CONNECTIONS ON HIGHER ORDER FRAME BUNDLES

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ABSTRACT

In the paper we present the analysis of connections on frame bundles of higher order contact, with special emphasis on the question of local flatness.

1.Introduction

The motivation for the research presented in this paper comes from the theory of continuous distributions of defects in continuous material bodies, in particular, the proposed generalization of the theory of continuous distributions of dislocations of simple elastic bodies to incorporate the higher order defects.[2,3,4] Recognizing that a definite G-structure can always be associated with the uniform elastic body the fundamental problem of this theory is the question of local integrability of such a structure. In purely mathematical terms this is equivalent to determining the existence of locally flat G-connections.

In this short paper we concentrate our efforts on studying the connections on frame bundles of order 2 and higher. We analyze both the form and the structure of these connections using as the fundamental concepts the notions of the fundamental form[6], the standard horizontal space of a frame[3] and, introduced here, the concept of the characteristic manifold of a connection. We discuss the conditions under which a connection on a bundle of frames of higher order becomes simple and locally flat. In this context we show the interplay between the simplicity, local integrability and the vanishing of the torsion. Although, our analysis, for most part, is presented in the general case of the semi-holonomic frame bundles, some interesting observations about the holonomic case are also made.

2.Canonical Forms

Let $\mathcal{M}$ be an n-dimensional connected smooth manifold. Denote by $\hat{\mathrm{H}}^k(\mathcal{M})$ the space of all k-order frames of $\mathcal{M}$. Respectively, let $\tilde{\mathrm{H}}^k(\mathcal{M})$ be the space of all *holonomic* k-frames of $\mathcal{M}$. While $\tilde{\mathrm{H}}^k(\mathcal{M})$ is the space of k-order jets of all local diffeomorphisms of $\mathbb{R}^n$ into $\mathcal{M}$ with the source at the origin and the target anywhere in $\mathcal{M}$, $\hat{\mathrm{H}}^k(\mathcal{M})$ can be thought of (recursively) as the space of first jets of all local sections of $\hat{\mathrm{H}}^{k-1}(\mathcal{M})$.[7] For example, let $\mathrm{f}:\mathrm{U}\to\mathrm{H}^1(\mathcal{M})$ be a differentiable map of a neighborhood of the origin of $\mathbb{R}^n$ into $\mathrm{H}^1(\mathcal{M})$ and such that $\pi^1\circ\mathrm{f}:\mathrm{U}\to\mathcal{M}$ is a local diffeomorphism where $\pi^1:\mathrm{H}^1(\mathcal{M})\to\mathcal{M}$ is the

standard projection. The first jet of f at 0 can be considered a (non-holonomic) 2-frame of $\mathcal{M}$ at $\pi^1(\mathrm{f}(0))$. If, in addition, f is such that the first jet of $\pi^1 \circ \mathrm{f}$ at 0 is equal to f(0) the corresponding 2-frame is called *semi-holonomic*. Extending this definition recursively to an arbitrary k-order we obtain the set of all semi-holonomic frames of $\mathcal{M}$, say $\mathrm{H}^k(\mathcal{M})$. Hence, as mentioned in the Introduction, we shall be dealing only with semi-holonomic frames.

The space $\mathrm{H}^k(\mathcal{M})$ (also $\tilde{\mathrm{H}}^k(\mathcal{M})$) is a principal bundle over $\mathcal{M}$. Its structure group G^k is the fibre at 0 of $\mathrm{H}^k(I\!R^n)$, i.e. the group of first jets at the origin of all local sections of $\mathrm{H}^{k-1}(I\!R^n)$ satisfying the condition of semi-holonomicity. The structure group of the bundle of holonomic frames $\tilde{\mathrm{G}}^k$ is the set of k-jets of all origin preserving local diffeomorphisms of $I\!R^n$. In particular, $\mathrm{G}^1 = \tilde{\mathrm{G}}^1 = \mathrm{GL}(n, I\!R)$. Given two, different order, frame bundles over $\mathcal{M}$, say $\mathrm{H}^k(\mathcal{M})$ and $\mathrm{H}^m(\mathcal{M})$, where $k > m$ there exists a natural projection $\pi^k_m : \mathrm{H}^k(\mathcal{M}) \to \mathrm{H}^m(\mathcal{M})$ making $\mathrm{H}^k(\mathcal{M})$ in to an affine bundle over $\mathrm{H}^m(\mathcal{M})$ the structure group of which is the kernel N^k_m of the induced epimorphism $\mu^k_m : \mathrm{G}^k \to \mathrm{G}^m$. It is easy to see that N^k_m is the normal subgroup of G^k, N^k_{k-1} is canonically isomorphic to the abelian vector group of all multilinear $I\!R^n$ – valued k-forms on $I\!R^n$ and that G^k is the semi-direct product of G^r and N^k_r for any r > k. Similarly, $\tilde{\mathrm{H}}^k(\mathcal{M})$, which is a subbundle of $\mathrm{H}^k(\mathcal{M})$, is an affine bundle over $\tilde{\mathrm{H}}^m(\mathcal{M})$. Its structure group $\tilde{\mathrm{N}}^k_m$ contains symmetric multilinear $I\!R^n$-valued k-forms on $I\!R^n$.

