Material uniformity and the concept of the stress space

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Abstract The notion of the stress space, introduced by Schaefer [14], and further developed by Kröner [7] in the context of materials free of defects, is revisited. The comparison between the Geometric Theory of Material Inhomogeneities and the Stress Space approach is discussed. It is shown how to extend Kröner's approach to the case of the material body with inhomogeneities (defects).

1 Introduction

The work presented in this note is a continuation of the earlier work by Ciancio *et al* [3]. Its main objective is to investigate the relation between the Geometric Theory of Material Inhomogeneities (Epstein and Elżanowski [4], Wang and Truesdell [16]) and the description of the continuous distribution of defects based on the concepts of the intermediate configuration and the stress space (Bilby [2], Kröner [6], [7], Lee [9], Stojanovic [15]).

We are particularly interested in describing effectively the residual stresses associated with the presence of material inhomogeneities (defects). To this end, we employ the Bilby-Kröner-Lee multiplicative decomposition of the deformation gradient (and the concept of the intermediate configuration) as well as the notions of the stress and strain spaces of Schaefer [14] and Kröner [7]. Using the language of modern Differential Geometry we show that the multiplicative decomposition of the deformation gradient, exemplifying the elasto-plastic material behavior, leads

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The authors would like to dedicated this paper to Krzysztof Wilmański on the occasion of his 70th birthday.

to the introduction of the uniformity tensor which plays the role similar to that of the material isomorphism of the Geometric Theory of Material Inhomogeneities. When discussing the role of the uniformity tensor, and its uniqueness, we show the importance of the concept of the intermediate configuration.

In employing the notion of the stress space we follow Kröner's idea (see Kröner [7]) of the non-holonomic transformation between the spaces of strain and stress. This allows us to introduce the residual stress metric, the Ricci tensor of which is interpreted as the residual stress tensor. The said non-holonomic transformation, known as the residual stress function, represents a constitutive law relating the residual stress to the material strain of the intermediate configuration of the inhomogeneous material body. We show how to reconcile the introduction of the residual stress function with the existence of the uniformity tensor.

The paper has the following layout. In the next section we introduce the basic notions of Continuum Mechanics and the Geometric Theory of Material Inhomogeneities. In Section 3 we discuss the Bilby-Kröner-Lee multiplicative decomposition of the deformation gradient introducing the concept of the uniformity tensor and the notion of the material strain. In Section 4 the construction of the stress space is presented. The paper is concluded by a couple of examples in Section 5.

2 Hyperelastic unifomity

We start by reviewing some basic concepts of Continuum Mechanics and the Geometric Theory of Material Inhomogeneities restricting our presentation to hyperelastic materials only.

2.1 Configurations and the Cauchy metric

In Continuum Mechanics the **material body** is usually represented by a connected 3-dimensional smooth oriented manifold M with a piece-wise smooth boundary ∂M. However, as the issues discussed in this paper are of the local nature only, it is sufficient to consider M as a connected, open domain in $\mathbb{R}^3$ with coordinates $\{X^I\}$, $I = 1,2,3$. We assume that the **physical space** our body is placed in is the 3-dimensional Euclidean vector space E^3 equipped with the (flat) Euclidean metric $\mathbf{h}$. Given a global Cartesian coordinate system $\{x^i\}$, $i = 1,2,3$, in E^3, de facto allowing us to identify E^3 with $\mathbb{R}^3$, the metric $\mathbf{h}$ takes the form $h_{ij}dx^i dx^j$ where the standard summation convention is enforced. A **configuration** of the body M, often called its **placement**, is an (differentiable) embedding $\phi : M \to \mathbb{R}^3$. Its **deformation gradient** at a point $X \in M$ is a linear isomorphism from the tangent space $T_X M$ into the tangent space $T_{\phi(X)}\mathbb{R}^3$. Namely,

$$\mathbf{F}(X) \equiv \phi_*(X) : T_X M \to T_{\phi(X)}\mathbb{R}^3, \tag{1}$$

where ϕ_* denotes the tangent map of ϕ. The deformation gradient at a material point, say $Y \in M$, is represented (in the given coordinates systems on M and E^3) by the non-singular matrix of partial derivatives of ϕ, that is,

$$F^i_I(Y) = \frac{\partial \phi^i}{\partial X^I}(Y) \equiv \phi^i_{,I}(Y), \tag{2}$$

where $\phi^i(X^1, X^2, X^3) = x^i$, $i = 1, 2, 3$. The material equivalence of the special metric **h**, relative to the placement ϕ the body is at, is the right **Cauchy-Green deformation tensor C** obtained by the pull-back of the Euclidean metric **h** to the body manifold M. That is,

$$\mathbf{C} \equiv \phi^* \mathbf{h}, \tag{3}$$

where ϕ^* denotes the pull-back map. The matrix $C_{IJ} = h_{ij}\phi^i_{,I}\phi^j_{,J}$ evaluated at the point X is the coordinate representation of the tensor **C**.

