$z^n = r^n e^{in\theta} = r^n \cos n\theta + ir^n \sin n\theta \tag{3.2.30}$$

for the complex variable $z = x + iy$ and represent the solution

$$u(x, y) = 2\mathrm{Im}\left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n}\right) = 2\mathrm{Im}\log(1 + z). \tag{3.2.31}$$

If we now write $1 + z = \varrho e^{i\psi}$ then the solution takes the form

$$u(x, y) = 2\psi = 2\arctan\frac{y}{1 + x}, \tag{3.2.32}$$

where ψ is the angle between the line through (x, y) and $(-1, 0)$, and the x-axis. One can easily show that the solution given by (3.2.32) has the correct boundary value.

3.3. The Green's Function

Consider the Poisson equation

$$-\triangle u = f(x, y), \tag{3.3.1}$$

that is, the inhomogeneous counterpart of the Laplace's equation. As in the one-dimensional case, we will use the Green's function method to solve relevant boundary value problems. By the Green's function we mean, the same way as before, the solution in which the inhomogeneity $f(x, y)$ (the forcing term) is a concentrated unit impulse; the delta function. The solution to the equation with the general forcing term is then obtained by using the method of superposition.

We denote the delta function concentrated at position $\xi \in \mathbb{R}^{\mathbf{2}}$ by

$$\delta_\xi(\mathbf{x}) \equiv \delta(\mathbf{x} - \xi). \tag{3.3.2}$$

As in the one-dimensional case, we interpret it as a linear functional on the space $C(\Omega)$ of all continuous scalar-valued function on the domain $\Omega \subset \mathbb{R}^2$, satisfying

$$\int\int_\Omega \delta_\xi(\mathbf{x}) f(\mathbf{x}) d\xi = \begin{cases} f(\xi), & \xi \in \Omega \\ 0, & \xi \notin \overline{\Omega} \end{cases} \tag{3.3.3}$$

for any function $f(\mathbf{x})$ and any domain Ω. If we view the delta function as the product of two one-dimensional delta functions

$$\delta_\xi(\mathbf{x}) = \delta_\xi(x)\delta_\eta(y), \quad \xi = (\xi, \eta), \quad \mathbf{x} = (x, y), \tag{3.3.4}$$

and if the domain Ω is a rectangle, say $\Omega = \{a < x < b, c < y < b\}$, then

$$\int\int_\Omega \delta_\xi(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} = \int_a^b \int_c^d \delta_\xi(x)\delta_\eta(y) f(x, y) dy dx = \int_a^b \delta_\xi(x) f(x, \eta) dx = f(\xi, \eta), \tag{3.3.5}$$

provided $(\xi, \eta) \in \Omega$. If $(\xi, \eta) \notin \overline{\Omega}$ the integral is 0.

Alternatively, the delta function can be viewed as the limit of concentrated unit densities $g_n(x, y)$ such that

$$\lim_{n\to\infty} g_n(x, y) = 0, \quad \text{for} \quad (x, y) \neq (0, 0), \tag{3.3.6a}$$

and

$$\int\int_\Omega g_n(x, y) dx dy = 1. \tag{3.3.6b}$$

A good example of such a sequence are the so-called *radial Gauss distributions*

$$g_n(x,y) = \frac{e^{\frac{-(x^2+y^2)}{n}}}{n\pi}. \tag{3.3.7}$$

We leave to the reader to show that

$$\int\int_{\mathbb{R}^2} e^{-(x^2+y^2)/n} dxdy = n\pi.$$

Finding the Green's function amounts to solving the equilibrium problem subject to a concentrated forcing. For the Poisson equation, this takes the form

$$-\triangle u = \delta_\xi(x)\delta_\eta(y), \quad (x,y) \in \Omega \tag{3.3.8}$$

supplemented by homogeneous boundary conditions. As we shall see later, the only acceptable boundary conditions are pure Dirichlet or mixed boundary conditions as the pure Neumann boundary value problem has a non unique solution. The Green's function solution to (3.3.8) - with appropriate boundary condition - is usually denoted as

$$G(\mathbf{x},\xi) = G(x,y;\xi,\eta). \tag{3.3.9}$$

As in the one-dimensional case, the self-adjointness of the boundary value problem implies that the Green's function is symmetric under the change of pairs of arguments, i.e.,

$$G(x,y;\xi,\eta) = G(\xi,\eta;x,y). \tag{3.3.10}$$

Once the Green's function is available, the solution to the general Poisson boundary value problem

$$-\triangle u = f(x,y), \quad u_{|\partial\Omega} = 0, \tag{3.3.11}$$

is constructed by using the superposition principle. In other words, representing

$$f(x,y) = \int\int_\Omega \delta_x(\xi)\delta_y(\eta) f(\xi,\eta) d\xi d\eta \tag{3.3.12}$$

and using the linearity the solution to the boundary value problem (3.3.11) is given by the superposition. That is, regarding the forcing term as the combination of point impulses, and the Green's functions as the corresponding solutions, we write the solution as

$$u(x,y) = \int\int_\Omega G(x,y;\xi,\eta) f(\xi,\eta) d\xi d\eta. \tag{3.3.13}$$

It is rather unfortunate that most Green's functions cannot be evaluated explicitly. However, using a bit of intuition and the fact that the general solution to the inhomogeneous linear differential equation is a combination of a particular solution and the general solution to its homogeneous counterpart, we shall be able to gain some inside into the form of the Green's function of the Poisson equation.

Let us, once again, consider the Poisson equation with the forcing unit impulse

$$-\triangle u = \delta(x-\xi, y-\eta). \tag{3.3.14}$$

The general solution to such a linear inhomogeneous equation can always be written as

$$u(x,y) = u_p(x,y) - u_h(x,y), \tag{3.3.15}$$

where u_h denotes an arbitrary *harmonic function* (the general solution to the Laplace's equation) while u_p is a particular solution to the Poisson equation, irrespective of any boundary conditions. As usually, we shall use the solution u_h to accommodate any particular boundary conditions imposed.

We propose as a particular solution

$$u_p(x,y) = \frac{a}{2}\log[(x-\xi)^2 + (y-\eta)^2]. \tag{3.3.16}$$

First of all this function is radially symmetric about the point at which the unit external impulse is applied, a characteristic expected from a solution modelling a homogeneous medium. Also, as we have discovered earlier in (3.2.26) it solves the Laplace's equation everywhere except at (ξ, η), at which it exhibits a singularity. If this function is to work as a particular solution to our inhomogeneous problem we must be able to determine the correct constant a. In other words, we must find a such that

$$-\triangle \log[(x-\xi)^2 + (y-\eta)^2] = \frac{2}{a}\delta(x-\xi, y-\eta). \tag{3.3.17}$$

Interpreting the delta function as a linear functional on a space of sufficiently regular functions $g(x,y)$ with L^2-inner product allows us to evaluate the inner product

$$< -\triangle \log[(x-\xi)^2 + (y-\eta)^2]; g >= \frac{2}{a} < \delta(x-\xi, y-\eta); g > . \tag{3.3.18}$$

By the definition of the delta function (3.3.3), the right hand side equals to $g(\xi,\eta)$, . Remembering that $\log[(x-\xi)^2+(y-\eta)^2]$ satisfies Laplace's equation everywhere but at the center, the right hand side

$$\begin{aligned}
&< -\triangle \log[(x-\xi)^2+(y-\eta)^2]; g > \\
&= \int\int_{D_\epsilon} (-\triangle \log[(x-\xi)^2+(y-\eta)^2]) g(x,y) dxdy \\
&= g(\xi,\eta) \int\int_{D_\epsilon} (-\triangle \log[(x-\xi)^2+(y-\eta)^2]) dxdy, \qquad (3.3.19)
\end{aligned}$$

where D_ϵ is a small disk of radius ϵ centered at (ξ,η). Using the fact that $-\triangle = -\nabla\cdot\nabla$, and the Green's formula (3.4.6) we get that

$$\begin{aligned}
&\int\int_{D_\epsilon} \triangle \log[(x-\xi)^2+(y-\eta)^2] dxdy = \int\int_{D_\epsilon} (\nabla\cdot\nabla \log[(x-\xi)^2+(y-\eta)^2] dxdy \\
&= \oint_{\partial D_\epsilon} \frac{\partial \log[(x-\xi)^2+(y-\eta)^2]}{\partial \mathbf{n}} ds = \oint_{\partial D_\epsilon} \frac{1}{r} ds = \int_0^{2\pi} d\theta = 2\pi. \qquad (3.3.20)
\end{aligned}$$

When evaluating the one before last integral we benefited from the fact that the arc length on a circle of radius ϵ is $ds = \epsilon d\theta$. The last result shows that

$$< -\triangle \log[(x-\xi)^2+(y-\eta)^2]; g >= -2\pi g(\xi,\eta), \qquad (3.3.21)$$

proving that

$$-\triangle \log[(x-\xi)^2+(y-\eta)^2] = -2\pi\delta(x-\xi, y-\eta). \qquad (3.3.22)$$

This allows us to use

$$u_p(x,y) = -\frac{1}{4\pi} log[(x-\xi)^2+(y-\eta)^2] \qquad (3.3.23)$$

as a particular solution of the inhomogeneous Poisson equation (3.3.14). The general solution to the equation (3.3.14) is

$$u(x,y) = -\frac{1}{4\pi} log[(x-\xi)^2+(y-\eta)^2] + u_h(x,y), \qquad (3.3.24)$$

where $u_h(x,y)$ is an arbitrary harmonic function. Note that postulating Dirichlet boundary conditions forces the solution u_h to satisfy

$$u_h(x,y) = \frac{1}{2\pi} log[(x-\xi)^2+(y-\eta)^2] \quad \text{at all} \quad (x,y)\in\partial\Omega. \qquad (3.3.25)$$

In conclution, we have just showed that

THEOREM 3.2. *The Greens's function for any positive definite boundary value problem for the Poisson equation has the form*

$$G(x,y;\xi,\eta) = -\frac{1}{4\pi} log[(x-\xi)^2 + (y-\eta)^2] + u_h(x,y) \tag{3.3.26}$$

where $u_h(x,y)$ is the harmonic function that has the same boundary value as the logarithmic potential $\frac{1}{2\pi} log[(x-\xi)^2 + (y-\eta)^2]$.

3.4. Minimum Principles

The main goal of this section is to show that the solution to the Poisson equation (3.3.1), subject to proper boundary conditions, is the unique minimizer of the corresponding "energy" functional. This requires presenting the Poisson equation in the self-adjoint form, relative to the appropriate inner product and suitable boundary conditions.

Notice first that the Laplacian

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

as a linear operator on a scalar-valued function $u(x,y)$, can be viewed as a superposition of two operations: the gradient and the divergence. Indeed, using the multi-variable calculus presentation of the divergence as a formal dot product of the gradient operator with the vector field, i.e.,

$$\nabla \cdot (v_1, v_2) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y},$$

we obtain that

$$\nabla \cdot \nabla u = \nabla \cdot (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

This very fact, and the fact that in one-dimension the adjoint of the derivative is its negative (1.3.5), suggest that the adjoint of the gradient is minus the divergence.

The gradient operator $\nabla : U \to \mathbf{V}$ maps a suitable space of scalar-valued functions (*scalar fields*) U into a space of vector-valued functions (*vector fields*)

$\mathbf{V}$. According to the original definition of the adjoint (1.3.1), the operator $\nabla^* : \mathbf{V} \to U$ is the adjoint of the gradient if

$$< u; \nabla^* \mathbf{v} >=\ll \nabla u; \mathbf{v} \gg, \tag{3.4.1}$$

for all $u \in U$, $\mathbf{v} \in \mathbf{V}$, where $< \cdot;\cdot >$, and $\ll \cdot;\cdot \gg$ denote the inner products in U and $\mathbf{V}$, respectively. We assume that both inner products refer to L^2-inner products. Namely,

$$< u; w >\equiv \int\int_\Omega u(x,y)w(x,y)dxdy, \quad \ll \mathbf{v}; \mathbf{w} \gg\equiv \int\int_\Omega \mathbf{v}(x,y)\cdot\mathbf{w}(x,y)dxy. \tag{3.4.2}$$

The right hand side of the identity (3.4.1) calculates as

$$\ll \nabla u; \mathbf{v} \gg= \int\int_\Omega (\nabla u \cdot \mathbf{v})dxdy. \tag{3.4.3}$$

Recalling *Green's formula*

$$\int\int_\Omega \nabla\cdot(u\mathbf{v})dxdy = \oint_{\partial\Omega} u(\mathbf{v}\cdot\mathbf{n})ds, \tag{3.4.4}$$

where $\mathbf{n}$ is the outer normal to the boundary $\partial\Omega$, and *Leibniz's rule*

$$\nabla\cdot(u\mathbf{v}) = u\nabla\cdot\mathbf{v} + \nabla u\cdot\mathbf{v}, \tag{3.4.5}$$

we conclude that

$$\int\int_\Omega (\nabla u\cdot\mathbf{v})dxdy = \oint_{\partial\Omega} u(\mathbf{v}\cdot\mathbf{n})ds - \int\int_\Omega u(\nabla\cdot\mathbf{v})dxdy. \tag{3.4.6}$$

If the choice of scalar and vector fields, u and $\mathbf{v}$, is such that the boundary integral vanishes,

$$\oint_{\partial\Omega} u(\mathbf{v}\cdot\mathbf{n})ds = 0, \tag{3.4.7}$$

then according to (3.4.3) and (3.4.6)

$$\ll \nabla u; \mathbf{v} \gg= -\int\int_\Omega u(\nabla\cdot\mathbf{v})dxdy =< u; -\nabla\cdot\mathbf{v} > . \tag{3.4.8}$$

This shows that, subject to the boundary constraint (3.4.7), the adjoint of the gradient is minus the divergence, $\nabla^* = -\nabla\cdot$. What remains to be established is what boundary conditions on u and/or $\mathbf{v}$ guarantee the vanishing of the boundary

integral (3.4.7). Clearly the boundary integral vanishes if u satisfies *Dirichlet boundary conditions*, that is if

$$u \equiv 0 \quad \text{on} \quad \partial\Omega. \tag{3.4.9}$$

This can be interpreted, for example, as that the membrane of a drum is attached rigidly to the edge of the drum. In heat conduction, this boundary condition corresponds to fixing the temperature at the boundary of the region Ω. We may also require that

$$\mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega, \tag{3.4.10}$$

which simply states that $\mathbf{v}$ is tangent to $\partial\Omega$ at each point, and so there is no flux through the boundary.

In the context of the Poisson equation we choose $\mathbf{v} = \nabla u$, and the boundary condition (3.4.10) translates into the *Neumann boundary conditions*

$$\nabla u \cdot \mathbf{n} = \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega. \tag{3.4.11}$$

When thermal equilibrium problem is considered, this means that that Ω is thermally insulated. In the case of a membrane, the Neumann conditions represent the edge of the membrane left free. Both types of boundary conditions can obviously be mixed so that on one part of the boundary, say N, the Neumann conditions are imposed, while on the other part D, disjoint from N, but such that $\partial\Omega = N \cup D$, the Dirichlet conditions are assumed. Moreover, all these boundary conditions can be generalized to the inhomogeneous form, i.e., a fixed displacement (temperature) at the boundary and/or prescribing a flux through the boundary.