To be able to introduce the notion of the *fundamental form* on a frame bundle let us recall[6,8] first that given the (semi-holonomic) k-frame p^k there exists an isomorphism, called the *admissible isomorphism*, $h^{k-1} : \mathrm{H}^{k-1}(I\!R^n) \to \mathrm{H}^{k-1}(\mathcal{M})$ such that $\mathrm{p}^k = j^1 h^{k-1}(e^{k-1})$ where e^{k-1} denotes the identity of the group G^{k-1}. Indeed, e.g., for any holonomic k-frame p^k there exists a local, about the origin of $I\!R^n$, diffeomorphism $f : \mathrm{U} \subset I\!R^n \to \mathcal{M}$ such that $\mathrm{p}^n = j^k f(0)$. The corresponding isomorphism h^{k-1} is then defined by the condition that $j^{k-1} f \circ f = h^{k-1} \circ j^{k-1} id$ where, $j^{k-1} f : \mathcal{M} \to \tilde{\mathrm{H}}^k(\mathcal{M})$. The isomorphism h^{k-1} induces a linear isomorphism $\tilde{h}^{k-1} : T_{e^{k-1}}\mathrm{H}^{k-1}(I\!R^n) \to T_{\pi^k_{k-1}(\mathrm{p}^k)}\mathrm{H}^{k-1}(\mathcal{M})$. Since $\mathrm{H}^{k-1}(I\!R^n) = I\!R^n \times \mathrm{G}^{k-1}$ we have that $T_{e^{k-1}}\mathrm{H}(I\!R^n) = I\!R^n \oplus \mathfrak{g}^{k-1}$. Here $\mathfrak{g}^{k-1}$ is the Lie algebra of the structure group G^{k-1}. Generalizing the concept of the solder form one defines the *fundamental form* on $\mathrm{H}^k(\mathcal{M})$ as the $I\!R^n \oplus \mathfrak{g}^{k-1}$-valued 1-form θ^k such that given the k-frame p^k and the tangent vector $\xi \in T_{\mathrm{p}^k}\mathrm{H}^k(\mathcal{M})$

$$\tilde{h}^{k-1}(\theta^k(\xi)) = T\pi^k_{k-1}(\xi). \tag{1}$$

where $T\pi^k_{k-1}$ denotes the tangent map.

The form θ^k is equivariant with respect to the right action of G^k on $\mathrm{H}^k(\mathcal{M})$ and the action ρ^k of G^k on the tangent space $T\mathrm{H}^k(B)$. The latter being just an extension of the natural action of GL(n,$I\!R$) on $I\!R^n$. Namely,

$$\theta^k(\mathfrak{R}_{\mathrm{g}^k}(\xi)) = \rho^k((\mathrm{g}^k)^{-1})\theta^k(\xi) \tag{2}$$

for any $g^k \in G^k$ and any tangent vector ξ where $\mathfrak{R}_{g^k}$ represents the right action by the group element g^k. The adjoint action ρ^k of the structure group G^k on $I\!R^n \oplus \mathfrak{g}^{k-1}$ is such that for any $\mathfrak{X}^{k-1} \in \mathfrak{g}^{k-1}$ and any $g^k \in G^k$

$$\rho^k(g^k)\mathfrak{X}^{k-1} = \mathfrak{a}d^k(\mu^k_{k-1}(g^k))\mathfrak{X}^{k-1}. \tag{3}$$

On the other hand, for any $v \in I\!R^n$

$$\rho^k(g^k)(v,0) = (\mu^k_1(g^k)v, \lambda^k(g^k, v)) \tag{4}$$

for some mapping $\lambda^k : G^k \times I\!R^n \to \mathfrak{g}^{k-1}$ such that $T\mu^{k-1}_{k-2} \circ \lambda^k \equiv \lambda^{k-1} \circ \{\mu^k_{k-1} \times id_{I\!R^n}\}$. For a fixed $g^k \in G^k$ $\lambda^k(g^k, \cdot) : I\!R^n \to \mathfrak{g}^{k-1}$ is linear. It is identically zero if, and only if, $g^k \in G^1$. Moreover,

$$\lambda^k(g^k_2 g^k_1, v) = \lambda^k(g^k_2, \mu^k_1(g^k_1)v) + \mathfrak{a}d^k(T\mu^k_{k-1}(g^k_2))\lambda^k(g^k_1, v) \tag{5}$$

for any $g^k_1, g^k_2 \in G^k$.

We also note that the fundamental form θ^k decomposes canonically into the sum of 1-forms with values in the subalgebras of $I\!R^n \oplus \mathfrak{g}^{k-1}$. In particular, $\theta^k = \theta_{-1} + \theta_k$ where θ_{-1} is just a projection of θ^k onto $I\!R^n$ while θ_k takes values in $\{0\} \oplus \mathfrak{g}^{k-1}$. Furthermore, as for any $r < k$ the group G^k can be represented as the semidirect product of $G^r \equiv \mu^k_r(G^k)$ and the kernel N^k_r of the epimorphism μ^k_r, we can write

$$\theta^k = \theta_{-1} + \theta_r + \theta^k_r \tag{6}$$

where $\pi^{k*}_r \theta_r = \mu^k_{r*}\theta_k$ and where θ^k_r takes values in $\mathfrak{n}^{k-1}_{r-1}$, the algebra of the Lie group N^{k-1}_{r-1}. As a result of the equivariance of the fundamental form θ^k, Eq.2, we get that

$$\theta_{-1}(\mathfrak{R}_{g^k *}(\xi)) = \mu^k_1((g^k)^{-1})\theta_{-1}(\xi) \tag{7}$$

and that

$$\theta_k(\mathfrak{R}_{g^k *}(\xi)) = \mathfrak{a}d^k(\mu^k_{k-1}((g^k)^{-1}))\theta_k(\xi) + \lambda^k((g^k)^{-1}, \theta_{-1}(\xi)). \tag{8}$$

for any vector $\xi \in TH^k(\mathcal{M})$.