2.2 Material uniformity

Recall, that the material is called **hyperelastic** if its constitutive response is completely determined by a single scalar-valued function, say W, called the **elastic energy density** (per unit reference volume υ_0[1]). We assume that W is a function of a material point and the deformation gradient at this point, that is, $W = W(X, \mathbf{F}(X))$.

The material body M is considered **uniform** if it is made of the same material at all points. In mathematical terms, this means that for any pair of material points, say X and Y, there exists a linear isomorphism, referred to as a **material isomorphism**

$$K^Y_X : T_X M \to T_Y M, \tag{4}$$

between the corresponding tangent spaces such that

$$W(Y, \mathbf{F}(Y)) K^{Y*}_X d\upsilon_0(Y) = W(X, \mathbf{F}(Y) K^Y_X) d\upsilon_0(X) \tag{5}$$

holds for all possible deformation gradients **F**, where K^{Y*}_X denotes the pullback of a 3-form by the mapping K^Y_X. Equivalently, the material body is considered uniform, if there exist material isomorphisms

$$P_X : T_{X_0} M \to T_X M, \tag{6}$$

called the **implants**, from a fixed point $X_0 \in M$ to every point $X \in M$ such that $K^Y_X = P_Y \circ P^{-1}_X$ and the equation (5) holds. The said fixed point X_0 can be arbitrarily chosen. In fact, it is often convenient to think about its tangent space $T_{X_0} M$ as a

[1] Although we do not explicitly utilize here the concept of the reference configuration, we assume that assigning coordinates to the body manifold $M \subset \mathbb{R}^3$ is equivalent to selecting its reference configuration; see Epstein and Elżanowski [4].

standing alone vector space V with the orthogonal frame $\{\mathbf{e}_i\}$, $i = 1,2,3$, and its own metric being the the Euclidean (flat) metric $\mathbf{h}$ at the origin.

Having the implants P_X available we can "pull-back" the Euclidean volume element of the **archetype** V to the material manifold M. Indeed, let

$$\upsilon_P(X) \equiv P_X^{-1}(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) \tag{7}$$

and let $J_P : M \longrightarrow \mathbb{R}$ be a real-valued function such that

$$\upsilon_P(X) = J_P(X)\upsilon_0(X) \tag{8}$$

at every $X \in M$. Let $GL(V)$ be the group of all linear automorphism of the archetype V and define a real-valued function $\widehat{W} : M \times GL(V) \longrightarrow \mathbb{R}$ by

$$\widehat{W}(X,\mathbf{A}) \equiv J_P^{-1}(X)W(X,\mathbf{A}P_X^{-1}), \tag{9}$$

for all $X \in M$ and any $\mathbf{A} \in GL(V)$. It should now be easy to see that the uniformity condition (5) is equivalent to

$$\widehat{W}(X,\mathbf{A}) = \widehat{W}(Y,\mathbf{A}) \tag{10}$$

for all $\mathbf{A} \in GL(V)$ and any pair of material points X and Y. In other words, the material body M is uniform if its **archetypical energy density** function $\widehat{W}$ is material point independent. Consequently, the strain energy density function of the uniform material body M is such that

$$W(X,\mathbf{F}(X)) = J_P(X)\widehat{W}(\mathbf{F}(X)P_X) \tag{11}$$

for any (non-singular) deformation gradient $\mathbf{F}$ and some archetypical energy function $\widehat{W}$ obeying the relation (10). For the clarity and the simplicity of our presentation we assume here that that archetypical energy $\widehat{W}$ has the trivial isotropy group[2].