To complete our presentation of the boundary value problem for the Poisson equation as a self-adjoint problem we must determine whether or not the boundary value problem is positive definite. We already know that the operator

$$-\triangle = \nabla^* \circ \nabla : U \to U$$

is self-adjoint. To show that it is positive definite we must identify these boundary conditions for which, according to (1.3.11), $\nabla u = 0$ if and only if $u \equiv 0$. Relaying on the elementary fact that the gradient ∇u of a C^1 function $u(x, y)$ defined on a connected domain Ω vanishes if and only if u is constant, we establish

that only the homogeneous Dirichlet or homogeneous mixed boundary conditions are acceptable. Indeed, only these boundary conditions are forcing a constant function, defined on a connected domain, to vanish identically.

In conclusion, we have properly formulated the Poisson and Laplace's equations in positive definite self-adjoint form

$$-\triangle u = \nabla^* \circ \nabla u = f,$$

including either homogenous Dirichlet or homogeneous mixed boundary conditions. Hence, we can present now the minimization principle for the Poisson and Laplace's equations.

THEOREM 3.3. *Let $u(x, y)$ be the unique minimizer of the Dirichlet integral*

$$\mathfrak{D}[u] \equiv \frac{1}{2}\|\nabla u\|^2 - < u; f >= \int\int_\Omega (\frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 - fu)dxdy \tag{3.4.12}$$

among all C^1 functions that satisfy either homogeneous Dirichlet or mixed boundary conditions. Then, $u(x, y)$ is the solution to the Poisson equation (3.3.1).

Although we are not going to prove the Dirichlet minimization principle, we will prove here, using a version of Green's formula, that any solution to our boundary value problem for the Poisson equation is unique. First, we show that the only harmonic function satisfying the homogeneous Dirichlet or mixed boundary conditions is the zero function. Starting from (3.4.6) and assuming that $\mathbf{v} = \nabla u$ we obtain that

$$\int\int_\Omega [u\triangle u + \|\nabla u\|^2]dxdy = \oint_{\partial\Omega} u\frac{\partial u}{\partial \mathbf{n}}ds.$$

Supposing that u solves Laplace's equation and either itself vanishes or its normal derivative vanishes on $\partial\Omega$, we deduce from the previous identity that

$$\int\int_\Omega \|\nabla u\|^2 dxdy = 0.$$

As the integrant is nonnegative and continuous, the only way for the integral to vanishes is if $\|\nabla u\|^2 \equiv 0$. This implies that $\nabla u = 0$. Therefore, u (a C^1 function) must be constant. Hence, for the trivial boundary conditions and connected domain Ω, the only possible choice is $u \equiv 0$.

Suppose now that u_1 and u_2 are two solutions to the same inhomogeneous boundary value problem for the Poisson equation. Their difference $u = u_1 - u_2$ satisfies homogeneous boundary conditions and, by linearity, solves the Laplace's equation. As we have just established, $u = 0$, and hence the solutions to the Poisson equation are the same, $u_1 = u_2$.

REMARK 3.4. It should be stressed here that Theorem 3.3 does not guarantee the existence of the minimizer. It only states that if the minimizer exists then its solves the given boundary value problem.

REMARK 3.5. The Dirichlet integral (3.4.12) is minimized under the assumption that the boundary conditions are homogeneous. In fact, the minimization principle also applies to inhomogeneous boundary conditions (all due to the fact that the kernel of the gradient operator is trivial). When the inhomogeneous Neumann boundary conditions are considered, the minimization principle can also be applied but with some modification. Namely, the Dirichlet integral must be supplemented by

$$\int_N uq\, ds, \quad \text{where} \quad \frac{\partial u}{\partial \mathbf{n}} = q \quad \text{on} \quad N \subset \partial\Omega. \tag{3.4.13}$$

However, the Neumann boundary value problem, in contrast with the Dirichlet and the mixed boundary value problems, is not positive definite. As we pointed out in the one-dimensional case - Section 1.3 - the solution is not unique, i.e., if $u(x, y)$ is a solution, so is $u(x, y) + c$, for any constant c. This contributes to the instability of the problem. If we view $u(x, y)$ as the vertical displacement of the membrane, it takes no force to move the entire membrane up or down retaining its equilibrium status. In the case of thermal equilibrium, when the Nemannn conditions prescribe the heat flux through the boundary, the same heat flux can be observed at different levels of temperature.

CHAPTER 4

Vibrations of One-Dimensional Medium

4.1. The Wave Equation

We shall consider now Newton's Law of motion for a one-dimensional medium, say a string. This means that we will be investigating solutions to a partial differential equation

$$\varrho(x)\frac{\partial^2 u}{\partial t^2} = -K[u] = \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial u}{\partial x}\right), \quad 0 < x < l, \tag{4.1.1}$$

where $\kappa(x) > 0$ is a material characteristic such as the tension at x of the transversally vibrating string or a stiffness of a longitudinally stretched bar. In fact, equation (4.1.1) describes a number of other phenomena such as sound waves in a column of air or water waves.

Since our equation is a partial differential equation of second order in time we must, in order to get a unique solution, provide the initial displacement as well as the initial velocity:

$$u(0,x) = u_0(x), \quad \frac{\partial u}{\partial t}(0,x) = v_0(x). \tag{4.1.2}$$

Moreover, we must impose some suitable boundary conditions. This means selecting such boundary conditions that the underlying (spatial) differential operator

$$K = -D(\kappa(x)D) = \frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial u}{\partial x}\right)$$

is self-adjoint. For example, homogeneous Neumann boundary conditions correspond to a free end while a Dirichlet boundary conditions describe a fixed end.

Assuming that the medium is homogeneous implies that both the density and the tension are position independent. This simplifies the equation (4.1.1) so that we obtain the one-dimensional *wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \tag{4.1.3}$$

where the constant

$$c = \sqrt{\frac{\kappa}{\varrho}}. \tag{4.1.4}$$

As in all the instances considered so far, let us try first a separable solution

$$u(t, x) = v(x) \cos \omega t. \tag{4.1.5}$$

Substituting it into the wave equation (4.1.3) and cancelling common factors we obtain the following ordinary differential equation

$$c^2 \frac{d^2 v}{dx^2} + \omega^2 v = 0. \tag{4.1.6}$$

If $\omega \neq 0$, the solutions are the trigonometric functions

$$\cos \frac{\omega}{c} x, \quad \sin \frac{\omega}{c} x.$$

The corresponding solutions to the wave equation are

$$\cos \omega t \cos \frac{\omega}{c} x, \quad \cos \omega t \sin \frac{\omega}{c} x.$$

Two additional solutions can be obtained by replacing cosine by the sine function in the original ansatz (4.1.5). These four fundamental solutions represent standing $2\pi c/\omega$-periodic wave solutions vibrating with the frequency ω. Due to the linearity of the wave equation, we may use now the superposition principle to write the general solution in the form of the trigonometric Fourier series. The boundary conditions, which have been neglected so far, will specify particular eigenvalues and the corresponding fundamental frequencies.

To this end, consider a string of a fixed length l with both ends fixed. In other words, let us consider the Dirichlet boundary conditions

$$u(t, 0) = u(t, l) = 0.$$

In this case, the eigenvalues and the eigenfunctions of the eigenvalue problem (4.1.6) are

$$\omega_n = \frac{n\pi c}{l}, \quad v_n(x) = \sin \frac{n\pi x}{l}, \quad n = 1, 2, 3, \dots$$

Therefore, the general solution takes the form

$$u(t, x) = \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} + b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \right]. \tag{4.1.7}$$

This can be re-written as

$$\sum_{n=1}^{\infty} d_n \cos\left(\frac{n\pi ct}{l} + \delta_n\right) \sin\frac{n\pi x}{l}. \tag{4.1.8}$$

Thus, the solution is a linear combination of the various modes vibrating with the frequencies

$$\omega_n = \frac{n\pi c}{l} = \frac{n\pi}{l}\sqrt{\frac{\kappa}{\varrho}}, \quad n = 1, 2, 3, \ldots. \tag{4.1.9}$$

The Fourier coefficients a_n and b_n can be uniquely determined by the initial condition (4.1.2). Indeed, differentiating our series solution term by term, we obtain that

$$u(0,x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} = u_0(x), \quad u_t(0,x) = \sum_{n1}^{\infty} b_n \frac{n\pi c}{l} \sin\frac{n\pi x}{l} = v_0(x). \tag{4.1.10}$$

The corresponding Fourier coefficients of the initial displacement $u_0(x)$, are

$$a_n = \frac{2}{l}\int_0^l u_0(x) \sin\frac{n\pi x}{l} dx, \quad n = 1, 2, 3, \ldots$$

while the scaled Fourier coefficients of the initial velocity $v_0(x)$ take the form

$$b_n = \frac{2}{n\pi c}\int_0^l v_0(x) \sin\frac{n\pi x}{l} dx, \quad n = 1, 2, 3, \ldots.$$

4.2. d'Alembert's Solution

Let first re-write the wave equation (4.1.3) as

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = (\partial_t^2 - c^2\partial_x^2)u = 0, \tag{4.2.1}$$

where a subscript indicates partial differentiation. Presenting the *wave operator*

$$\Box \equiv \partial_t^2 - c^2\partial_x^2 \tag{4.2.2}$$

as a product (superposition) of two first order linear differential operator

$$\Box u = (\partial_t^2 - c^2\partial_x^2)u = (\partial_t + c\partial_x)(\partial_t - c\partial_x)u \tag{4.2.3}$$

suggests that any solution to the simpler first order equation

$$(\partial_t \pm c\partial_x)u = u_t \pm cu_x = 0 \tag{4.2.4}$$

is a solution to the wave equation (4.1.3). Equations (4.2.4) are simple enough to be integrated explicitly. Indeed, consider the solution $u(t,x)$ to the *transport (advection) equation*[1]

$$u_t + cu_x = 0. \tag{4.2.5}$$

We may ask ourselves if there exists a curve $x(t)$ such that the solution, when restricted to this curve, is constant. If such a curve exists then

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t,x(t)) = u_t + \frac{\mathrm{d}x}{\mathrm{d}t}u_x = 0. \tag{4.2.6}$$

Comparing (4.2.6) with the advection equation (4.2.5) we conclude easily that the solution $u(t,x)$ is constant on any strait line $\xi = x_0 - ct$, where x_0 denotes the initial point at $t = 0$. In fact, any function

$$u(t,x) = f(x - ct) \tag{4.2.7}$$

where $f(\xi)$ is a completely arbitrary function, solves the advection equation (4.2.5). Consequently, by virtue of (4.2.3), it solves also the wave equation (4.1.3). The same can obviously be said about solutions to the sister transport equation

$$u_t - cu_x = 0. \tag{4.2.8}$$

In this case, however, the solution

$$u(t,x) = g(x + ct), \tag{4.2.9}$$

where $g(\eta)$, $\eta = x + ct$, is again an arbitrary function. The new variables ξ and η are called *characteristic variables* and the corresponding strait lines the *characteristics* of the wave equation.

Thus, we have found two different classes of solutions to the wave equation. They are known as *travelling waves* moving with with the same speed c in opposite directions.

Linearity of the wave equation implies that a combination of solutions is again a solution. Therefore, given $f(x - ct)$ and $g(x + ct)$ as solutions we know that $u(t,x) = f(x - ct) + g(x + ct)$ is a solution. In fact, every solution to the wave

[1]This type of equation describes a transport process in which the changes in the value of $u(t,x)$ are due to movement of the medium. This is in contrast to the diffusion process.

equation can be represented as a combination of travelling waves propagating in opposite directions.

THEOREM 4.1. *The general solution to the wave equation* (4.1.3) *is a combination of right and left travelling wave solutions*

$$u(t,x) = f(x-ct) + g(x+ct) \tag{4.2.10}$$

for some functions $f(\xi)$ *and* $g(\eta)$ *of the characteristic variables* $\xi = x - ct$ *and* $\eta = x + ct$.

PROOF. Let us first rewrite the wave equation in terms of the characteristic variables ξ and η. Set $u(x,t) = w(\xi(t,x), \eta(t,x))$. Then

$$u_{xx} = w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}, \quad u_{tt} = c^2[w_{\xi\xi} - 2w_{\xi\eta} + w_{\eta\eta}].$$

This implies that

$$\Box u = -4c^2 w_{\eta\xi}.$$

Solving the wave equation reduces to solving the second order partial differential equation

$$\frac{\partial^2 w}{\partial\xi\partial\eta} = 0.$$

Integrated this equation first with respect to ξ and then with respect to η, results in

$$w(\xi\eta) = f(\xi) + g(\eta) = f(x-ct) + g(x+ct).$$

Some attention need to be paid to the domain of definition of the functions involved and the differentiability assumptions. □

Initial Value Problem on an Infinite Domain.

We have just shown that the general solution to the wave equation depends on two arbitrary functions of characteristic variables. What is missing are any initial as well as boundary conditions. Let us first consider an infinite string. In order to solve the initial value problem

$$\Box u = 0, \quad u(0,x) = f(x), \quad u_t(0,x) = g(x), \quad -\infty < x < \infty, \tag{4.2.11}$$

we must find functions $p(\xi)$ and $q(\eta)$ such that

$$u(0,x) = p(x) + q(x) = f(x), \quad \text{and} \quad u_t(0,x) = -cp'(x) + cq'(x) = g(x).$$

Differentiating the first equation, multiplying it by c, and subtracting the second equation from the first leaves us with

$$2p'(x) = f'(x) - \frac{1}{c}g(x).$$

Hence,

$$p(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(y)dy + a,$$

where a is an integration constant. Consequently, according to the first equation,

$$q(x) = f(x) - p(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(y)dy - a.$$

Substituting both functions back into (4.2.10), we obtain that

$$u(x,t) = p(\xi) + q(\eta) = \frac{f(\xi) + f(\eta)}{2} + \frac{1}{2c}\int_\xi^\eta g(y)dy,$$

where ξ and η are the characteristic variables.

We have derived *d'Alembert's formula* for the solution to the wave equation for the infinite one-dimensional medium:

$$u(t,x) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy. \tag{4.2.12}$$

Note that we have not yet imposed any boundary conditions. As the medium is infinite we simply require that the solution is bounded throughout.

Example 4.2. Let us consider the initial value problem (4.2.11) where

$$u(0,x) = e^{-x^2}, \quad u_t(0,x) = 0.$$

In other words, we consider the situation when the motion of the string is purely the result of the initial displacement. According to (4.2.12)

$$u(t,x) = \frac{1}{2}e^{-(x-ct)^2} + \frac{1}{2}e^{-(x+ct)^2}.$$

The initial displacement gets partitioned into two waves travelling in opposite directions with the same speed and the shape half as tall as the initial one.