3.Connections on Frame Bundles

DEFINITION 1. *Let* $\mathrm{p}^k \in \mathrm{H}^k(\mathcal{M})$ *and let* h^{k-1} *denote the corresponding admissible isomorphism. The* **standard horizontal space of the frame** p^k *is the n-dimensional vector space* $\mathcal{SH}(\mathrm{p}^k) \equiv \tilde{h}^{k-1}(I\!R^n, 0)$.[3]

Suppose now that $q : \mathrm{H}^{k-1}(\mathcal{M}) \to \mathrm{H}^k(\mathcal{M})$ is a local section and let p^k be in the image of q. Given $\xi \in \mathcal{SH}(\mathrm{p}^k)$ $\quad q^*\theta^k(\xi) = \theta^k(q_*(\xi)) \in I\!R^n \oplus \{0\}$ as $\tilde{h}^{k-1}(\theta^k(q_*(\xi)) = T\pi^k_{k-1}(q_*(\xi)) = \xi$ by Eq. 1. Note that this is true irrespective of the section q as long as p^k belong to its image. Therefore we have:

PROPOSITION 1.[3] *Let* p^k *be a k-frame.* $\xi \in \mathcal{SH}(\mathrm{p}^k)$ *if, and only if, given the section* $q : \mathrm{H}^{k-1}(\mathcal{M}) \to \mathrm{H}^k(\mathcal{M})$ *such that* p^k *is in the image of* q, $\quad q^*\theta_k(\xi) \equiv 0$.

To get some true insight into the structure and the form of connections on the bundle of k-frames (holonomic or not) we start by recalling the construction of an arbitrary k-connection ω^k in terms of the so-called $\mathcal{E}$-connection[8]. We adapt, however, this presentation to our particular needs. First, we note that the local section $q : \mathrm{H}^r(\mathcal{M}) \to \mathrm{H}^k(\mathcal{M})$ is *invariant* (G^r*-invariant*) if for any $\mathrm{p}^r \in \mathrm{H}^r(\mathcal{M})$ and every $\mathrm{g}^r \in \mathrm{G}^r$

$$\mathrm{q}(\mathfrak{R}_{\mathrm{g}^r}(\mathrm{p}^r)) = \mathfrak{R}_{\nu^k_r(\mathrm{g}^r)}(\mathrm{q}(\mathrm{p}^r)) \tag{9}$$

where ν^k_r is the canonical embedding of G^r into G^k. Let $\varepsilon^{k+1} : \mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^{k+1}(\mathcal{M})$ be a G^1-invariant section. It defines a G^1 reduction of the bundle $\mathrm{H}^{k+1}(\mathcal{M})$ given by the image $\varepsilon^{k+1}(\mathrm{H}^1(\mathcal{M}))$. We shall denote it by $\mathcal{M}_{\omega^k}$. The projection of $\mathcal{M}_{\omega^k}$ to the bundle $\mathrm{H}^k(\mathcal{M})$, that is $\mathcal{N}_{\omega^k} \equiv \pi^{k+1}_k(\varepsilon^{k+1}(\mathcal{M}_{\omega^k}))$, is also a G^1 reduction. This, in turn, induces the G^1-invariant partial section $q^k : \mathcal{N}_{\omega^k} \to \mathcal{M}_{\omega^k}$. The connection ω^k on $\mathrm{H}^k(\mathcal{M})$ is then defined by selecting as its horizontal space at $\mathrm{p}^k \in \mathrm{H}^k(\mathcal{M})$ $\quad \mathcal{SH}(q^k(\mathrm{p}^k))$ if $\mathrm{p}^k \in \mathcal{N}_{\omega^k}$ and $T\mathfrak{R}_{\mathrm{n}^k_1}\mathcal{SH}(q^k(\mathrm{p}^k))$ for any other k-frame, where n^k_1 denotes the appropriate element of the affine group N^k_1. The G^1-invariant submanifold $\mathcal{N}_{\omega^k}$ of $\mathrm{H}^k(\mathcal{M})$, fundamental for the construction of the connection ω^k, will be called its *characteristic manifold.*

We are now in the position to represent the k-connection ω^k through the fundamental form θ^{k+1}.

THEOREM 1. *Let* ω^k *be a k-connection on the bundle of semi-holonomic k-frames* $\mathrm{H}^k(\mathcal{M})$ *and let* ε^{k+1} *denotes its generating* $\mathcal{E}$*-connection with* $\mathcal{N}_{\omega^k}$ *as its characteristic manifold. Then, for any* $\mathrm{p}^k \in \mathcal{N}_{\omega^k}$ *and any* $\mathrm{g}^k \in \mathrm{G}^k$

$$\omega^k(\mathfrak{R}_{\mathrm{g}^k}(\mathrm{p}^k))(\mathfrak{R}_{\mathrm{g}^k *}\xi) = \tilde{q}^{k*}\theta_{k+1}(T\mathfrak{R}_{\mathrm{g}^k}\xi) - \lambda^k((\mathrm{g}^k)^{-1}, \tilde{q}^{k*}\theta_{-1}(\xi))$$

where $\xi \in T_{\mathrm{p}^k}\mathcal{N}_{\omega^k}$ *and* $\tilde{q}^k$ *denotes the* G^k*-equivariant extension of the* G^1*-invariant partial section* q^k *induced by the* $\mathcal{E}$*-connection* ε^{k+1}.