2.3 Material connections

It is normally assumed that the material isomorphisms, consequently the implants, are locally smoothly distributed on M. This implies that the materially uniform body M can be equipped with the (locally smooth) global material **uniform frame field**

$$\mathbf{p}_J(X) \equiv P_X(\mathbf{e}_J) = P_J^I \frac{\partial}{\partial X^I}, \quad I = 1,2,3. \tag{12}$$

[2] Note, that if the isotropy group of $\widehat{W}$ is nontrivial the material implants P_X are not necessarily uniquely defined. Indeed, suppose that the archetypical energy function $\widehat{W}$ has a continuous isotropy group, say $G \subset GL(V)$. Then, given an implant P_X, $P_X\mathbf{G}$ is also an implant as long as $\mathbf{G} \in G$, Epstein and Elżanowski [4].

The material isomorphisms K_X^Y, or equivalently the material implants P_X, establish a long distance parallelism on M as $K_X^Y(\mathbf{p}_J(X)) = \mathbf{p}_J(Y)$, $j = 1,2,3$. Such a parallelism defines a **material connection**, say ω, the curvature of which vanishes identically. Indeed, as evident from the definition of the global uniform frame field (12), the corresponding parallel transport is curve independent. The torsion of the connection ω provides the measure of the non-integrability of the material frame field $\mathbf{p}_J$, $J = 1,2,3$. This, in turn, is accepted as a “measure” of the local **non-homogeneity** of the given material body. More precisely, the hyperelastic body M, as defined by the strain energy density function W, is considered locally **homogeneous** provided there exists[3] a material connection ω such that its torsion vanishes identically.

As we have mentioned earlier, the implant maps induce the uniform material frame field (12). They also induce the corresponding **uniform material metric g** defined by the pull-back of the Euclidean metric of the archetype. That is,

$$\mathbf{g} \equiv P_X^{-1^*}\mathbf{h} \tag{13}$$

or

$$g_{IJ} = (P_X^{-1})_I^K (P_X^{-1})_J^L h_{KL} \tag{14}$$

in the corresponding local coordinate systems. The availability of the metric **g** allows one to consider the corresponding Levi-Civita connection ω_g, that is the connection in which the material frame field $\mathbf{p}_I$, $I = 1,2,3$, is parallel and **g**-orthonormal. It can be shown that the curvature of the connection ω_g is defined by the torsion of the material connection ω, Wang and Truesdell [16].

3 The multiplicative decomposition of the deformation gradient

In this section we will look at the uniformity and the homogeneity of a material body from a somewhat different perspective, that is, using the concept of an intermediate configuration. To this end, suppose that we are given the material body M in a configuration $\phi : M \to \mathbb{R}^3$. Its deformation gradient $\mathbf{F}$ can be viewed as a two-point tensor field on M, i.e., the tangent bundle mapping $\phi_* : TM \to T\mathbb{R}^3$ over the base mapping (configuration) $\phi : M \to \mathbb{R}^3$. Let

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p \tag{15}$$

be the Bilby-Kröner-Lee multiplicative decomposition (**BKL-decomposition**, in short) of the deformation gradient where $\mathbf{F}^e$ is understood as the elastic part of

[3] If the isotropy group G of the archetypical energy function $\widehat{W}$ is nontrivial and continuous, different material parallelisms, and different material connections are possible, all gauged by the isotropy group G. However, if the isotropy group G is discrete the corresponding material connection is unique, as implied by the local smoothness of the distribution of implants. In this instance and, in particular, when the group G is trivial, the torsion of ω may be considered the true measure of the local non-homogeneity, Epstein and Elżanowski [4].

the deformation gradient while $\mathbf{F}^p$ is its inelastic (plastic) component (see for example Bilby [2], Kröner [6] and Lee [9]). Assume, that every time the deformation gradient of a material configuration is available[4] we have means of identifying its BKL-decomposition. Interpreting the relation (15) as the composition of (tangent) bundle maps it is only natural (at least from the mathematical stand point) to consider the **intermediate configuration** $\widehat{\phi} : M \to \mathbb{R}^3$ as the base map for the bundle map $\mathbf{F}^p$. Indeed, if $\mathbf{F}^e$ and $\mathbf{F}^p$ are to represent bundle maps one is required to introduce a configuration $\widehat{\phi}$, which we assume to be a (differential) embedding, such that the composition

$$\widehat{\psi} \equiv \phi \circ \widehat{\phi}^{-1} \tag{16}$$

is well defined and the bundle maps $\mathbf{F}^e$ and $\mathbf{F}^p$ are based over $\widehat{\psi}$ and $\widehat{\phi}$, as illustrated by the following diagram

$$\begin{array}{ccccc} TM & \xrightarrow{\mathbf{F}^p} & T\widehat{\phi}(M) & \xrightarrow{\mathbf{F}^e} & T\phi(M) \\ \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow{\widehat{\phi}} & \widehat{\phi}(M) & \xrightarrow{\widehat{\psi}} & \phi(M) \end{array}$$