EXAMPLE 4.3. Consider the wave equation with the characteristic speed $c = 1$, and the initial condition build out of two pulses concentrated at two different points. For example, let

$$u(0,x) = e^{-x^2} + e^{-(x+1)^2}, \quad u_t(0,x) = 0.$$

The solution

$$u(t,x) = \frac{1}{2}e^{-(x-t)^2} + \frac{1}{2}e^{-(x+1-t)^2} + \frac{1}{2}e^{-(x+t)^2} + \frac{1}{2}e^{-(x+1+t)^2}$$

consists of four different pulses travelling through the medium. The first and the second wave propagate to the left while the third and the fourth propagate to the right. The first and the last wave run away from each other However, the second wave and and the third wave collide[2] at $t = 1/2$ but eventually emerge from this collision unchanged. Indeed,

$$u(1,x) = \frac{1}{2}e^{-(x-1)^2} + \frac{1}{2}e^{-x^2} + \frac{1}{2}e^{-(x+1)^2} + \frac{1}{2}e^{-(x+2)^2}.$$

The derivation of the general solution in the d'Alemebrt's form was possible only because we were able to show that, in general, signals propagate only along characteristics. A nice way to see how this works is to consider the initial condition in the form of a delta function concentrated at a point, say a,

$$u(0,x) = \delta(x-a), \quad u_t(0,x) = 0. \tag{4.2.13}$$

The corresponding solution takes the form

$$u(t,x) = \frac{1}{2}\delta(x-ct-a) + \frac{1}{2}\delta(x+ct-a), \tag{4.2.14}$$

and it does not vanish only at $x = ct + a$ and $x = -ct + a$ respectively. These are the two characteristics emerging from the initial point $(0,a)$ on the *phase plane* $t-x$. The displacement induced at the position $x = a$ at time $t = 0$ partitions itself, and a half of it travels along one characteristic while the other half propagates along the other characteristic. Both signals advance at the same speed, although

[2]This is obviously a bit inaccurate as both pulses are not compactly supported (are nonzero at all points of the real axis rather than only on a finite interval). They interact at all times. On the other hand, as the exponential functions approach zero very rapidly, the bulk of the wave is concentrated around the tallest point. Our description may therefore be considered visually accurate.

in the opposite directions, and there are no other signals propagating anywhere else. Observing any particular point of the string one notices that once the wave passes through, the point returns to its original state of zero displacement. To see that this is not always the case let us investigate the initial condition of no displacement but nonzero velocity. In particular, let

$$u(0,x)=0, \quad u_t(0,x)=\delta(x-a). \tag{4.2.15}$$

The d'Alembert solution (4.2.12) takes the form

$$\begin{aligned} u(t,x) &= \frac{1}{2c}\int_{x-ct}^{x+ct}\delta_a(y)dy = \sigma_{-ct+a}(x)-\sigma_{ct+a}(x) \\ &= \begin{cases} \frac{1}{2c}, & a-ct<x<a+ct, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{4.2.16}$$

The solution consists of a constant displacement of magnitude $1/2c$ expanding in both directions at the constant speed c. In contrast to the concentrated initial displacement case, the string once displaced will never return to its initial state of no displacement but it will remain displaced by a fixed amount of $1/2c$.

Boundary Value Problem on a Semi-Infinite Interval.

Suppose that a string is semi-infinite. That is, it extends over the region $0 \le x < \infty$ with its left end at $x = 0$ fixed. This forms the following initial-boundary value problem for the motion of the semi-infinite string:

$$\begin{cases} u_{tt}=c^2u_{xx}, \quad 0<x<\infty, \quad t>0, \\ u(x,0)=f(x), \\ u_t(x,0)=g(x), \\ u(0,t)=0. \end{cases} \tag{4.2.17}$$

According to Theorem 4.1 the general solution of the wave equation is a combination of travelling waves, that is

$$u(x,t)=F(x-ct)+G(x+ct),$$

where the functions F and G are constructed from the initial conditions:

$$F(z)=\frac{1}{2}f(z)-\frac{1}{2c}\int_0^z g(s)ds, \tag{4.2.18a}$$

$$G(z) = \frac{1}{2}f(z) + \frac{1}{2c}\int_0^z g(s)ds. \tag{4.2.18b}$$

Note that the initial conditions $f(x)$ an $g(x)$ are defined only for $x \geq 0$. This implies that that the wave equation solution (4.2.12) is valid only as long as $x - ct$ and $x + ct$ remain nonnegative. The value of $x + ct$ is always nonnegative. However, the value of $x - ct$ is nonnegative only if $x \geq ct$. Therefore, whenever $x < ct$ the solution $u(x, t)$ is not represented by the formula (4.2.12). On the other hand, even when $x < ct$ the solution $u(x, t)$ must be the sum of two waves, one travelling to the right and the other travelling to the left. The wave represented by the function $G(x + ct)$ is still valid, but we shall try to replace $F(x - ct)$ by a different travelling wave $\widehat{F}(x - ct)$. Since the string is fixed at that end, the boundary condition $u(0, t) = 0$ implies that

$$u(0, t) = \widehat{F}(-ct) = G(ct) = 0.$$

This condition must be satisfied at any time $t > 0$. It shows that the function $\widehat{F}$ is defined by the relation

$$\widehat{F}(z) = -G(-z)$$

valid for all $z < 0$, where $z = -ct$. It follows from (4.2.18b) that the value of the solution $u(x, t)$ when $x - ct < 0$ is

$$u(x, t) = \frac{1}{2}\left(f(x + ct) - f(ct - x)\right) + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)ds. \tag{4.2.19}$$

Putting the two cases (4.2.12) and (4.2.19) together we conclude that the solution to the initial-boundary value problem (4.2.17) for the semi-infinite string with one end fixed, is piecewise defined in the following form:

$$u(x, t) = \begin{cases} \frac{1}{2}\left(f(x + ct) + f(x - ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds, & x \geq ct, \\ \\ \frac{1}{2}\left(f(x + ct) - f(ct - x)\right) + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)ds, & x < ct. \end{cases} \tag{4.2.20}$$

Remark 4.4. Another way to approach the initial-boundary value problem for the semi-infinite string with one end fixed is to replace it by the infinite problem where the initial condition is odd, and therefore satisfies the boundary

condition $u(0,t)=0$. For example, replace the initial-boundary value problem

$$\begin{cases} u_{tt}=c^2u_{xx}, \quad 0<x<\infty, \quad t>0, \\ u(x,0)=f(x), \\ u_t(x,0)=0, \\ u(0,t)=0, \end{cases} \tag{4.2.21}$$

by the corresponding initial value problem for the infinite string with $u_t(x,0)=0$ and $u(x,0)=f_o(t)$, where

$$f_o(t)=\begin{cases} f(x) & \text{if} \quad x\geq 0, \\ -f(-x) & \text{if} \quad x<0, \end{cases}$$

is the *odd extension* of $f(x)$. It is straightforward to verify that the d'Alembert solution

$$u(x,t)=\frac{1}{2}\left(f_o(x+ct)+f_o(x-ct)\right),$$

satisfies the initial and boundary conditions of (4.2.21).

4.3. Nonlinear Waves

When investigating the wave equation we came across the transport equation (4.2.5) the solution to which proved to be the wave $f(x-ct)$ propagating at constant speed c along characteristics $x-ct=x_0$.

In this section we would like to investigate some properties of solutions to the nonlinear version of this one-way wave equation. First, let us look at slightly more complicated, but still linear, version of (4.2.5), that is

$$u_t+c(x)u_x=0. \tag{4.3.1}$$

The wave speed depends now on the position of the wave. It is, however, not unreasonable to investigate how the concept of the characteristic generalizes to this case. By analogy to the uniform case, we define the *characteristic curve*, as the solution to the ordinary differential equation

$$\frac{dx}{dt}=c(x). \tag{4.3.2}$$

We can show (the same way as we showed it before) that the solutions to the wave equation equation (4.3.1) are constant along characteristic curves. Indeed, restricting the solution to a characteristic and differentiating it with respect to time t we obtain that

$$\frac{d}{dt}u(t,x(t)) = u_t + u_x\frac{dx}{dt} = u_t + c(x(t))u_x = 0.$$

Since $u(t,x(t))$ solves the equation (4.3.1), we conclude that $u(t,x(t))$ is constant. Therefore, $u(t,x(t)) = u(0,x(0))$ along the characteristic $x(t)$ starting at $x(0)$. Note that in contrast to the uniform linear case the characteristic curves are not necessarily strait lines.

EXAMPLE 4.5. Consider the equation

$$u_t + xu_x = 0.$$

The characteristic curves are the solutions to the differential equation

$$\frac{dx}{dt} = x.$$

Integrating the characteristic equation, we find that

$$x(t) = x_0e^t.$$

Since the solution is constant on characteristics, in order to determine the value of the solution at an arbitrary point (t,x) we must find the characteristic passing through this point. In particular, we must find the initial point x_0 this characteristic starts from. This should not be difficult. From the equation of the characteristic we get that

$$x_0 = xe^{-t}.$$

Given the initial condition $u(0,x) = u_0(x)$ we deduce that the solution

$$u(t,x) = u_0(xe^{-t}).$$

Indeed, differentiating our function with respect to t and x and substituting into the original differential equation confirms that we have found the solution.

One of the simples, yet very important, nonlinear differential equations is the nonlinear advection equation

$$u_t + a(u)u_x = 0. \tag{4.3.3}$$

The coefficient $a(u)$ is again interpreted as the wave speed, this time dependent of the size of the wave.

REMARK 4.6. If the coefficient $a(u)$ in the equation (4.3.3) is such that $a(u) = f'(u)$, for some function $f(u)$, then the transport equation takes the form of the *conservation law*

$$u_t + f(u)_x = 0. \tag{4.3.4}$$

A conservation law is an equation which accounts for all the ways in which a particular quantity can change. Suppose that we have a one-dimensional medium. Let $u(t, x)$ measure the density at position x of some quantity at time t. The total amount of that quantity in the finite segment $[a, b]$ is given by the integral

$$\int_a^b u(t, x)dx.$$

The rate at which this amount changes in time is computed by the derivative

$$\frac{d}{dt}\int_a^b u(t, x)dx. \tag{4.3.5}$$

Assuming that our quantity can enter the segment $[a, b]$ only at either of its ends, the rate at which it enters is given by the *flux function*. Let $f(t, x)$ denote the rate at which the quantity flows past position x at time t. The net rate at which the quantity enters the segment $[a, b]$ is measured by

$$f(t, a) - f(t, b). \tag{4.3.6}$$

Assuming that there is no other way in which the given quantity can enter the medium, we get an integral form of the conservation law:

$$\frac{d}{dt}\int_a^b u(t, x)dx = f(t, a) - f(t, b). \tag{4.3.7}$$

This is equivalent to

$$\int_a^b u_t(t, x)dx = -\int_a^b f_x(t, x)dx, \tag{4.3.8}$$

provided we can differentiate under the integral and the flux function is regular enough. If u_t and f_x are continuous, and the integral conservation law is valid for any finite segment of the medium, then the integral law (4.3.8) is equivalent to a conservation law in differential form (4.3.4).

We define the *characteristic curves* $x(t)$ as solutions to the ordinary differential equation

$$\frac{dx}{dt} = a(u(t, x(t)). \tag{4.3.9}$$

As the right hand side is solution dependent we cannot determine the characteristic curve without knowing the solution. We can, however, show that the solution is still constant on characteristics. Assume that $x(t)$ is the characteristic. Restrict the solution $u(t, x)$ to $x(t)$, and differentiate $u(t, x(t))$ along the characteristic. Thus,

$$\frac{d}{dt}u(t, x(t)) = u_t(t, x(t)) + \frac{dx}{dt}u_x(t, x(t)) = u_t(t, x(t)) + a(u(t, x(t)))u_x(t, x(t)) = 0,$$

since $u(t, x)$ is the solution to equation (4.3.3). This implies, in turn, that the characteristic curves are strait lines

$$x = a(u)t + x_0, \tag{4.3.10}$$

where u is the value of the solution on the characteristic as determined by the value of the initial condition at x_0. Indeed, given the initial condition $u_0(x)$, we draw through each point $(0, x)$ the characteristic line $\xi = a(u_0(x))t + x$. The solution $u(t, x(t)) = u_0(x)$ on the entire line.

Although all characteristic curves are strait lines they slopes $a(u)$ are dependent on the size of the wave. Thus the characteristic slopes depend on the form of the initial condition and the properties of the nonlinearity $a(u)$. In fact, they are parallel only if the initial condition is constant. Otherwise they may intersect at some future time t_c. When characteristics cross the solution is no longer well defined, not at least in the classical sense. Indeed, at the same point in space and the same time the solution is supposed to have two different values, one corresponding to each characteristic.

EXAMPLE 4.7. Consider the initial value problem for the *inviscid* Burgers' equation

$$u_t + uu_x = 0, \quad u(0,x) = u_0(x). \tag{4.3.11}$$

The equation (4.3.11) can also be written as the conservation law

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0. \tag{4.3.12}$$

The characteristic curves take the form

$$x = ut + x_0 = u_0(x_0)t + x_0. \tag{4.3.13}$$

Consider now two different characteristic curves emerging at $t = 0$ from x_0 and $x_0 + \triangle x$, respectively. These two strait lines will cross if at some future time $t = t_c$

$$u_0(x_0)t_c + x_0 = u_0(x_0 + \triangle x)t_c + x_0 + \triangle x.$$

This yields

$$t_c = -\frac{\triangle x}{u_0(x_0 + \triangle x) - u_0(x_0)}.$$

Taking the limit of the right had side when $\triangle x \to 0$ we obtain that the characteristic line emanating from $(0, x)$ intersects with another characteristic at critical time

$$t_c = -\frac{1}{u_0'(x)}. \tag{4.3.14}$$

Moreover, since $t_c > 0$, the characteristics intersect only if the initial condition $u_0(x)$ is a decreasing function. If, on the other hand, the initial condition is nondecreasing, the characteristics never cross and the solution is well defined for all times. The characteristics emanating from the x-axis create a fan of strait lines. This solution represents a *rarefaction wave* which gradually spreads out as time progresses.

For the general nonlinear wave equation (4.3.3) the critical time at which a characteristic intersect with another characteristic is given by

$$t_c = -\frac{1}{a'(u_0)u_0'(x)}. \tag{4.3.15}$$

Thus, given the initial condition $u_0(x)$ the classical solution exits only for times

$$0 < t < \min_x t_c. \tag{4.3.16}$$

The critical time t_c depends on the monotonicity of the initial condition as well as on the monotonicity of the wave speed $a(u_0)$. This, in the case of the conservation law, translates into convexity of the flux $f(x)$.

4.4. Classification of Linear Partial Differential Equations

We have already come across three different linear, second order, partial differential equations for functions in two variables. These are:

(1) The heat equation: $u_t - \gamma u_{xx} = 0$,
(2) Laplace equation: $u_{xx} + u_{yy} = 0$,
(3) The wave equation: $u_{tt} - c^2 u_{xx} = 0$.

They represent three fundamentally different, as we explain shortly, classes of partial differential equations; *parabolic*, *elliptic*, and *hyperbolic*. As we saw it already, they have distinct types of solutions, possessing completely different properties, and different numerical and analytical solution techniques. Hyperbolic and parabolic equations describe dynamical processes while equations of equilibrium such as Laplace equation involve only spatial variables and are elliptic. Also, elliptic partial differential equations lead to boundary value problems whereas hyperbolic and parabolic involve solving initial-boundary value problems. Although the classification we discuss here concerns only planar differential equations, the terminology and, if fact, the fundamental properties of solutions carries over to higher dimensions.

Consider the most general linear, second order partial differential equation in two variables. Namely,

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = f(x, y), \tag{4.4.1}$$

where all coefficients may, in general, be functions of both variables, x and y. This equation is said to be *homogeneous* if and only if $f \equiv 0$. Also, in order to consider only genuine second order equations, we assume that at least one of the coefficients A, B, C is nonzero.

In order to determine the *type* of the differential equation we calculate its *discriminant*

$$\Delta(x, y) \equiv B^2 - 4AC. \tag{4.4.2}$$

The terminology which we use to classify partial differential equations is motivated by the underlining classification of planar conic sections. That is, at a point (x,y), a linear, second order equation is called:

- elliptic if $\Delta(x, y) < 0$,
- parabolic if $\Delta(x, y) = 0$,
- hyperbolic if $\Delta(x, y) > 0$.