Proof. As implied by Eq. 8 the 1-form on the right hand side is equivariant. What remains to be shown is that both sides are identical on the characteristic manifold of the connection ω^k. Thus, let $\mathrm{p}^k \in \mathcal{N}_{\omega^k}$ then $\omega^k(\mathrm{p}^k)(\xi) = 0$ if, and only if, $\xi \in \mathcal{SH}(q^k(\mathrm{p}^k))$. On the other hand if $\mathrm{p}^k \in \mathcal{N}_{\omega^k}$ so does $\mathrm{p}\mathrm{g}^k$ for any $\mathrm{g}^k \in \nu_1^k(\mathrm{G}^1)$. However, $\lambda^k((\mathrm{g}^k)^{-1}, \cdot)$ is identically zero for any $\mathrm{g}^k \in \mathrm{GL}(\mathrm{n}, \mathbf{R}^\mathrm{n}) \oplus \{0\}$. Also, $q^{k*}\theta_{k+1}(T\mathfrak{R}_{\mathrm{g}^k}\xi) = 0$ if, and only if $\xi \in \mathcal{SH}(q^k(\mathrm{p}^k))$ as attested by the Proposition 1 ♣

To get even more detailed description of k-connections let us compare the standard horizontal spaces corresponding to two different $(k+1)$-frames over the same k-frame. Hence, let us take $\hat{\mathrm{p}}^{k+1}, \mathrm{p}^{k+1} \in \mathrm{H}^{k+1}(\mathcal{M})$ such that p^k is their projection onto $\mathrm{H}^k(\mathcal{M})$. This implies that there exists $\mathrm{n}_k^{k+1} \in \mathrm{N}_k^{k+1}$ such that $\hat{\mathrm{p}}^{k+1} = \mathrm{p}^{k+1}\mathrm{n}_k^{k+1}$. Moreover, there exists the admissible local isomorphism $\alpha^k : \mathrm{H}^k(I\!R^n) \to \mathrm{H}^k(I\!R^n)$ preserving the neutral element and such that $\mathrm{n}^{k+1} = j^1\alpha^k(e^k)$. Also, there is the admissible local isomorphism $h^k : \mathrm{H}^k(I\!R^n) \to \mathrm{H}^k(\mathcal{M})$ such that $j^1h^k(e^k) = \mathrm{p}^{k+1}$ (see Definition 1). The composition $h^k \circ \alpha^k$ is then an admissible local isomorphism the first jet of which at e^k gives the $(k+1)$-frame $\hat{\mathrm{p}}^{k+1}$. According to Definition 1 $(\widetilde{h^k \circ \alpha^k})(v, 0) \in \mathcal{SH}(\hat{\mathrm{p}}^{k+1})$ for any $(v, 0) \in I\!R^n \oplus \mathfrak{g}^k$. Recalling the definition of the fundamental form and that of the action ρ^{k+1} of the group G^{k+1} on the tangent space of $\mathrm{H}^k(\mathcal{M})$ we obtain $\widetilde{h^k \circ \alpha^k}(v, 0) = \tilde{h}^k \circ \rho^{k+1}((\mathrm{n}_k^{k+1})^{-1})(v, 0) = \tilde{h}^k(\tilde{\pi}_1^{k+1}(\mathrm{n}_k^{k+1})v, \lambda^k((\mathrm{n}_k^{k+1})^{-1}, v)) = \tilde{h}^k(v, 0) + \tilde{h}^k(0, \lambda^k((\mathrm{n}_k^{k+1})^{-1}, v)) = \tilde{h}^k(v, 0) + h_*^k(\lambda^k((\mathrm{n}_k^{k+1})^{-1}), v) = \tilde{h}^k(v, 0) + \widetilde{\lambda^k((\mathrm{n}_k^{k+1})^{-1}, v)}$ for every $(v, 0) \in I\!R^n \oplus \mathfrak{g}^k$ where, $\widetilde{\lambda^k(\cdot, \cdot)}$ denotes a vertical vector at p^k corresponding to the Lie algebra element $\lambda^k(\cdot, \cdot)$. All of this shows:

LEMMA 1. *Given two, in general different, $(k+1)$-frames $\hat{\mathrm{p}}^{k+1}, \mathrm{p}^{k+1}$ over the same k-frame p^k the standard horizontal space of $\hat{\mathrm{p}}^{k+1}$ is the $\mathfrak{g}^k$-translate, through λ^k, of the standard horizontal space of p^k.*

Consequently, the previous statement about the decomposition of the connection form (Theorem 1) can be made even more precise:

THEOREM 2. *Let ω^k be a k-connection with $\mathcal{N}_{\omega^k}$ as its characteristic manifold. Let $l_1^k : \mathrm{H}^k(\mathcal{M}) \to \mathrm{N}_1^k$ be an equivariant mapping, i.e. $l_1^k(\mathrm{p}^k\mathrm{n}_1^k) = l_1^k(\mathrm{p}^k)\mathrm{n}_1^k$ for any k-frame p^k and any $\mathrm{n}_1^k \in \mathrm{N}_1^k$ while $l_1^k(\mathrm{p}^k\mathrm{g}) = \mathrm{g}^{-1}l_1^k(\mathrm{p}^k)\mathrm{g}$ for any $\mathrm{g}^1 \in \mathrm{G}^1$. Assume that l_1^k is such that $\mathrm{p}^k l_1^k(\mathrm{p}^k)^{-1} \in \mathcal{N}_{\omega^k}$ for every $\mathrm{p}^k \in \mathrm{H}^k(B)$. Also, let $q : \mathcal{N}_{\omega^k} \to \mathrm{H}^{k+1}(\mathcal{M})$ be the G^1-equivariant section such that $\omega^k = q^*\theta_{k+1}$ when restricted to $\mathcal{N}_{\omega^k}$. Then*

$$\omega^k(\mathrm{p}^k)(\xi) = \tilde{q}^*\theta_{k+1}(\xi) - \lambda^k(l_1^k(\mathrm{p}^k)^{-1}, \theta_{-1}(\tilde{q}_*\xi)) \tag{10}$$

for any $\mathrm{p}^k \in \mathrm{H}^k(\mathcal{M})$ and $\xi \in T_{\mathrm{p}^k}\mathrm{H}^k(\mathcal{M})$. Moreover, there is a one-to-one correspondence between linear connections on $\mathrm{H}^k(\mathcal{M})$ and pairs of mappings $(\tilde{q}, l_1^k)$.

Proof. Given the pair $(\tilde{q}, l_1^k)$ where $\tilde{q} : \mathrm{H}^k(\mathcal{M}) \to \mathrm{H}^{k+1}(\mathcal{M})$ is an equivariant section and where $l_1^k : \mathrm{H}^k(\mathcal{M}) \to \mathrm{N}_1^k$ is an equivariant mapping the k-connection is uniquely

defined. On the other hand, given the connection ω^k the mapping l_1^k is uniquely defined, modulo the G^1 action, from the equation: $\pi_k^{k+1*}\omega^k - \theta_{k+1} = \pi_k^{k+1*}\lambda^k((l_1^k)^{-1}, \theta_{-1})$. Once l_1^k is available the equivariant section $\tilde{q}$ can be obtained from the condition that $\omega^k|_{(l_1^k)^{-1}(0)} = \tilde{q}|_{(l_1^k)^{-1}(0)}\theta_{k+1}$. We remark here that $\lambda^1 \equiv 0$ and that for $k = 2$ we get the known expression of Garcia for a 2-connection[5].♣

Any k-connection ω^k on $\mathrm{H}^k(\mathcal{M})$ induces, through a projection, a (k-1)-connection $proj_1\omega^k$ on $\mathrm{H}^{k-1}(\mathcal{M})$. Namely, for any $\xi \in T\mathrm{H}^k(\mathcal{M})$

$$\mu_{k-1*}^k\omega^k(\xi) = \pi_{k-1}^{k*}proj_1\omega^k(\xi). \tag{11}$$

If $\mathcal{N}_{\omega^k}$ is the characteristic manifold of ω^k then the characteristic manifold of $proj_1\omega^k$ is the projection of $\mathcal{N}_{\omega^k}$, i.e. $\mathcal{N}_{proj_1\omega^k} = \pi_{k-1}^k(\mathcal{N}_{\omega^k})$. Indeed, suppose that ε^{k+1} is the $\mathcal{E}$-connection generating ω^k. Then, $\mathcal{N}_{\omega^k} = \pi_k^{k+1}(\varepsilon^{k+1}(\mathrm{H}^1(\mathcal{M})))$ and there exists the partial section $q^k : \mathcal{N}_{\omega^k} \to \varepsilon^{k+1}(\mathrm{H}^1(\mathcal{M}))$ such that for any $\mathrm{p}^k \in \mathcal{N}_{\omega^k}$ the horizontal space of ω^k at p^k is $\mathcal{SH}(q^k(\mathrm{p}^k))$, that is the kernel of $q^{k*}\theta_{k+1}$. Let now q^{k-1} be a partial section on $\pi_{k-1}^k(\mathcal{N}_{\omega^k})$ with the property that $q^{k-1} \circ \pi_{k-1}^k = \pi_k^{k+1} \circ q^k$. Recalling that the projections π_k^{k+1} and π_{k-1}^k, when restricted to the characteristic manifolds, are one-to-one and invoking the definition of the standard horizontal space, as well as Proposition 1, we get:

LEMMA 2. *The standard horizontal space of a projection of a frame is a projection of the standard horizontal space, i.e. if* $\mathrm{p}^{k+1} \in \mathrm{H}^{k+1}(\mathcal{M})$ *then* $\pi_{k-1*}^k\mathcal{SH}(\mathrm{p}^{k+1}) = \mathcal{SH}(\pi_k^{k+1}(\mathrm{p}^{k+1}))$. *Thus, the characteristic manifold of the projected connection* $proj_1\omega^k$ *is the projection of the characteristic manifold* $\mathcal{N}_{\omega^k}$.

This is obviously also true for a projection of a k- connection to any r-order frame bundle, as long as $0 < r < k$.