REMARK 4.8. Note that as the coefficients of the partial differential equation are allowed to vary from one point to another, the type of the equation can change respectively. One such example is the *Tricomi equation*

$$yu_{xx} = u_{yy}. \tag{4.4.3}$$

Its discriminant is $\Delta = 4y$. Therefore, its type changes with y.

APPENDIX A

Normed and Inner Product Vector Spaces

A.1. Inner Product Vector Spaces

The concepts of a norm and an inner product on a vector space formalize, and the same time generalize, the Euclidean notions of length and angle. In this chapter we make a brief and necessarily rather sketchy presentation of these concepts, and show how they appear in the context of vector spaces, both finite and infinite-dimensional. We start our presentation by introducing the notion of an inner product on a vector space.

Let V denote a real vector space. An *inner product* on V is a real-valued function

$$\langle \cdot ; \cdot \rangle : V \times V \to \mathbb{R} \tag{A.1.1}$$

which is bilinear (linear in each argument), symmetric and positive definite. Being positive definite means that

$$\langle \mathbf{v}; \mathbf{v} \rangle \geq 0, \quad \text{for any vector} \quad \mathbf{v} \in V, \tag{A.1.2}$$

and

$$\langle \mathbf{v}; \mathbf{v} \rangle = 0 \quad \text{if and only if} \quad \mathbf{v} = \mathbf{0}. \tag{A.1.3}$$

A vector space with a given inner product is called an *inner product vector space*. Be aware that any particular vector space may admit several different inner products.

Example A.1. Consider $V = \mathbb{R}^n$. The standard *Euclidean inner product* is defined as

$$\langle \mathbf{v}; \mathbf{w} \rangle = \sum_{i=1}^{n} v_i w_i, \tag{A.1.4}$$

where $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$. This is not the only possible inner product on $\mathbb{R}^n$, although most important. For example, let $c_1, \ldots, c_n$ be a set of

positive numbers. Define the *weighted inner product*

$$\langle \mathbf{v}; \mathbf{w} \rangle = \sum_{i=1}^{n} c_i v_i w_i. \tag{A.1.5}$$

EXAMPLE A.2. Let $[a, b]$ be a bounded, close interval on $\mathbb{R}$. Consider the vector space $C[a, b]$ of all continuous functions $f : [a, b] \to \mathbb{R}$, and define an inner product as

$$\langle f; g \rangle = \int_a^b f(x)g(x)dx. \tag{A.1.6}$$

Notice that the assumption that all our functions are continuous on $[a, b]$ is essential for the given product to be definite. Relaxing the continuity assumption and extending the selection of functions to include functions which are, for example, piecewise continuous creates problems. Indeed, consider a function h which is zero everywhere on $[a, b]$ except at a single point, at which it is 1. It it obvious that $\langle h; h \rangle = 0$ despite the fact that $h \neq 0$.

The inner product function spaces play essential role in the development of the theory of Fourier series and the solution to the boundary value problems.

Given the inner product vector space V there exists a *norm* associated with the given inner product. Such a norm is defined as the square root of an inner product of a vector with itself:

$$\| \mathbf{v} \| = \sqrt{\langle \mathbf{v}; \mathbf{v} \rangle}. \tag{A.1.7}$$

The positive definiteness of the inner product implies that the norm (A.1.7) is nonnegative, and that it vanishes if and only if $\mathbf{v} \in V$ is the zero vector.

In general, a *norm* on the real vector space V is a non-negative , real-valued function

$$\| \cdot \| : V \to \mathbb{R}^+ \tag{A.1.8}$$

which is homogeneous:

$$\| c\mathbf{v} \| = |c| \, \| \mathbf{v} \|, \quad \text{for any} \quad c \in \mathbb{R}, \quad \text{and} \quad \mathbf{v} \in V, \tag{A.1.9}$$

satisfies *triangle inequality*:

$$\| \mathbf{v} + \mathbf{w} \| \leq \| \mathbf{v} \| + \| \mathbf{w} \| \quad \text{for any} \quad \mathbf{v}, \mathbf{w} \in V, \tag{A.1.10}$$

and is positive definite:

$$\| \mathbf{v} \| = 0 \quad \text{if and only if} \quad \mathbf{v} = 0. \tag{A.1.11}$$

EXAMPLE A.3. The Euclidean inner product (A.1.4) induces on $\mathbb{R}^n$ the norm

$$\| \mathbf{v} \| = \langle \mathbf{v}; \mathbf{v} \rangle = \sqrt{\sum_{i=1}^{n} v_i^2}. \tag{A.1.12}$$

The corresponding weighted norm takes the form

$$\| \mathbf{v} \| = \sqrt{\sum_{i=1}^{n} c_i^2 v_i^2}. \tag{A.1.13}$$

The L^2-norm on $C[a, b]$ is:

$$\| f \| = \sqrt{\int_a^b f(x)^2 dx}. \tag{A.1.14}$$

On of the most important relations between the inner product of two vector and the associated norms of these vectors is the *Cauchy-Schwarz inequality*.

THEOREM A.4. *Every inner product satisfies the Cauchy-Schwarz inequality*

$$|\langle \mathbf{v}; \mathbf{w} \rangle \leq \| \mathbf{v} \| \| \mathbf{w} \|, \quad \mathbf{v}, \mathbf{w} \in V. \tag{A.1.15}$$

Equality holds if and only if the vectors $\mathbf{v}$ *and* $\mathbf{w}$ *are parallel*[1].

PROOF. First, observe that if any of the vectors $\mathbf{v}, \mathbf{w}$ is zero the inequality is trivially satisfied as both sides vanish. We, therefore, assume that both vectors are different from the zero vector. Second, consider the function

$$f(t) = \| \mathbf{v} + t\mathbf{w} \|^2$$

where $t \in \mathbb{R}$ is an arbitrary scalar. Using the bilinearity of the inner product and the definition of the norm (A.1.7) we obtain that

$$f(t) = \| \mathbf{w} \|^2 t^2 + 2\langle \mathbf{v}; \mathbf{w} \rangle t + \| \mathbf{v} \| .$$

[1]Two vectors are considered parallel if there exists a scalar such that one vector is the multiple of the other. According to this convention the zero vector is parallel to every other vector.

The function $f(t)$ in nonnegative and it attains a minimum as its leading coefficient is positive. The minimum is taken at a point at which the derivative $f'(t)$ vanishes. Namely,

$$f'(t) = 2 \parallel \mathbf{w} \parallel^2 t + 2\langle \mathbf{v}; \mathbf{w} \rangle^2 = 0,$$

when

$$t = -\frac{\langle \mathbf{v}; \mathbf{w} \rangle}{\parallel \mathbf{w} \parallel^2}.$$

Substituting this into the definition of the function we obtain that

$$\parallel \mathbf{v} \parallel^2 \geq \frac{\langle \mathbf{v}; \mathbf{w} \rangle^2}{\parallel \mathbf{w} \parallel^2}, \quad \text{or equivalently} \quad \parallel \mathbf{v} \parallel^2 \parallel \mathbf{w} \parallel^2 \geq \langle \mathbf{v}; \mathbf{w} \rangle^2.$$

Taking the square root of both sides completes the proof. □

Two vectors $\mathbf{v}, \mathbf{w} \in V$, of an inner product vector space V, are called *orthogonal* if $\langle \mathbf{v}; \mathbf{w} \rangle = 0$. In $\mathbb{R}^n$ (with the Euclidean inner product) orthogonality means geometric perpendicularity. In spaces of functions such geometric analogies are not available. Also, being orthogonal with respect to one inner product does not imply being orthogonal with respect to another inner product.

EXAMPLE A.5. Consider the space $C[0, \pi]$ with the L^2-inner product. There, the function $\sin x$ is orthogonal to $\cos x$. Indeed,

$$\langle \sin x; \cos x \rangle = \int_0^\pi \sin x \cos x dx = \frac{1}{2} \sin^2 x \Big|_0^\pi = 0.$$

EXAMPLE A.6. Let $P^2[0, 1]$ be the space of all polynomials of degree not bigger than 2 defined on the interval $[0, 1]$. It is elementary to show that the polynomials 1 and x are orthogonal with respect to the standard inner product,

$$\langle a_0 + a_1 x + a_2 x^2; b_0 + b_1 x + b_2 x^2 \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2.$$

However, the same two polynomials are not orthogonal in the L^2-inner product as

$$\langle 1; x \rangle = \int_0^1 x dx = \frac{1}{2} x^2 \Big|_0^1 = \frac{1}{2}.$$

One of the consequences of the Cauchy-Schwarz inequality is the triangle inequality (A.1.10).

THEOREM A.7. *The norm associated with an inner product satisfies the triangle inequality*

$$\| \mathbf{v} + \mathbf{w} \| \leq \| \mathbf{v} \| + \| \mathbf{w} \| \tag{A.1.16}$$

for every pair of vectors $\mathbf{v}, \mathbf{w} \in V$. *The equality holds if and only if* $\mathbf{v}$ *and* $\mathbf{w}$ *are parallel.*

PROOF. Consider $\| \mathbf{v}+\mathbf{w} \|^2$ and use the Cauchy-Schwarz inequality (A.1.15) to show that

$$\begin{aligned} \| \mathbf{v} + \mathbf{w} \|^2 &= \| \mathbf{v} \|^2 + 2\langle \mathbf{v}; \mathbf{w} \rangle + \| \mathbf{w} \|^2 \\ &\leq \| \mathbf{v} \|^2 + 2 \| \mathbf{v} \| \| \mathbf{w} \| + \| \mathbf{w} \|^2 = \left(\| \mathbf{v} \| + \| \mathbf{w} \| \right)^2 . \end{aligned}$$

Taking the square root of both sides of the inequality completes the proof. □

A.2. Normed Vector Spaces

An inner product vector space has the associated inner product norm, as we showed in the previous section. On the other hand, a vector space may be equipped with a norm which does not come from any inner product.

REMARK A.8. Given a norm in a vector space we may define the notion of a *distance* between vectors:

$$d(\mathbf{v}, \mathbf{w}) = \| \mathbf{v} - \mathbf{w} \| . \tag{A.2.1}$$

Realize that this function possess all the properties we expect from a function measuring distance. It is symmetric, vanishes if and only if $\mathbf{v} = \mathbf{w}$, and satisfies the triangle inequality

$$d(\mathbf{v}, \mathbf{w}) \leq d(\mathbf{v}, \mathbf{u}) + d(\mathbf{u}, \mathbf{w}) \tag{A.2.2}$$

for any choice of vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.

Let us look now at some norms which are not associated with any inner product.

EXAMPLE A.9. Consider again $\mathbb{R}^n$. The 1-*norm* of a vector is defined as:

$$\| \mathbf{v} \|_1 = |v_1| + \cdots + |v_n|. \tag{A.2.3}$$

The p-*norm*, for any integer $p \geq 1$, is

$$\| \mathbf{v} \|_p = \sqrt[p]{\sum_{i=1}^{n} |v_i|^p}. \tag{A.2.4}$$

Taking p to ∞ yields ∞-*norm*

$$\| \mathbf{v} \|_\infty = \sup_{1 \leq i \leq n} \{|v_i|\}. \tag{A.2.5}$$

We leave as an exercise to verify that these are indeed well defined norms[2].

The norms in $\mathbb{R}^n$ have they counterparts in the space of functions.

EXAMPLE A.10. Consider once again $C[a, b]$, the space of all continuous functions on a closed, bounded interval $[a, b]$. The L^p-*norm* is defined as

$$\| f \|_p = \left(\int_a^b |f(x)|^p x \right)^{\frac{1}{p}}. \tag{A.2.6}$$

Respectively, the L^∞-*norm* is defined as

$$\| f \|_\infty = \sup_{a \leq x \leq b} \{|f(x)|\}. \tag{A.2.7}$$

The L^2-norm is the only norm associated with an inner product.

Having all these different norms available one may wounder if they are equivalent in some sense. The answer to this question depends on whether the vector space is finite-dimensional like $\mathbb{R}^n$ or infinite-dimensional like the space $C[a, b]$. That is, any two norms in a finite-dimensional space are equivalent in the following sense:

THEOREM A.11. *Let $\| \cdot \|_1$ and $\| \cdot \|_2$ be any two norms in $\mathbb{R}^n$. Then, there exist positive constants c and C such that*

$$c \| \mathbf{v} \|_1 \leq \| \mathbf{v} \|_2 \leq C \| \mathbf{v} \|_1 \tag{A.2.8}$$

for every $\mathbf{v} \in \mathbb{R}^n$.

The proof of this fact can be found in [**Lang**].

[2]The triangle inequality for the p-norm, known as the *Hölder inequality*, is non-trivial, [**Taylor**].

EXAMPLE A.12. Let us compare the 1-norm and the ∞-norm on $\mathbb{R}^n$. To this end, note first that $\| \mathbf{v} \|_\infty = \sup\{|v_i|\} \leq |v_1| + \cdots + |v_n| = \| \mathbf{v} \|_1$. Thus, we may choose in (A.2.8) $C = 1$. Also, $\| \mathbf{v} \|_\infty = \sup\{|v_i|\} \geq \frac{1}{n} (|v_1| + \cdots + |v_n|) = \frac{1}{n} \| \mathbf{v} \|_1$. Therefore, $c = \frac{1}{n}$ and we have that

$$\frac{1}{n} \| \mathbf{v} \|_1 \leq \| \mathbf{v} \|_\infty \leq \| \mathbf{v} \|_1 . \tag{A.2.9}$$

The important consequence of the equivalence of norms in any finite-dimensional vector space, for example $\mathbb{R}^n$, is that the convergence of sequences is norm independent.

We say that the sequence $\{\mathbf{v}^{(k)}\} \subset V$ *converges in the norm* to $\hat{\mathbf{v}} \in V$ if the sequence of scalars $\| \mathbf{v}^{(k)} - \hat{\mathbf{v}} \|$ converges to 0. It is evident from Theorem A.11 that the convergence with respect to one norm implies the convergence with respect to any other norm. Also, the convergence in the norm implies the convergence of the individual components, i.e., $v_i^{(k)} \to \hat{v}_i$. The converse is obviously true as well. This is, in general, not true in infinite-dimensional spaces.

EXAMPLE A.13. Consider the space $C[0, 1]$. The sequence of continuous functions

$$f_n(x) = \begin{cases} -nx + 1, & 0 \leq x \leq \frac{1}{n} \\ 0, & \frac{1}{n} \leq x \leq 1 \end{cases}$$

is such that its L^∞-norm

$$\| f_n \|_\infty = \sup_{0 \leq x \leq 1} \{|f_n(x)|\} = 1.$$

On the other hand, the L^2-norm is

$$\| f_n \|_2 = \left(\int_0^{\frac{1}{n}} (1 - nx)^2 dx \right)^{\frac{1}{2}} = \frac{1}{\sqrt{3n}}.$$

It approaches zero when $n \to \infty$. Therefore, there is no constant C such that

$$\| f_n \|_\infty \leq C \| f_n \|_2$$

for this choice of functions, and so for all functions in $C[0, 1]$. This proves that the norms L^∞ and L^2 are not equivalent in $C[0, 1]$. Notice, however, that

$$\| f \|_2 \leq \| f \|_\infty \tag{A.2.10}$$

for any $f \in C[0,1]$. The convergence in the L^∞-norm implies the convergence in the L^2-norm but not vice versa.

A.3. Complex Vector Spaces

In this section we present a few basic facts about *complex vector spaces* that is the vector spaces in the definition of which the set of real scalars is replaced by the set of complex numbers $\mathbb{C}$. In this set of notes, where we deal with real quantities like the measure of deformation or temperature, we use complex numbers and complex vector spaces primarily to simplify presentation of periodic phenomena. There are, however, physical theories, e.g., quantum mechanics, where complex valued functions are intrinsic as they describe basic physical quantities.

The fundamental example of the complex vector space is the space $\mathbb{C}^n$ consisting of n-tuples of complex numbers $(u_1, u_2, \cdots, u_n)$, where $u_1, \ldots, u_n \in \mathbb{C}$. We can write any vector $\mathbf{u} \in \mathbb{C}^n$ as a linear combination of two real vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, namely $\mathbf{u} = \mathbf{x} + i\mathbf{y}$. Its complex conjugate $\overline{\mathbf{u}}$ is obtained by taking the complex conjugate of its coordinates, that is $\overline{\mathbf{u}} = \mathbf{x} - i\mathbf{y}$.

Most vector space concepts curry over from the real case to the complex realm. The only notable exception is the concept of the inner product. Motivated by the desire to have the real associated norm[3] on a complex vector we define the inner product on $\mathbb{C}^n$ as

$$\mathbf{v} \cdot \mathbf{w} = v_1\overline{w}_1 + v_2\overline{w}_2 + \cdots + v_n\overline{w}_n, \tag{A.3.1}$$

where $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$. This construction is known as the *Hermitian inner product* on $\mathbb{C}^n$. For example, if

$$\mathbf{v} = (i, -1), \quad \text{and} \quad \mathbf{w} = (1 + i, i),$$

then

$$\mathbf{v} \cdot \mathbf{w} = i(1 - i) + (-1)(-i) = 1 + 2i.$$

However,

$$\mathbf{w} \cdot \mathbf{v} = (1 + i)(-i) + i(-1) = 1 - 2i,$$

[3]Complex numbers cannot be ordered. Therefore, it does not make any sense to have a complex number $z > 0$.

which shows that the Hermitian dot product is not symmetric. It conjugates, however, under reversal of arguments. Indeed,

$$\mathbf{w} \cdot \mathbf{v} = \overline{\mathbf{v} \cdot \mathbf{w}}, \tag{A.3.2}$$

for any $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$. Also, the dot product is "*sesquilinear*", rather than bilinear, as

$$(c\mathbf{v}) \cdot \mathbf{w} = c(\mathbf{v} \cdot \mathbf{w}), \quad \text{while} \quad \mathbf{v} \cdot (c\mathbf{w}) = \overline{c}(\mathbf{v} \cdot \mathbf{w}). \tag{A.3.3}$$

On the other hand, the associated norm looks the same as in the real case:

$$\| \mathbf{v} \| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{|v_1|^2 + \cdots + |v_n|^2}, \tag{A.3.4}$$

and it is positive for any $\mathbf{v} \neq \mathbf{0}$. $|\cdot|$ denotes here the complex modulus.

The general definition of an inner product on a complex vector space is based on the Hermitian dot product on $\mathbb{C}^n$ and it states that an inner product on a complex vector space V is a complex-valued function $\langle \cdot ; \cdot \rangle : V \times V \to \mathbb{C}$ which is sesquilinear, positive definite and such that

$$\langle \mathbf{v}; \mathbf{w} \rangle = \overline{\langle \mathbf{w}; \mathbf{v} \rangle}$$

for any pair of vectors $\mathbf{v}, \mathbf{w} \in V$.

One can show that the Cauchy-Schwarz inequality (A.1.15), in which the absolute value is replaced by the complex modulus, holds for any inner product complex vector space.

EXAMPLE A.14. Let $\mathbb{C}[-\pi, \pi]$ denote the space of all complex valued continuous function on the interval $[-\pi, \pi] \subset \mathbb{R}$. The Hermitian L^2-inner product is defined as

$$\langle f; g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx. \tag{A.3.5}$$

The associated with this inner product norm is

$$\| f \| = \left(\int_{-\pi}^{\pi} |f(x)|^2 dx \right)^{\frac{1}{2}}. \tag{A.3.6}$$

For example, the set of complex exponential e^{ikx}, where k is any integer, is an orthonormal system of functions. Indeed,

$$\langle e^{ikx}; e^{imx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} e^{-imx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-m)x} dx = \begin{cases} 1, & k = m, \\ 0, & k \neq m, \end{cases} \tag{A.3.7}$$

where we utilized the fact that $e^{ik\pi} = (e^{i\pi})^k = (\cos \pi + i \sin \pi)^k = (-1)^k$. The orthogonality property (A.3.7) of complex exponentials is of significants in the Fourier theory, as we will see in the next chapter.

APPENDIX B

Fourier Theory

This chapter serves as a brief introduction to the Fourier theory and related topics.

B.1. Fourier Series

The basic object of study of the Fourier theory is an infinite trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos kx + b_k \sin kx \right], \tag{B.1.1}$$

called a *Fourier series*. In general, such a series may not converge without additional assumptions about its coefficients. We would like to know if a Fourier series can converge to a function, and whether a function can be represented by a Fourier series.

A *Fourier series of the function* $f(x)$ is a Fourier series (B.1.1) where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx, \qquad k = 0, 1, 2, \ldots, \tag{B.1.2a}$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx, \qquad k = 1, 2, 3, \ldots, \tag{B.1.2b}$$

and where we assume that the integrals are well defined. In fact, as we will see later, these integrals are well defined for a quite a broad class of functions $f(x)$. This obviously does not guarantee the convergence of the series, and certainly not to the function $f(x)$. The choice of the coefficients is, however, dictated by our desire to be able to represent a function by a Fourier series. Indeed, suppose that $f(x)$ is well approximated over the interval $[-\pi, \pi]$ by a Fourier series, that is

$$f(x) \cong \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos kx + b_k \sin kx \right] \tag{B.1.3}$$

for some choice of the coefficients a_k and b_k. Multiply both sides of (B.1.3) by $\cos lx$ and use integration by parts to show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\cos kx\cos lxdx = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin kx\cos lxdx = 0 \quad \text{if} \quad k \neq l,, \tag{B.1.4a}$$

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 lxdx = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2 lxdx = 1. \tag{B.1.4b}$$

This implies immediately the choice of the Fourier coefficients (B.1.2).

EXAMPLE B.1. Consider the function $f(x) = x$. Computing its Fourier coefficients directly we obtain that:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}xdx = 0, \qquad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}x\cos kxdx = 0, \tag{B.1.5}$$

and

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi}x\sin kxdx = \frac{1}{\pi}\left[-\frac{x\cos kx}{k} + \frac{\sin kx}{k^2}\right]|_{-\pi}^{\pi} = \frac{2}{k}(-1)^{k+1}. \tag{B.1.6}$$

The vanishing of the coefficients a_k is a consequence of the fact that the function x is odd while $\cos kx$ is an even function. Therefore, the Fourier series of $f(x) = x$ is

$$2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin kx}{k}. \tag{B.1.7}$$

It is not an elementary exercise to determine the convergence of this series. But, even if we overcome this obstacle and determine that it converges, one does not know what it converges to. For example, it certainly does not converge to $f(x) = x$. Indeed, at $x = \pi$ the series (B.1.7) converges to 0, as every term in the series vanishes at π, while $f(\pi) = \pi \neq 0$.

As our example shows the convergence of Fourier series is not a simple matter. The standard tests used to analyze the convergence of power series fail. Also, power series always converge at least at 0 or on an interval (possibly infinite) centered at 0. Fourier series, on the other hand, may converge on rather unusual sets. Moreover, as all components of a Fourier series are 2π periodic the series will, in general, converge to 2π periodic functions. The power series and the Fourier series differ also by what they converge to. Indeed, if a power series

converges on an interval it converges to an infinitely differentiable function as all its derivative series converge to the corresponding derivatives. Such functions are called *analytic*. A Fourier series, on the other hand, may converge to discontinuous functions and even to such a "bizarre" function as the Dirac delta function.

We will look first at the issue of periodicity of the limit of a Fourier series. Given a function $f(x)$ for $-\pi < x \leq \pi$ let us define its *periodic extension* as a function $\tilde{f}(x)$ defined everywhere and such that $\tilde{f}(x + 2\pi) = \tilde{f}(x)$, and $\tilde{f}(x) = f(x)$ for $-\pi < x \leq \pi$. To this end, let $x \in \mathbb{R}$. Thus, there exists a unique integer m such that $(2m-1)\pi < x \leq (2m+1)\pi$. We therefore postulate that

$$\tilde{f}(x) = f(x - 2m\pi) \qquad \text{for} \qquad (2m-1)\pi < x \leq (2m+1)\pi.$$

Such a function is obviously unique, 2π periodic, and coincides with $f(x)$ on $-\pi < x \leq \pi$. In most cases such a periodic extension is not a continuous function. It is, however, "piecewise continuous".

DEFINITION B.2. A function $f(x)$ is called *piecewise continuous* on the interval $[a, b]$ if it is defined and continuous except possibly a finite number of point. Moreover, at each "point of discontinuity", say x_i, the left and the right hand limits

$$f_+(x_i) \equiv \lim_{x \to x_i^+} f(x), \qquad f_-(x_i) \equiv \lim_{x \to x_i^-} f(x),$$

exist. A function defined on all of $\mathbb{R}$ is piecewise continuous if it is piecewise continuous on every bounded interval.

Note that according to this definition a piecewise discontinuous function may not be defined at a "point of discontinuity". Even if it is defined its value $f(x_i)$ is not necessarily equal to either right or left hand limit. Such a point x_i is called a *jump discontinuity* and we say that the function $f(x)$ experiences a jump of the magnitude

$$[f(x_i)] \equiv f_+(x_i) - f_-(x_i). \tag{B.1.8}$$

If the function $f(x)$ is piecewise continuous its (formal) Fourier series is well defined as the function is integrable on any closed interval. The convergence of

such a series, even if the function is continuous on $[-\pi, \pi]$ interval, is still not guaranteed.

DEFINITION B.3. A function $f(x)$ is called *piecewise* C^1 on the interval $[a, b]$ if it is piecewise continuous on $[a, b]$, and continuously differentiable on $[a, b]$ except possibly a finite number of points. If x_i is such a point then the left and right hand limits

$$f'_+(x_i) \equiv \lim_{x \to x_i^+} f'(x), \qquad f'_-(x_i) \equiv \lim_{x \to x_i^-} f'(x), \tag{B.1.9}$$

exist.

Note that a piecewise C^1 function may have points at which either the function experiences a jump but the left and right hand derivatives exist or it is continuous but its derivative experiences a jump. The periodic extension of the function $f(x) = x$ has jump discontinuities of the first type at $\pi + 2m\pi$ while the absolute value $|x|$ is continuous everywhere but its derivative experiences a jump at 0. We are now in the position to state the convergence theorem for Fourier series. The proof will be presented in Section B.4.

THEOREM B.4. *Let $\tilde{f}(x)$ be a periodic extension of a piecewise C^1 function on $[-\pi, \pi]$, then its Fourier series converges at all x to:*

$$\tilde{f}(x) \qquad \textit{whenever } \tilde{f} \textit{ is continuous,}$$

$$\frac{1}{2}\left[\tilde{f}_+(x) + \tilde{f}_-(x)\right] \qquad \textit{if } x \textit{ is a point of discontinuity.}$$

In other words, if in the definition of the periodic extension we replace the values at any discontinuity point x by

$$\tilde{f}(x) = \frac{1}{2}\left[\tilde{f}_+(x) + \tilde{f}_-(x)\right],$$

the Fourier series of a piecewise C^1 periodic extension $\tilde{f}(x)$ will converge to $\tilde{f}(x)$ everywhere.

EXAMPLE B.5. Let us revisit example B.1, but let the periodic extension of $f(x) = x$ be such that

$$\tilde{f}((2m+1)\pi) = 0, \qquad \text{for any integer m.}$$

Then the Fourier series

$$2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin kx}{k}$$

converges everywhere to the "new" periodic extension, and the convergence problem we previously encountered at π is solved. Indeed, for any $x = (2m+1)\pi$ all terms in our Fourier series vanish.

EXAMPLE B.6. Let $f(x) = |x|$. Observe that as $|x|$ is an even function its periodic extension is continuous. Its Fourier coefficients can easily be computed:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|dx = \frac{2}{\pi}\int_0^{\pi}xdx = \pi,$$

$$a_k = \frac{2}{\pi}\int_0^{\pi}x\cos xdx = -\frac{4}{k^2\pi}, \quad \text{if } k \text{ is odd, and otherwise 0.}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\sin xdx = 0,$$

as the function $|x|\sin x$ is odd. Therefore, the Fourier series of $f(x) = |x|$ is

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos(2k+1)}{(2k+1)^2}. \tag{B.1.10}$$

According to Theorem B.4, it converges to the periodic extension of $f(x) = |x|$. In particular, if $x = 0$ we obtain that

$$\frac{\pi}{2} = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}. \tag{B.1.11}$$

This series can be used to obtain an approximation to the number π.

As we have noted in Examples B.5 and B.6 the coefficients of the Fourier cosine series of the function $f(x) = x$, and the sine coefficients of the function $f(x) = |x|$, are 0. This is not a coincidence, but rather a consequence of the fact that x is an odd function while $|x|$ is an even function. Indeed, the following proposition, whose proof is elementary and will be left to the reader, generalizes our observations.

PROPOSITION B.7. *If $f(x)$ is an even function then its Fourier series sine coefficients b_k all vanish, and f can be represented by a Fourier cosine series*

$$f(x) \cong \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx. \tag{B.1.12}$$

If $f(x)$ is odd, then its Fourier series cosine coefficients a_k all vanish. Thus, f can be represented by a Fourier sine series

$$f(x) \cong \sum_{k=1}^{\infty} a_k \sin kx. \tag{B.1.13}$$

Conversely, a convergent Fourier cosine (sine) series always represents an even (odd) function.

B.2. Differentiation and Integration of Fourier Series

Knowing that power series can be differentiated and integrated term by term, and that these two operations do not change (except for the end points of the interval of convergence) the convergence of these series, it make sense to investigate whether a similar property holds for Fourier series. The main difference between these two cases is that the power series converge to analytic functions, and hence can be freely differentiated and integrated, while the Fourier series may converge to functions of very different degrees of smoothness. Thus, investigating differentiation and integration of Fourier series we must pay careful attention to the regularity of its limits.

Integration. When attempting to integrate a Fourier series we are faced with the fundamental problem that, in general, the integral of the periodic function is not periodic. However, a Fourier series consists mostly of sine and cosine terms which when integrated are also period functions. The only term we are evidently going to have a problem with is the free term $a_0/2$. Hence, we will first try to integrate Fourier series with the zero constant term:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx = 0.$$

This shows that a function has no constant term in its Fourier series if its *average* on the interval $[-\pi, \pi]$ is zero. The periodic zero average functions are the once

that remain periodic upon integration. Indeed, using the definition of a periodic function one can easily confirm the following property:

LEMMA B.8. *Let $f(x)$ be 2π periodic. Then, the integral*

$$g(x) \equiv \int_0^x f(x)dx$$

is a 2π periodic function if and only if f has zero average on the interval $[-\pi, \pi]$.