If the k-connection ω^k is such that its horizontal distribution is locally integrable then in addition to its projection it induces locally yet another, and in general different, $(k-1)$-connection. Indeed, let $\mathfrak{l}^k : U \subset \mathcal{M} \to \mathrm{H}^k(\mathcal{M})$ be such a section. Thus, there exists the local section $\mathfrak{p}^1 : U \to \mathrm{H}^k(\mathcal{M})$ and the map $\varepsilon_{\mathfrak{l}^k}^k : \mathfrak{p}^1(U) \to \mathrm{H}^k(\mathcal{M})$ such that for any $\mathrm{y} \in U$ $\mathfrak{l}^k(\mathrm{y}) = \varepsilon_{\mathfrak{l}^k}^k(\mathfrak{p}^1(\mathrm{y}))$. We extend the mapping $\varepsilon_{\mathfrak{l}^k}^k$ to the G^1-equivariant section $\tilde{\varepsilon}_{\mathfrak{l}^k}^k$. As Theorem 2 asserts such an equivariant section, together with the characteristic manifold $\mathcal{N}_{i_1\omega^k} \equiv \tilde{\varepsilon}_{\mathfrak{l}^k}^k(\mathfrak{p}^1(U)\mathrm{G}^1) = \pi_{k-1}^k[\mathfrak{l}^k(\mathrm{U})\mathrm{G}^1]$ defines a $(k-1)$-connection which we denote by $i_1\omega^k$.

DEFINITION 2. *The* **induced connection** *of the locally integrable k-connection* ω^k, *the horizontal distribution of which is locally induced by the section* $\mathfrak{l}^k : \mathrm{U} \to \mathrm{H}^k(\mathcal{M})$, *is the* $(k-1)$*-connection* $i_1\omega^k$ *such that* $\pi_{k-1}^k[\mathfrak{l}^k(\mathrm{U})\mathrm{G}^1]$ *is its characteristic manifold and* $q^{k-1} : \pi_{k-1}^k[\mathfrak{l}^k(\mathrm{U})\mathrm{G}^1] \to \mathfrak{l}^k(\mathrm{U})\mathrm{G}^1$ *as its generating section.*

In general $\mathrm{N}_{i_1\omega^k} \neq \mathrm{N}_{proj_1\omega^k}$. However, as the section $\mathfrak{l}^k : \mathrm{U} \subset \mathcal{M} \to \mathrm{H}^k(\mathcal{M})$ defines locally the k-connection ω^k the section $\pi_{k-1}^k \circ \mathfrak{l}^k$ defines its projection $proj_1\omega^k$. This, in turn,

enables one to define the G^1-invariant section $\tilde{\varepsilon}^{k-1}_{\pi^k_{k-1}} \circ \mathfrak{l}^k$ inducing the $(k-2)$-connection $i_1 proj_1 \omega^k$ with $\pi^k_{k-2}[\mathfrak{l}^k(\mathrm{U})\mathcal{G}^1]$ as its characteristic manifold. The space $\pi^{k-1}_{k-2}\mathcal{N}_{i_1\omega^k}$ is the characteristic manifold of the projection of $i_1\omega^k$ proving:

PROPOSITION 2. *Let ω^k the locally integrable k- connection. Then, locally*

$$i_1 proj_j \omega^k = proj_j i_1 \omega^k$$

for any $j < k$.

4.Prolongations of Connections

DEFINITION 3. *Given the k-connection ω^k let ε^{k+1} be its generating $\mathcal{E}$-connection and $\tilde{q}^k : \mathrm{H}^k(\mathcal{M}) \to \mathrm{H}^{k+1}(\mathcal{M})$ the corresponding equivariant section. The* **prolongation** *of ω^k is the $(k+1)$-connection $\mathcal{P}(\omega^k)$ such that its horizontal space at any $\mathrm{p}^{k+1} \in q^k(\mathcal{N}_{\omega^k})$ is the q^k-lift of the horizontal space of ω^k, i.e. for any $\mathrm{p}^k \in \mathcal{N}_{\omega^k}$*

$$hor_{q^k(\mathrm{p}^k)}\mathcal{P}(\omega^k) = q^k_*(hor_{\mathrm{p}^k}\omega^k).$$

The following facts are easy consequences of the definition of prolongation.

PROPOSITION 3.

a. *Given the k-connection ω^k there is only one prolongation $\mathcal{P}(\omega^k)$.*
b. $proj_1\mathcal{P}(\omega^k) = \omega^k$.
c. *The connection $\omega^{k+1} = \mathcal{P}(proj_1\omega^{k+1})$ if, and only if, $\mathcal{N}_{\omega^{k+1}} = q^k(\mathcal{N}_{proj_1\omega^{k+1}})$.*

DEFINITION 4. (Yuen[8]) *The k-connection ω^k is called* **simple** *if it is the $(k-1)$-prolongation of some linear connection ω^1; $\omega^k = \mathcal{P}^{k-1}(\omega^1)$.*

It appears that any simple k-connection can be characterized by the "position" of its horizontal distribution relative to its characteristic manifold. Indeed, we have:

PROPOSITION 4. *If ω^k is a simple connection then its horizontal distribution is tangent to its characteristic manifold at all points.*

Proof. It is enough to point out that if the 2-connection ω^2 is the prolongation (simple) of some linear connection ω^1 then, by the definition of a simple connection, $hor_{q^1(\mathrm{p}^1)}\omega^2 = q^1_*(hor_{\mathrm{p}^1}\omega^1)$ for any $\mathrm{p}^1 \in \mathcal{N}_{\omega^1}$. However, according to Proposition 3(c) $q^1_*(\mathrm{H}^1(\mathcal{M})) = q^1_*(\mathcal{N}_{\omega^1}) = \mathcal{M}_{\omega^1} = \mathcal{N}_{\omega^2}$. Therefore, the definition of the prolongation implies immediately that $hor\mathcal{P}(\omega^1)|_{\mathcal{N}_{\omega^2}} \subset T\mathcal{N}_{\omega^2}$. Applying this argument recursively proves the original claim. ♠

In fact, somewhat more general statement can be made.

THEOREM 3. *The connection ω^k on the bundle of k- frames (holonomic or non-holonomic) $\mathrm{H}^k(\mathcal{M})$ is the $(k-s)$-prolongation of its $proj_{k-s}\omega^k$ if, and only if, its horizontal distribution is tangent to the G^s-reduction of the bundle of frames $\mathrm{H}^k(\mathcal{M})$ induced by the characteristic manifold $\mathcal{N}_{\omega^k}$, i.e. it is tangent to $\mathcal{N}_{\omega^k}\mathrm{N}^s_{s-1}$. In particular, ω^k is simple if, and only if, its horizontal distribution is tangent to its characteristic manifold.*

Proof. The condition is obviously necessary as easily attested by the definition of the prolongation of connection and Proposition 4. Also, as the projection of the characteristic manifold of a connection is the characteristic manifold of the projected connection $\mathcal{N}_{proj_{k-s}\omega^k} = \pi^k_s(\mathcal{N}_{\omega^k})$. Therefore, the horizontal distribution of $proj_{k-s}\omega^k$ is tangent to $\mathcal{N}_{proj_{k-s}\omega^k}\mathrm{N}^s_{s-1} = \mathrm{H}^s(\mathcal{M})$. consequently, the sequence of invariant sections $\{\tilde{q}^l\}_{l=s,\ldots,k-1}$, corresponding to the sequence of prolongations of $proj_{k-s}\omega^k$ to $\mathrm{H}^k(\mathcal{M})$, maps the horizontal distribution of the $(k-s)$-projection of ω^k onto the horizontal distribution of ω^k, satisfying conditions of Definition 3.♠

If the horizontal distribution of ω^k is locally integrable Theorem 3 has particularly far reaching consequences.

COLLORARY 1. *Suppose that the locally integrable 2- connection ω^2 is the prolongation of its projection ω^1. Let $i_1\omega^2$ be the corresponding induced (locally) linear connection. Then $i_1\omega^2 = \omega^1$. In fact, for any integrable connection $\omega^{k+1} = \mathcal{P}^{k-1}(\omega^1)$ if, and only if, $i_1\omega^{k+1} = proj_1\omega^{k+1}$.*

Proof. If the connection $\omega^2 = \mathcal{P}(\omega^1)$ then by Theorem 3 $hor\omega^2 \subset T\mathcal{N}_{\omega^2}$. On the other hand, as ω^2 is locally integrable, i.e. locally generated by the section $\mathfrak{l}^2 : U \subset \mathcal{M} \to \mathrm{H}^2(\mathcal{M})$, $hor\omega^2|_{\mathfrak{l}^2(U)} = T\mathfrak{l}^2(U)$. Therefore, $\mathfrak{l}^2(U) \subset \mathcal{N}_{\omega^2}$ and $\mathfrak{l}^2(U)\ \mathrm{G}^1 = \mathcal{N}_{\omega^2}$ as any characteristic manifold is a G^1-reduction. This, in fact, concludes the proof as the characteristic manifold of $i_1\omega^2$ is, by the definition of the induced connection, $\pi^2_1(T(\mathfrak{l}^2(U)\ \mathrm{G}^1)) = \pi^2_1(\mathcal{N}_{\omega^2})$, the characteristic manifold of the projection ω^1. It is easy to see that the same argument applies for any k.♣

Applying the above argument recursively one can easily conclude the following:

COLLORARY 2. *Let the k-connection ω^k be the simple connection, i.e. $\omega^k = \mathcal{P}^{k-1}(proj_{k-1}\omega^k)$. Then the horizontal distribution of ω^k is locally integrable if, and only if, the horizontal distribution of $proj_{k-1}\omega^k$ is locally integrable.*

Finally, we are ready to try to determine under what conditions a k-connection is locally equivalent to the standard connection on $\mathbb{R}^n \times \mathrm{G}^k$, i.e. is locally flat. To this end, let us recall that it had been shown by Yuen[8] that:

THEOREM 4. *The k-connection ω^k is locally flat if, and only if, it is simple, is curvature free and has a vanishing torsion, i.e. $\omega^k = \mathcal{P}^{k-1}(proj_{k-1}\omega^k)$, $\Omega_{\omega^k} = 0$ and $\Theta_{\omega^k} = 0$ where the* **curvature** *Ω_{ω^k} is the $\mathfrak{g}^k$-valued 2-form $\mathrm{d}\omega^k|_{hor\omega^k}$ while the* **torsion** *Θ_{ω^k} is the $\mathbb{R}^n \oplus \mathfrak{g}^{k-1}$-valued 2-form $\mathrm{d}\theta^k|_{hor\omega^k}$.*

Note that the curvature and torsion of the jth-projection of ω^k are defined respectively by the following identities[1]:

$$\pi_j^{k*}\Theta_{proj_j\omega^k} = \mu_{j-1*}^{k-1}\Theta_{\omega^k}, \tag{12}$$

$$\pi_j^{k*}\Omega_{proj_j\omega^k} = \mu_{j-1*}^{k-1}\Omega_{\omega^k}. \tag{13}$$

Therefore, if the connection ω^k has a vanishing torsion and/or curvature then $proj_j\omega^k$ has the same characteristics.