Furthermore, simple integration by parts shows that:

THEOREM B.9. *If $f(x)$ is a piecewise continuous, 2π periodic function, and has zero average, then its Fourier series can be integrated term by term to produce the Fourier series of*

$$g(x) = \int_0^x f(y)dy \cong m + \sum_{k=1}^{\infty} \left[\frac{a_k}{k} \sin kx - \frac{b_k}{k} \cos kx \right], \tag{B.2.1}$$

where

$$m \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)dx$$

is the average value of $g(x)$.

Note that integrating the Fourier series (B.2.1) on the interval $[-\pi, \pi]$, and observing that the average of any odd function is zero we obtain that

$$m = \sum_{k=1}^{\infty} \frac{b_k}{k}. \tag{B.2.2}$$

This provides a convenient alternative derivation of the sine coefficients of a Fourier series.

EXAMPLE B.10. Let us consider again the function $f(x) = x$. This function is odd hence, has zero average. Integrating its Fourier series (B.1.7) from 0 to x we obtain the series

$$2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} (1 - \cos kx).$$

The constant term of this series is the average of $x^2/2$. As in (B.2.2):

$$m = \sum_{k=1}^{\infty} \frac{b_k}{k} = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{x^2}{2} dx = \frac{\pi^2}{6}.$$

This immediately yields the Fourier series of the function x^2:

$$x^2 \cong \frac{\pi^2}{3} - 4\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\cos kx}{k^2}.$$

As x^2 is an even function its Fourier series converges everywhere to its periodic extension.

Differentiation. As much as integration is making functions nicer, differentiation does the opposite. Therefore, in order to secure the convergence of the derivative of a Fourier series we must start with the sufficiently nice function; but how nice. If the Theorem B.4 is to be applicable the derivative $f'(x)$ must be piecewise C^1. This means that the function $f(x)$ must be at least continuous and piecewise C^2.

THEOREM B.11. *If $f(x)$ is continuous, piecewise C^2 and 2π periodic, then the term by term differentiation of its Fourier series produces the Fourier series for the derivative*

$$g(x) = f'(x) \cong \sum_{k=1}^{\infty} k\left[b_k \cos kx - a_k \sin kx\right]. \tag{B.2.3}$$

EXAMPLE B.12. Consider again $f(x) = |x|$. Differentiating its Fourier series (B.1.10) we obtain

$$\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin kx}{k}. \tag{B.2.4}$$

On the other hand, the derivative of the absolute value $|x|$ can be represented as a difference of two step functions

$$\sigma(x) - \sigma(-x) = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

A simple calculation shows that the Fourier series of the step function $\sigma(x)$ is

$$\frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\sin kx}{k}. \tag{B.2.5}$$

It is now easy to see that the derivative of the Fourier series of $|x|$ is indeed the Fourier series of the difference $\sigma(x)-\sigma(-x)$. Moreover, according to Theorem B.4 it converges to it.

REMARK B.13. So far, we have defined a Fourier series on the interval $[-\pi, \pi]$ only. This procedure can obviously be easily adopted to any other 2π length interval. In many applications the functions we deal with are defined on intervals of other lengths. Therefore, it would help to show how the formulas we developed so far change if we change the length of the interval. First, let us note that any symmetric interval $[-l, l]$ can be rescaled to the interval $[-\pi, \pi]$ by the simple change of variables

$$y = \frac{\pi}{l}x.$$

The rescaled function

$$\widehat{f}(y) \equiv f\left(\frac{l}{\pi}y\right),$$

which lives on $[-\pi, \pi]$ interval, has the standard Fourier series

$$\widehat{f}(y) \cong \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos ky + b_k \sin ky\right].$$

Going back to the original variable x we deduce that

$$f(x) \cong \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos \frac{k\pi x}{l} + b_k \sin \frac{k\pi x}{l}\right], \tag{B.2.6}$$

where, thanks to the our change of variables, the Fourier coefficients a_k and b_k have the modified formulas

$$a_k = \frac{1}{l}\int_{-l}^{l} f(x) \cos \frac{k\pi x}{l} dx, \qquad b_k = \frac{1}{l}\int_{-l}^{l} f(x) \sin \frac{k\pi x}{l} dx. \tag{B.2.7}$$

According to our convergence Theorem B.4 the given Fourier series converges to the $2l$ periodic extension of $f(x)$ with the midpoint values at jump discontinuities, provided $f(x)$ is piecewise C^1 on $[-l, l]$.

If the function $f(x)$ is defined on an arbitrary interval $[a, b]$ then the first step in the process of developing its Fourier series is to rescale it into a symmetric interval. This can easily be done by the translation

$$y = x - \frac{1}{2}(a + b).$$

The new interval $\left[\frac{1}{2}(a-b), \frac{1}{2}(b-a)\right]$ is symmetric of length $b-a$, and the latter procedure can be utilized.

B.3. Complex Fourier Series and the Delta Function

A natural, and often very convenient, generalization of Fourier series is its complex counterpart, where exponential functions with complex exponents replace sine and cosine functions. Indeed, using Euler's formula

$$e^{\mathrm{i}kx} = \cos kx + \mathrm{i}\sin kx, \tag{B.3.1}$$

we can represent trigonometric functions sine and cosine as

$$\cos kx = \frac{e^{\mathrm{i}kx} + e^{-\mathrm{i}kx}}{2}, \quad \text{and} \quad \sin kx = \frac{e^{\mathrm{i}kx} - e^{-\mathrm{i}kx}}{2\mathrm{i}}. \tag{B.3.2}$$

Substituting these relations into a Fourier series of a piecewise continuous (real or complex valued) function we obtain a new Fourier series representation

$$\sum_{k=1}^{\infty} c_{-k} e^{-\mathrm{i}kx} + \sum_{k=0}^{\infty} c_k e^{\mathrm{i}kx} = \sum_{k=-\infty}^{k=\infty} c_k e^{\mathrm{i}kx}, \tag{B.3.3}$$

where

$$c_k = \frac{1}{2}(a_k + \mathrm{i}b_k), \qquad \text{if} \quad k \le 0, \tag{B.3.4a}$$

and

$$c_k = \frac{1}{2}(a_k - \mathrm{i}b_k), \qquad \text{if} \quad k \ge 0. \tag{B.3.4b}$$

This and Euler's formula (B.3.1) show, in fact, that the *complex Fourier coefficients* can be evaluated directly as

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-\mathrm{i}kx} dx. \tag{B.3.5}$$

Remark B.14. This result is deeper than it seems. It is a consequence (as in the real case) of the *orthonormality* of the complex exponential functions with respect to the Hermitian inner product

$$\langle f; g\rangle \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)} dx, \tag{B.3.6}$$

where $\overline{g(x)}$ denotes the conjugate. Indeed, evoking Euler's formula (B.3.1) one can easily show that

$$\left\langle e^{\mathrm{i}kx}; e^{\mathrm{i}lx} \right\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}(k-l)x}dx = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases} \tag{B.3.7}$$

Therefore, multiplying the complex Fourier series (B.3.3) by $e^{\mathrm{i}lx}$ and integrating term by term we obtain the formula for the complex Fourier coefficients (B.3.5).

EXAMPLE B.15. Let us develop the complex Fourier series for the function $f(x) = x$. Its complex Fourier coefficients are

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-\mathrm{i}kx}dx = \frac{(-1)^{k+1}}{\mathrm{i}k} = \frac{(-1)^k i}{k}.$$

Consequently, the complex Fourier series of $f(x) = x$ is

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k i}{k} e^{\mathrm{i}kx}.$$

The reader is asked to show, using once again Euler's formula (B.3.1), that this is exactly the same series as the sine Fourier series (B.1.7).

EXAMPLE B.16 (**Dirac delta function**). In this example we shall investigate a Fourier series representation of the *delta function* to prove that a Fourier series can converge to a *generalized function*, i.e., a function which although is a limit of a sequence of piecewise continuous functions is not a standard function itself. This example shows also the benefits of using complex Fourier series rather than its real counterpart. First, let us compute complex Fourier coefficients:

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(x)e^{-\mathrm{i}kx}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x)e^{-\mathrm{i}kx}dx = \frac{1}{2\pi}e^{-\mathrm{i}k0} = \frac{1}{2\pi}. \tag{B.3.8}$$

Therefore, the complex Fourier series of the delta function has the form

$$\frac{1}{2\pi}\sum_{k=-\infty}^{k=\infty} e^{\mathrm{i}kx}. \tag{B.3.9}$$

Observe that this series has the form of an infinite (in both directions) geometric series with the ratio $r = e^{\mathrm{i}x}$. This is in contrast to the real Fourier series of the

delta function which, as easily deduced directly from its definition, is

$$\frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \cos kx. \tag{B.3.10}$$

These are obviously two different representations of the same series. However, in order to show that such a series converges to the delta function we will use the complex version.

The complex Fourier series of the delta function (B.3.9) has a form of the geometric series, as we have indicated earlier. In order to determine its convergence let us consider its n^{th} partial sum

$$s_n(x) = \frac{1}{2\pi}\sum_{k=-n}^{k=n} e^{\mathrm{i}kx}.$$

This is the $2n+1$ partial sum of the geometric series with the initial term $a = e^{-\mathrm{i}nx}$ and the ratio $r = e^{\mathrm{i}x}$. It can be therefore computed exactly that

$$\begin{aligned} s_n(x) &= \frac{1}{2\pi}\sum_{k=-n}^{k=n} e^{\mathrm{i}kx} = \frac{1}{2\pi}e^{-\mathrm{i}nx}\left[\frac{e^{\mathrm{i}(2n+1)x}-1}{e^{\mathrm{i}x}-1}\right] = \frac{1}{2\pi}\left[\frac{e^{\mathrm{i}(n+1)x}-e^{-\mathrm{i}nx}}{e^{\mathrm{i}x}-1}\right] \\ &= \frac{1}{2\pi}\left[\frac{e^{-\mathrm{i}\frac{x}{2}}(e^{\mathrm{i}(n+1)x}-e^{-\mathrm{i}nx})}{e^{-\mathrm{i}\frac{x}{2}}(e^{\mathrm{i}x}-1)}\right] = \frac{1}{2\pi}\left[\frac{e^{\mathrm{i}(n+\frac{1}{2})x}-e^{-\mathrm{i}(n+\frac{1}{2})x}}{e^{\mathrm{i}\frac{x}{2}}-e^{-\mathrm{i}\frac{x}{2}}}\right] \\ &= \frac{1}{2\pi}\frac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}}, \end{aligned} \tag{B.3.11}$$

where the representation (B.3.2) of the sine function in terms of complex exponentials was utilized. The sequence of partial sums $s_n(x)$ converges at $x = 0$ to infinity, as easily attested from the (B.3.9). Moreover,

$$\int_{-\pi}^{\pi} s_n(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-n}^{k=n} e^{\mathrm{i}kx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-n}^{k=n}\left[\cos kx + \mathrm{i}\sin kx\right]dx = 1 \tag{B.3.12}$$

as required for the convergence to the delta function. At any other point x the sequence $s_n(x)$ does not converge to zero. In fact, one can see from (B.3.11) that it oscillates faster and faster. It appears, however, when added up over a large domain, that the oscillations cancel out. The Fourier series of the delta function

$\delta(x)$ does not converge to its periodic extension $\widetilde{\delta}(x)$ point-wise. It may be shown that it converges in a *weak sense*, i.e., in the sense of some integral average.

B.4. Convergence of the Fourier Series

In this section we present basic convergence results for Fourier series. A proper, although inevitably limited, presentation of these fundamental results requires some familiarity with the concepts and methods of advanced mathematical analysis. In our effort to make this presentation as self contained as possible we introduce the necessary tools as they become necessary for the analysis of the convergence of Fourier series. We start our presentation by analyzing the *pointwise convergence* of a Fourier series.

Pointwise Convergence.

Proof of Theorem B.4. Our objective is to show that, given a 2π periodic piecewise C^1 function $f(x)$, the limit of the sequence of partial Fourier sums $s_n(x)$ is the arithmetic average of the left hand and the right hand limits of this function, i.e.,

$$\lim_{n\to\infty} s_n(x) = \frac{1}{2}\left[f_+(x) + f_-(x)\right].$$

Consider

$$\begin{aligned} s_n(x) &= \sum_{k=-n}^{n} c_k e^{\mathrm{i}kx} = \sum_{k=-n}^{n} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-\mathrm{i}ky}dy\right) e^{\mathrm{i}kx} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) \sum_{k=-n}^{n} e^{\mathrm{i}k(x-y)}dy, \end{aligned}$$

where the formula (B.3.5) for the complex Fourier coefficients was used. Using the summation formula (B.3.11) and the periodicity of the functions involved we calculate that

$$s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\frac{\sin(n+\frac{1}{2})(x-y)}{\sin\frac{1}{2}(x-y)}dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+y)\frac{\sin(n+\frac{1}{2})y}{\sin\frac{1}{2}y}dy,$$

where to obtain the last integral we changed the variables from y to $x+y$. If we could now show that

$$\lim_{n\to\infty}\frac{1}{\pi}\int_{0}^{\pi} f(x+y)\frac{\sin(n+\frac{1}{2})y}{\sin\frac{1}{2}y}dy = f_+(x), \tag{B.4.1a}$$

and

$$\lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{0} f(x+y)\frac{\sin(n+\frac{1}{2})y}{\sin\frac{1}{2}y}dy = f_{-}(x) \tag{B.4.1b}$$

the proof would be complete. To this end, recalling (B.3.12) allows us to claim that

$$\frac{1}{\pi}\int_{0}^{\pi}\frac{\sin(n+\frac{1}{2})y}{\sin\frac{1}{2}y}dy = 1.$$

Consequently, the statement (B.4.1a) can be replaced by the equivalent statement

$$\lim_{n\to\infty}\int_{0}^{\pi}\frac{f(x+y)-f_{+}(x)}{\sin\frac{y}{2}}\sin(n+\frac{1}{2})ydy = 0. \tag{B.4.2}$$

Let

$$g(y) \equiv \frac{f(x+y)-f_{+}(x)}{\sin\frac{y}{2}}. \tag{B.4.3}$$

Then, using the trigonometric identity for $\sin(n+\frac{1}{2})$, and *Riemann's Lemma*[1] we notice that

$$\begin{aligned}&\lim_{n\to\infty}\int_{0}^{\pi}\frac{f(x+y)-f_{+}(x)}{\sin\frac{y}{2}}\sin(n+\frac{1}{2})ydy\\ &=\lim_{n\to\infty}\frac{1}{\pi}\int_{0}^{\pi}\left(g(y)\sin\frac{y}{2}\right)\cos nydy+\lim_{n\to\infty}\frac{1}{\pi}\int_{0}^{\pi}\left(g(y)\cos\frac{y}{2}\right)\sin nydy = 0\end{aligned}$$

as long as the function $g(y)\sin\frac{y}{2}$ is piecewise continuous. This completes our proof provided we can show that $g(y)$ is piecewise continuous on $[0,\pi]$. Looking at the definition of the function $g(y)$ we can easily see that this is true except possibly at $y=0$. However,

$$\lim_{y\to 0^{+}} g(y) = \lim_{y\to 0^{+}}\frac{f(x+y)-f_{+}(x)}{\frac{y}{2}}\frac{\frac{y}{2}}{\sin\frac{y}{2}} = 2f'_{+}(x),$$

as the function $f(x)$ is piecewise C^1. This fact confirms that $g(y)$ is piecewise continuous. Identical arguments prove (B.4.1b). □

[1]Riemann's Lemma states that if a function $f(x)$ is piecewise continuous on $[a,b]$ then

$$\lim_{\alpha\to\infty}\int_{a}^{b} f(x)\cos\alpha x dx = 0,$$

and similarly if cosine is replaced by sine. For the proof of this fact see for example [**Lang**].

Uniform Convergence. Let us consider a finite-dimensional vector space V, e.g., $\mathbb{R}^n$. In fact, all finite-dimensional spaces of the same dimension are essentially identical (*isomorphic*), as we stated earlier. In order to be able to talk about convergence of sequences in vector spaces we need a measure of a distance between vectors. Hence, let V be a normed vector space with the norm $\|\cdot\| : V \to \mathbb{R}$ (A.1.8).

Definition B.17. Let W be an normed vector space. We say that a sequence $\mathbf{v}_n \in \mathrm{W}$ converges in the norm to $\mathbf{w} \in \mathrm{W}$ if $||\mathbf{v}_n - \mathbf{w}|| \to 0$ as $n \to \infty$.

In a finite-dimensional vector space this is equivalent to a convergence of individual components of $\mathbf{u}^k = (u_1^k, \ldots, u_n^k)$ to the corresponding components of $\mathbf{u} = (u_1, \ldots, u_n)$. This is one way of saying that in a finite-dimensional space all norms are equivalent (see Theorem A.11), i.e., a convergence in one norm quarantines the convergence in any other norm. This, however, is not true in general in the infinite-dimensional vector spaces. All because of the fact that there are many, not necessarily equivalent, norms in such spaces - Example A.13. Also, in infinite dimensional vector spaces the convergence in the norm does not imply - as we will see later - *pointwise convergence*

$$\lim_{k\to\infty} u_i^k(x) = u_i(x), \quad i = 1, \ldots, \infty \quad \text{for all} \quad x, \tag{B.4.4}$$

of the sequence of values of the functions $u_i^k(x)$.

In addition to the convergence in the norm and the pointwise convergence mechanisms there exists yet another form of convergence; the *uniform convergence*.

Definition B.18. A sequence of real valued function $f_k(x)$ converges *uniformly* to a function $f(x)$ on an interval $I \subset \mathbb{R}$ if, given $\epsilon > 0$ there exists an integer n such that

$$|f_k(x) - f(x)| < \epsilon$$

for all $x \in I$ and all $k \geq n$.

The name uniform converges is a reflection of the fact that the sequence of functions converges "the same way" at all points of the interval I. The choice

of the integer n depends only on ϵ and not on a point $x \in I$. Note also, that although uniform convergence implies pointwise convergence the converse may not be true. For example, consider the sequence of functions $f_k(x) = \frac{1}{kx}$ on the interval $(0, 1)$. The sequence converges pointwise to $f(x) \equiv 0$ at all points $x \in (0, 1)$ but it does not converge uniformly on $(0, 1)$ as given ϵ there is no possible choice of the integer n. One can always select a point x so close to 0 that $\frac{1}{nx}$ is larger than ϵ.

The greatest advantage of uniform convergence is that it preserves continuity.

THEOREM B.19. *If a sequence of continuous functions $f_k(x)$ converges uniformly on I then the limit $f(x)$ is a continuous function on I.*

As we are interested here primarily in the convergence of series the following test proves to be particularly useful:

THEOREM B.20 (Weierstrass test). *Suppose that the function $f_k(x)$ are bounded, i.e.,*

$$|f_k(x)| \leq m_k \quad \textit{for all} \quad x \in I,$$

where m_k are positive constants. If, in addition, the series

$$\sum_{k=1}^{\infty} m_k$$

converges, then the series

$$\sum_{k=1}^{\infty} f_k(x)$$

converges uniformly on the interval I to a function $f(x)$. In particular, if the partial sums $s_n(x)$ are continuous functions so is the limit $f(x)$.

We advise the reader to prove that one can integrate and differentiate a uniformly convergent series term by term to obtain a uniformly convergent series, provided the series of derivatives is also uniformly convergent.

PROPOSITION B.21. *If the series*

$$\sum_{k=1}^{\infty} f_k(x) = f(x)$$

is uniformly convergent then

$$\int_a^x \left(\sum_{k=1}^{\infty} f_k(y) \right) dy = \sum_{k=1}^{\infty} \int_a^x f_k(y) dy = \int_a^x f(y) dy$$

is uniformly convergent. Also, if the series of derivatives

$$\sum_{k=1}^{\infty} f_k'(x) = g(x)$$

is uniformly convergent then the original series converges uniformly and $g(x) = f'(x)$.

Let us now examine a uniform convergence of a complex Fourier series

$$\sum_{k=-\infty}^{\infty} c_k e^{-\mathrm{i}kx}. \tag{B.4.5}$$

Since x is real the magnitude

$$|e^{-\mathrm{i}kx}| = |\cos kx + \mathrm{i} \sin kx| = \sqrt{\cos^2 kx + \sin^2 kx} = 1,$$

and

$$|c_k e^{\mathrm{i}kx}| \leq |c_k| \quad \text{for all} \quad x.$$

Applying the Weierstrass test (Theorem B.20) we immediately deduce that

THEOREM B.22. *If the coefficients* $c_k = \frac{1}{2}(a_k \pm ib_k)$ *of a complex Fourier series* (B.4.5) *are such that*

$$\sum_{k=-\infty}^{\infty} |c_k| = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} \sqrt{a_k^2 + b_k^2} < \infty \tag{B.4.6}$$

then the Fourier series converges uniformly to a continuous function $f(x)$. *Moreover, the coefficients* c_k *are equal to the Fourier coefficients of the function* $f(x)$.

Although Theorem B.22 gives conditions guaranteeing the convergence of a Fourier series, it does not tell if the limit $f(x)$ is the original function $\tilde{f}(x)$ for which the Fourier series was derived. Indeed, knowing that the coefficients are derived by integration one may suspect that the functions $f(x)$ and $\tilde{f}(x)$ differ at a finite, or even a countable, number of point. In fact, $\tilde{f}(x)$ may not be continuous. However, it can be shown that if a function is periodic and regular

enough its Fourier series converges uniformly to the very function it got derived from.

THEOREM B.23. *Let $f(x)$ be 2π periodic and piecewise C^1. If $f(x)$ is continuous on an interval (a, b) then its Fourier series converges uniformly to $f(x)$ on any subinterval $[a + \delta, b - \delta]$ where $\delta > 0$.*

In other words, as long as we stay away from the discontinuities of $\tilde{f}(x)$ its Fourier series converges uniformly.

The uniform convergence of a Fourier series, as sanctioned by Theorem B.22, requires that the Fourier coefficients c_k approach 0 when $k \to \infty$. However, the convergence of the Fourier coefficients to zero is only a necessary condition. The coefficients need to converge fast enough to guarantee that the sum is finite. As we know from elementary calculus these coefficients must converge to zero faster then $\frac{1}{k}$. If, for example,

$$|c_k| \leq \frac{M}{|k|^\beta} \qquad \text{for all } k \text{ sufficiently large, where } \beta > 1, \tag{B.4.7}$$

then using the standard ratio test we can conclude that the series of coefficients c_k converges absolutely. This yields, according to the uniform convergence test (B.4.6), the uniform convergence of the Fourier series.

Note also that the faster the Fourier coefficients converge to zero the smoothes the limit function become. Indeed, suppose that for some positive integer n

$$|c_k| \leq \frac{M}{|k|^{\beta+n}} \qquad \text{for all } k \text{ sufficiently large, where } \beta > 1. \tag{B.4.8}$$

Then, we can differentiate the Fourier series up to n-times obtaining always, according to Proposition B.21, a uniformly convergent series.

THEOREM B.24. *If the Fourier coefficients are such that*

$$\sum_{k=-\infty}^{\infty} k^n |c_k| < \infty, \tag{B.4.9}$$

for some nonnegative integer n, then the Fourier series converges to a 2π periodic function $\tilde{f}(x)$ which is n times continuously differentiable. Moreover, for any $m \leq n$, the m times differentiated Fourier series converges uniformly to the corresponding derivative $\tilde{f}^{(m)}(x)$.

Hence, we may analyze the smoothness of a function by looking at how fast its Fourier coefficients approach zero. For example, if they converge to zero exponentially the function is infinitely differentiable but not necessarily analytic.

Convergence in the Norm. In order to be able to discuss this important aspect of convergence of Fourier series we need to take a short detour introducing the concept of a Hilbert space. The precise definition of a Hilbert space, which is rather technical, is beyond the scope of these notes. We will therefore present here a "working" version which will enable us to deal with Fourier series of functions such as, for example, the delta function.

DEFINITION B.25. A complex-valued function $f(x)$ is said to be *square-integrable* on the $[-\pi, \pi]$ if

$$||f|| \equiv \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx \right)^{\frac{1}{2}} < \infty. \tag{B.4.10}$$

In particular, but not only, every piecewise continuous function is square-integrable. Other, more singular, functions may also be square-integrable, e.g., x^{-r} is as long as $r < \frac{1}{2}$. Note also, that a function may not be square-integrable on some intervals while it is on some others.

The set of all square-integrable functions on $[-\pi, \pi]$, which is usually denoted as $\mathrm{L}^2[-\pi, \pi]$, is a complex vector space with the pointwise operations of addition and multiplication by a scalar. This is a strait forward consequence of the triangle inequality

$$|af(x) + bg(x)| \leq |a||f(x)| + |b||g(x)|.$$

Moreover, it is a normed vectors space with the L^2-norm given by (B.4.10) provided, we identify as one any two functions which differ at most on a *measure zero* set (for example, at a finite number of points of $[-\pi, \pi]$)[2]. L^2 space is a very special normed vector space as the L^2-norm is based on the (Hermitian) inner product

$$\langle f; g \rangle \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)} dx. \tag{B.4.11}$$

[2]Such an identification is necessary as otherwise the L^2-norm would be zero for other than the constant zero function. In fact, it would not be a norm according to its definition A.1.8.

Indeed, $\langle f; f\rangle = ||f||^2$. The inner product is well defined and finite for any two square-integrable functions thanks to the Cauchy-Schwarz inequality.

EXAMPLE B.26. Consider a square-integrable function $f(x)$ on $[-\pi, \pi]$. Its Fourier coefficients are inner products of f with the exponential functions:

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\mathrm{i}kx}dx = \langle f; e^{\mathrm{i}kx}\rangle.$$

Note also that as shown in (B.3.7)

$$||e^{\mathrm{i}kx}||^2 = \langle e^{\mathrm{i}kx}; e^{\mathrm{i}kx}\rangle = 1,$$

while

$$\langle e^{\mathrm{i}kx}; e^{\mathrm{i}lx}\rangle = 0 \qquad \text{if} \quad k \neq l.$$

We say that the infinite system of complex exponentials is an orthonormal system of functions (vectors) in $\mathrm{L}^2[-\pi, \pi]$, i.e., all elements have unit length and are mutually orthogonal as the scaler product of any two different elements vanish.

The sequence of square-integrable functions f_n converges in the L^2-norm to $g(x) \in \mathrm{L}^2[-\pi, \pi]$ if

$$||f_n(x) - g(x)||^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_n(x) - g(x)|^2 dx \to 0 \quad \text{as} \quad n \to \infty. \tag{B.4.12}$$

Hence, let us consider the convergence of the sequence of partial sums $s_n(x)$ of the Fourier series of a square-integrable function $f(x)$. As each element of the sequence $s_n(x)$ is obviously square-integrable we can evaluate $||f - s_n||^2$. Using the definition of the sequence $s_n(x)$, and the linearity and symmetry of the inner product we obtain that

$$\begin{aligned} ||f - s_n||^2 &= ||f||^2 - 2\langle f; s_n\rangle + ||s_n||^2 = ||f||^2 - 2\sum_{k=-n}^{n} \overline{c_k}\langle f; e^{\mathrm{i}kx}\rangle + ||s_n||^2 \\ &= ||f||^2 - 2\sum_{k=-n}^{n} \overline{c_k}c_k + ||s_n||^2 = ||f||^2 - 2\sum_{k=-n}^{n} \overline{c_k}c_k + \sum_{k=-n}^{n} |c_k|^2 \\ &= ||f||^2 - ||s_n||^2. \end{aligned} \tag{B.4.13}$$

As the left hand side is always nonnegative

$$\sum_{k=-n}^{n} |c_k|^2 \leq ||f||^2$$

for any integer n. We have therefore proved the following:

PROPOSITION B.27 (Bessel's inequality). *If the function f is square-integrable on $[-\pi, \pi]$ then its Fourier coefficients are such that*

$$\frac{1}{4}a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}\left(a_k^2 + b_k^2\right) = \sum_{-\infty}^{\infty} |c_k|^2 \leq ||f||^2. \tag{B.4.14}$$

This immediately implies that the individual Fourier coefficients (both complex and real) of a square-integrable function approach zero as $|k|$ approaches infinity. The convergence of the series (B.4.14) requires, in fact, more. The Fourier coefficients must tend to zero fast enough. If we postulate, as we did before in (B.4.7), that

$$|c_k| \leq \frac{M}{|k|^\beta} \qquad \text{for sufficiently large} \quad k, \tag{B.4.15}$$

then selecting $\beta > \frac{1}{2}$ guarantees the convergence already. Note this is a slower rate of decay than the one needed for the uniform convergence. One may have a Fourier series which converges in the L^2-norm but not uniformly.

If the Fourier series of a square-integrable function $f(x)$ converges in the L^2-norm to the very function it got derived from the Bessel's inequality becomes equality. Indeed, as evident from (B.4.13)

$$||f - s_n||^2 = ||f||^2 - ||s_n||^2 \tag{B.4.16}$$

for any Fourier partial sum $s_n(x)$. Therefore,

$$\lim_{n\to\infty} ||f - s_n|| = 0,$$

if and only if

$$||f||^2 = \lim_{n\to\infty} ||s_n||^2 = \sum_{k=-\infty}^{\infty} |c_k|^2. \tag{B.4.17}$$

This formula becomes a convenient criterion for the L^2-convergence of square-integrable functions.

PROPOSITION B.28 (Plancherel formula). *A Fourier series of a square integrable function $f \in L^2[-\pi, \pi]$ converges in the L^2-norm to $f(x)$ if and only if*

$$||f||^2 = \sum_{k=-\infty}^{\infty} |c_k|^2. \tag{B.4.18}$$

Before we finally show that the Fourier series of every L^2-function $f(x)$ converges in the L^2-norm to $f(x)$ let us note that one of the immediate consequences of the Plancheler formula is that a function is uniquely determined by its Fourier coefficients.

COROLLARY B.29. *Two functions $f, g \in L^2[-\pi, \pi]$ have the same Fourier coefficients if and only if $f = g$.*

The main result of the theory of Fourier series is the following convergence theorem.

THEOREM B.30. *Let $s_n(x)$ denote the n^{th} partial sum of the Fourier series of the square-integrable function $f(x) \in L^2[-\pi, \pi]$, then*

$$\lim_{n\to\infty} ||f - s_n|| = 0. \tag{B.4.19}$$

In other words, any square-integrable function on $[-\pi, \pi]$ can be uniquely represented by the infinite system[3] of complex exponentials $e^{\mathrm{i}kx}$, $k = 0, \pm 1, \pm 2, \ldots$.