Although Theorem 4 sets explicit sufficient and necessary conditions for the local flatness of connections we shall look at some special classes of connections, holonomic, locally integrable etc., to determine if these conditions could not be weaken.

DEFINITION 5. *The k-connection $\tilde{\omega}^k$ is called* **holonomic** *if it is a reduction of some k-connection to the holonomic frame bundle $\tilde{\mathrm{H}}^k(\mathcal{M})$.*

It was shown by Garcia[5]andYuen[8] that:

PROPOSITION 5. *Let ω^k be induced by the $\mathcal{E}$- connection $\varepsilon^k : \mathrm{H}^1(\mathcal{M}) \to \tilde{\mathrm{H}}^{k+1}(\mathcal{M})$ into the holonomic frame bundle. Then, ω^k has a vanishing torsion.*

This simple fact enables us to prove:

COLLORARY 3. *If the k-connection $\tilde{\omega}^k$ is holonomic and has the curvature zero then the induced connection $i_1\omega^k$ has vanishing torsion.*

Proof. Let $\mathfrak{l}^k : U \subset \mathcal{M} \to \tilde{\mathrm{H}}^k(\mathcal{M})$ define locally the horizontal distribution of ω^k. The corresponding $\mathcal{E}$-connection of $i_1\omega^k$ is a section into the holonomic k-frame bundle (see Definition 2). This, according to Proposition 5, guarantees the vanishing torsion of $i_1\omega^k$ ♣

Moreover,

PROPOSITION 6. *A curvature free k-connection cannot be prolonged (see Definition 3) into the holonomic $(k+1)$-frame bundle unless it is torsion-free.*

Proof. Let ω^k be curvature free and suppose that $\mathcal{P}(\omega^k)$ is its prolongation. Assume that it is holonomic, i.e. $\mathcal{P}(\omega^k) = \tilde{\mathcal{P}}(\omega^k)$. If $hor\omega^k$ is locally integrable so is $hor\tilde{\mathcal{P}}(\omega^k)$ (Collorary 2). Consequently, according to Collorary 3, $i_1\mathcal{P}(\omega^k)$ has vanishing torsion. However, $i_1\tilde{\mathcal{P}}(\omega^k) = proj_1\tilde{\mathcal{P}}(\omega^k) = \omega^k$ ♠

In fact, the same is true for any k-connection, integrable or not.

PROPOSITION 7. *A k-connection cannot be prolonged into the holonomic frame bundle* $\tilde{\mathrm{H}}^{k+1}(\mathcal{M})$ *unless has vanishing torsion.*

Proof. Suppose that ω^k has non-vanishing torsion and let $\tilde{\mathcal{P}}(\omega^k)$ be its prolongation into the holonomic frame bundle $\tilde{\mathrm{H}}^{k+1}(\mathcal{M})$. Also, $\mathcal{M}_{\omega^k} = \mathcal{N}_{\mathcal{P}(\omega^k)} \subset \tilde{\mathrm{H}}^{k+1}(\mathcal{M})$ as the prolongation is holonomic. This however means that the $\mathcal{E}$-connection inducing ω^k is a section into the holonomic frame bundle which implies (Proposition 5) that ω^k has vanishing torsion ♠

Finally, we are able to conclude by proving two theorems about locally flat connections. Some other interesting intermediate cases will be presented elsewhere. These require, however, somewhat deeper look at the structure of k-connections (Theorems 1 & 2) and the properties of their curvature and torsion forms.

THEOREM 5. *A simple k-connection* $\mathcal{P}^{k-1}(\omega^1)$ *is locally flat if, and only if,* ω^1 *is locally flat.*

Proof. If the prolongation $\mathcal{P}^{k-1}(\omega^1)$ is locally flat then obviously ω^1 is locally flat as $\omega^1 = proj_{k-1}\mathcal{P}^{k-1}(\omega^1)$. We also know, from Collorary 2, that ω^1 is curvature free if, and only if, its prolongations are curvature free. What remains to be shown is that if the torsion of ω^1 vanishes then any of its prolongations has vanishing torsion. This is, however, immediate by Collorary 1, Proposition 7 and the uniqueness of prolongations ♠

THEOREM 6. *Let the holonomic k-connection* $\tilde{\omega}^k$ *be simple and curvature free. Then, it is locally flat.*

Proof. If the connection $\tilde{\omega}^k = \mathcal{P}^{k-1}(proj_{k-1}\omega^k)$ is curvature free so is the linear connection $proj_{k-1}\omega^k$. It also has vanishing torsion as otherwise, according to Proposition 6, could not be prolonged into the holonomic frame bundle. Finally, as $proj_{k-1}\omega^k$ is locally flat so is its prolongation (Theorem 5)♣

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