PROOF OF THEOREM B.30. We shall provide here the prove for continuous functions only. Proving the validity of this statement for all square-integrable functions requires some extra work, see [**Carrier, Krook and Pearson**], and also [**Brown and Churchill**], and [**Kammler**]. According to Theorem B.23, if a function $f(x)$ is piecewise C^1 and continuous, its Fourier series converges to $f(x)$ everywhere, i.e.,

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{\mathrm{i}kx}.$$

[3]This property of an infinite system of elements of a Hilbert space is known as the *completeness* of such a system. As the exponentials functions are orthonormal in $\mathrm{L}^2[-\pi, \pi]$, this is the completeness of an orthonormal system of vectors.

Therefore, utilizing the fact that we are allowed to multiply and integrate uniformly convergent series term by term, we have

$$||f||^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \int_{-\pi}^{\pi} f(x)\overline{f(x)} = \sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi} f(x)\bar{c}_k e^{-\mathrm{i}kx}dx$$
$$= \sum_{k=-\infty}^{\infty}\bar{c}_k c_k = \sum_{k=-\infty}^{\infty}|c_k|^2,$$

and Plancherel's identity (B.4.18) holds proving the convergence of the Fourier series. □

B.5. Fourier Transform

The *Fourier transform* may be viewed as a generalization of the Fourier series to an infinite interval. That is, take a Fourier series on a symmetric interval $[-l, l]$, and consider taking the limit as the length $l \to \infty$. The result of such a process is a Fourier series on an infinite interval. The corresponding representation of a function is given the name of the Fourier transform. Indeed, let us look for the representation of a $f(x)$ by a rescaled complex Fourier series (B.3.3) on an interval $[-l, l]$ in the following form

$$\sum_{r=-\infty}^{\infty}\sqrt{\frac{\pi}{2}}\frac{\hat{f}(k_r)}{l}e^{ik_r x} \tag{B.5.1}$$

where the sum is over the discrete set of frequencies

$$k_r = \frac{r\pi}{l}, \quad r = 0, \pm 1, \pm 2, \ldots, \tag{B.5.2}$$

to incorporate all trigonometric functions of period $2l$. Therefore, based on the form of Fourier coefficients (B.3.5) we get that

$$\hat{f}(k_r) = \frac{1}{\sqrt{2\pi}}\int_{-l}^{l} f(x)e^{-ik_r x}dx. \tag{B.5.3}$$

This allows us to pass to the limit $l \to \infty$, and to get the infinite integral

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx \tag{B.5.4}$$

known as the *Fourier transform* of the function $f(x)$. If the function $f(x)$ is piecewise continuous and decays to 0 reasonably fast as $x \to \pm\infty$, the Fourier transform $\hat{f}(k)$ is defined for all frequencies $-\infty < k < \infty$.

REMARK B.31. Note that the discrete frequencies (B.5.2) used to represent the function $f(x)$ on the interval $[-l, l]$ are equally spaced as

$$\triangle k = k_{r+1} - k_r = \frac{\pi}{l}.$$

As $l \to \infty$, the spacing between frequencies $\triangle k \to 0$, and the frequencies become more and more densely distributed. That is why in the Fourier transform limit we anticipate that all frequencies participate in representing a function.

Taking into account the fact that the discrete frequencies k_r are equally spaced allows us to re-write the Fourier series (B.5.1) as

$$\frac{1}{\sqrt{2\pi}} \sum_{r=-\infty}^{\infty} \hat{f}(k_r) e^{ik_r x} \triangle k. \tag{B.5.5}$$

This has the form of a Riemann sum for the function $\hat{f}(k)e^{ikx}$. Assuming that it converges as $\triangle k \to 0$ we obtain

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk. \tag{B.5.6}$$

The series (B.5.5) becomes a *Fourier integral* which reconstructs the function $f(x)$ as a continuous superposition of complex exponential functions e^{ikx}. For example, if the function $f(x)$ is piecewise continuously differentiable everywhere and it decays reasonably fast as $|x| \to \infty$, then the inverse Fourier integral (B.5.6) converges to $f(x)$ at all points of continuity. At the jump discontinuities it converges to the midpoint $\frac{1}{2}[f_+(x) - f_-(x)]$. Indeed, it can be shown[4] that

THEOREM B.32. *If the function $f(x)$ is piecewise continuous and square-integrable (for $-\infty < x < \infty$), then its Fourier transform $\widehat{f}(k)$ is well defined and square-integrable. Moreover, if the right and left hand limits $f_-(x)$, $f_+(x)$ and $f'_-(x)$, $f'_+(x)$ exist, then the Fourier integral (B.5.6) converges to the average*

[4] A rigorous proof of a more general statement can be found in [**Carrier, Krook and Pearson**].

value $\frac{1}{2}[f_-(x) + f_+(x)]$. *In particular, if* $f(x)$ *is continuously differentiable at a point* x, *then the Fourier integral equals the value of* $f(x)$.

EXAMPLE B.33. Let us consider an exponentially decaying pulse

$$f(x) = e^{-a|x|}. \tag{B.5.7}$$

Its Fourier transform

$$\begin{aligned}\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(a-ik)x} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(a+ik)x} dx \\ &= \frac{1}{\sqrt{2\pi}} \left. \frac{e^{(a-ik)x}}{a-ik} \right|_{-\infty}^{0} - \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-(a+ik)x}}{a+ik} \right|_{0}^{\infty} = \sqrt{\frac{2}{\pi}} \frac{a}{k^2+a^2}.\end{aligned} \tag{B.5.8}$$

Notice that the Fourier transform of our pulse $f(x)$ which is real and even, is itself real and even. The inverse Fourier transform gives

$$e^{-a|x|} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a e^{ikx}}{k^2+a^2} dk = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a \cos kx}{k^2+a^2} dk. \tag{B.5.9}$$

Here the imaginary part of the integral vanishes as the integrant is odd.

We have been discussing the Fourier transform of the pulse function not without reason. First, it is interesting to notice that as $a \to 0$ the pulse approaches the constant function $g(x) \equiv 1$. Moreover, the limit of its Fourier transform (B.5.8) is

$$\lim_{a \to 0} \sqrt{\frac{2}{\pi}} \frac{2a}{k^2+a^2} = \begin{cases} 0, & k \neq 0 \\ \infty, & k = 0 \end{cases}. \tag{B.5.10}$$

Comparing this with the original construction (1.2.8) of the delta function as the limit of approximating functions we notice that

$$\delta(x) = \lim_{n \to \infty} \frac{n}{\pi(1+n^2x^2)} = \lim_{a \to 0} \frac{a}{\pi(a^2+x^2)}, \tag{B.5.11}$$

provided $n = 1/a$. This, in turn, allows us to write the Fourier transform of the constant function $f(x) \equiv 1$ as

$$\hat{f}(k) = \sqrt{2\pi}\delta(k). \tag{B.5.12}$$

Equivalently, this should mean that

$$\delta(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}dx. \tag{B.5.13}$$

The infinite integral on the right hand side does not converge[5]. However, from the definition of the delta function we have

$$\int_{-\infty}^{\infty} \delta(k)e^{-ikx}dk = e^{ik0} = 1, \tag{B.5.14}$$

which implies that the Fourier transform of the delta function is a constant function

$$\hat{\delta}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x)e^{-ikx}dx = \frac{e^{-ik0}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}. \tag{B.5.15}$$

To determine the Fourier transform of the delta function concentrated at an arbitrary position y, namely $\delta_y(x)$ we cite the following theorem:

THEOREM B.34.

- *If the Fourier transform of the function $f(x)$ is $\hat{f}(k)$, then the transform of $\hat{f}(x)$ is $f(-k)$.*
- *If the function $f(x)$ has the Fourier transform $\hat{f}(k)$, then the Fourier transform of the shifted function $f(x-y)$ is $e^{-iky}\hat{f}(k)$. By analogy, the transform of the product $e^{inx}f(x)$ is the shifted Fourier transform $\hat{f}(k-n)$.*

Therefore, according to (B.5.15) the Fourier transform of the delta function at $y \neq 0$ is

$$\hat{\delta}_y(k) = \frac{e^{-iky}}{\sqrt{2\pi}}. \tag{B.5.16}$$

Thus, combining the definitions of the Fourier transform and its inverse, as well as using the basic properties of the delta function leads to the identity

$$f(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{-ik(x-y)}dxdk \tag{B.5.17}$$

valid for suitable functions.

[5]It may, however, be interpreted in the context of generalized functions.

Differentiation and Integration.

As our main objective in dealing with Fourier transforms is to show how they can be used to solve differential equations we embark now on the analysis of the process of differentiation and integration of these functions.

Let us consider the Fourier transform of the derivative of $f(x)$

$$\widehat{f}'(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-ikx} dx, \tag{B.5.18}$$

assuming that $f'(x)$ is behaving well enough so the integral exists. Integrating by parts, and taking into account that $f(x)$ approaches 0 at infinities rapidly enough, we obtain that

$$\widehat{f}'(k) = \frac{ik}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \tag{B.5.19}$$

Thus, we have that

PROPOSITION B.35. *The Fourier transform of the derivative $f'(x)$ of a function is obtained by multiplying its Fourier transform by ik:*

$$\widehat{f}'(k) = ik\hat{f}(k). \tag{B.5.20}$$

Iterating the formula (B.5.20) yields that

COROLLARY B.36. *The Fourier transform of $f^{(n)}(x)$ is $(ik)^n \hat{f}(k)$.*

Consider now

$$g(x) = \int_{-\infty}^{x} f(y) dy.$$

We are interested in finding the Fourier transform $\hat{g}(k)$. To this end notice first that

$$\lim_{x \to -\infty} g(x) = 0, \quad \text{while} \quad \lim_{x \to +\infty} = \int_{-\infty}^{\infty} f(y) dy = c.$$

Therefore, consider the function $h(x) = g(x) - c\sigma(x)$ which decays to 0 at both $\pm\infty$. As the Fourier transform of the step function $\sigma(x)$ is

$$\hat{\sigma}(k) = \sqrt{\frac{2}{\pi}} \delta(k) - \frac{i}{k\sqrt{2\pi}} \tag{B.5.21}$$

we obtain that

$$\hat{h}(k) = \hat{g}(k) - c\sqrt{\frac{2}{\pi}} \delta(k) + \frac{ic}{k\sqrt{2\pi}}. \tag{B.5.22}$$

On the other hand,

$$h'(x) = f(x) - c\delta(x).$$

Applying Proposition B.35 we conclude after some manipulations that

$$\hat{g}(k) = \frac{\hat{f}(k)}{ik} + c\sqrt{\frac{\pi}{2}}\delta(k). \tag{B.5.23}$$

EXAMPLE B.37. Consider the boundary value problem

$$-u'' + \omega^2 u = h(x), \quad -\infty < x < \infty, \tag{B.5.24}$$

where $\omega > 0$, and the solution is assumed square-integrable implying that u approaches 0 at $\pm\infty$. Taking the Fourier transform of both sides of the differential equation renders an algebraic equation relating the Fourier transforms of u and h. Indeed,

$$k^2\widehat{u}(k) + \omega^2\widehat{u} = \widehat{h}(k). \tag{B.5.25}$$

The transformed equation can now be solved for the Fourier transform of the solution

$$\widehat{u}(k) = \frac{\widehat{h}(k)}{k^2 + \omega^2}. \tag{B.5.26}$$

Using the Fourier formula (B.5.6) we reconstruct the solution as

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\widehat{h}(k)e^{ikx}}{k^2 + \omega^2} dk. \tag{B.5.27}$$

One of the most important and interesting cases is the one in which $h(x) = \delta_y(x)$, that is when the forcing term is in the form of the impulse concentrated at y. The corresponding solution is the Green's function $u(x) = G(x, y)$. According to (B.5.26) and Theorem B.34, its Fourier transform with respect to x is

$$\widehat{G}(k, y) = \frac{e^{-iky}}{k^2 + \omega^2}. \tag{B.5.28}$$

Notice first that according to Example B.33 the inverse Fourier transform of the reciprocal of $k^2 + \omega^2$ is

$$\frac{e^{-\omega x}}{2\omega}.$$

Secondly, the exponential term in the numerator implies a shift. Therefore, the Green's function for our boundary-value problem is

$$G(x,y) = \frac{e^{-\omega|x-y|}}{2\omega}.$$

The Green's function satisfies the differential equation everywhere except at $x = y$ where it has a jump discontinuity of unit magnitude. It also satisfies the boundary conditions as it decays to 0 as $|x| \to \infty$. Invoking the general superposition principle for Green's function we obtain the solution to our boundary-value problem with an arbitrary forcing term as

$$u(x) = \int_{-\infty}^{\infty} G(x,y)h(y)dy = \frac{1}{2\omega}\int_{-\infty}^{\infty} e^{-\omega|x-y|}h(y)dy. \tag{B.5.29}$$

Note that the Green's function $G(x,y)$ depends only on the difference $x - y$ and that the solution $u(x)$ takes the form of the *convolution*. Namely,

$$u(x) = \int_{-\infty}^{\infty} g(x-y)h(y)dy = g(x) * h(x). \tag{B.5.30}$$

On the other hand, as we saw earlier in (B.5.26), the Fourier transform of the solution to our boundary value problem is a product of the Fourier transforms of the Green's function and the forcing term. Indeed, we have that:

THEOREM B.38. *The Fourier transform of the convolution $u(x) = g(x) * h(x)$ of two functions is, up to multiple, the product of their Fourier transforms*

$$\widehat{u}(k) = \sqrt{2\pi}\widehat{g}(k)\widehat{h}(k). \tag{B.5.31}$$

By symmetry, the Fourier transform of the product $h(x) = f(x)g(x)$ is a multiple of the convolution of their Fourier transforms

$$\widehat{h}(k) = \frac{1}{\sqrt{2\pi}}\widehat{f}(k) * \widehat{g}(k). \tag{B.5.32}$$

Bibliography

[Bleecker and Csordas] , D. Bleeker and G. Csordas, *Basic Partial Differential Equations*, International Press, Cambridge, Massachusetts, 1996.

[Boyce and DiPrima] , W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 5th edition, John Wiley & Sons Inc., New York, 1992.

[Brown and Churchill] , J. W. Brown, and R. V. Churchill, *Fourier Series and Boundary Value Problems*, McGraw-Hill, New York, 1993.

[Carrier, Krook and Pearson] , G. F. Carrier, M. Krook, and C. E. Pearson, *Functions of a Complex Variable, Theory and Technique*, McGraw-Hill, New York, 1966.

[Davis] , *Mathematics of Wave Propagation*, Princeton University Press, Princeton, NJ, 2000.

[Kammler] D. W. Kammler, *A First Course in Fourier Analysis*, Prentice Hall, Upper Saddler River, NJ, 2000.

[Keane] , M. K. Keane, *A Very Applied Course in Partial Differential Equations*, Prentice Hall, Upper Saddler River, NJ, 2002.

[Kevorkian] J. Kevorkian, *Partial Differential Equations*, Second Edition, Springer-Verlag, New York, 2000.

[Knobel] R. Knobel, *An Introduction to the Mathematical Theory of Waves*, American Mathematical Society, Providance, 2000.

[Lang] S. Lang, *Undergraduate Analysis*, Springer-Verlag, New York, 1983.

[Ogden] R. W. Ogden, *Non-Linear Elastic Deformations*, Dover Publications, Mineola, N.Y., 1997.

[Smoller] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York, 1983.

[Taylor] M. E. Taylor, *Partial Differential Equations (Basic Theory)*, Springer-Verlag, New York, 1996.

[Ziemer] W. P. Ziemer, *Weakly Differentiable Functions*, Springer-Verlag, New York, 1989.

Index