Thus, given the material point $X \in M$, $\mathbf{F}^p(X) \in T_{\widehat{\phi}(X)}\mathbb{R}^3$ while $\mathbf{F}^e(\widehat{\phi}(X)) \in T_{\phi(X)}\mathbb{R}^3$. Realize that despite the fact that the deformation gradient $\mathbf{F}$ is the tangent map of its base mapping (configuration) ϕ, the elements of the BKL-decomposition (15) are, in general, nonintegrable. That is, the inelastic part $\mathbf{F}^p$ is not necessarily the tangent map of the intermediate configuration[5] $\widehat{\phi}$ and the elastic part $\mathbf{F}^e$ is not the gradient of its base map $\widehat{\psi}$. Still, both elements of the BKL-decomposition, viewed as the bundle maps, are based over the corresponding base maps $\widehat{\phi}$ and $\widehat{\psi}$. All that implies that there exists a tangent bundle map $\mathbf{D} : TM \to TM$, over the identity map of M, such that

$$\mathbf{F}^p = \widehat{\phi}_* \circ \mathbf{D} \tag{17}$$

and

$$\mathbf{F}^e = \phi_* \circ (\widehat{\phi}_* \circ \mathbf{D})^{-1}. \tag{18}$$

We shall call the tensor $\mathbf{D}$ the **uniformity tensor**. Note that $\mathbf{D}$, when evaluated at a material point, say X, is effectively the implant map (6). Indeed, given the material point $X \in M$ the BKL-decomposition can be presented, using the language of the Geometric Theory of Material Inhomogeneities, as $\mathbf{F}^e = \mathbf{F}P_X$ and $\mathbf{F}^p = {P_X}^{-1}$, where P_X denotes the corresponding material isomorphism from the fixed material point X_0 (or the archetype V) to X; see for example Maugin and Epstein [12]. Invoking

[4] The strain tensor may be a better measure of the deformation, see Remark 1

[5] It is often argued that the (inelastic) intermediate configuration $\widehat{\phi}$ is defined uniquely by the material and the history of the deformation leading to the current configuration ϕ. See Lee and Agah-Tehrani [10] where the relaxation (unloading) of the material to the intermediate configuration is discussed.

the Euclidean parallelism in $\mathbb{R}^3$, which allows us to view P_X as a map from $T_X M$ to itself with $P_{X_0} = \mathbf{I}$, we can equate the uniformity tensor $\mathbf{D}(X)$ with P_X^{-1}.

It seems that the BKL-decomposition of the deformation gradient $\mathbf{F}$ and the intermediate configuration $\widehat{\phi}$ define completely[6] the uniformity structure of the material. Note however that given the deformation gradient $\mathbf{F}$ its BKL-decomposition is not necessarily uniquely defined. Indeed, replace the configuration $\widehat{\phi}$ with $\widetilde{\phi} = \beta \circ \widehat{\phi}$, where β is a diffeomorphism of the physical space E^3 (or simply a change of its coordinate system). Then, the plastic part of the deformation gradient corresponding to the new intermediate configuration $\widetilde{\phi}$ is given by

$$\mathbf{F}^p_\beta = \beta_* \circ \mathbf{F}^p \tag{19}$$

subsequently changing the form of the elastic part. The tensor $\mathbf{D}$ is not affected, however, by such a change of the intermediate configuration. On the other hand, if the intermediate configuration $\widehat{\phi}$ gets replaced by $\widehat{\phi} \circ \alpha$, where α can be interpreted, using the language of the Geometric Theory of Material Inhomogeneities, as the change of the archetype V[7], then all the elements of the BKL-decomposition do change. Indeed, the new uniformity tensor

$$\mathbf{D}_\alpha = \alpha_*^{-1} \circ \mathbf{D} \circ \alpha_* \tag{20}$$

and the new inelastic part of the deformation gradient is

$$\mathbf{F}^p_\alpha = \mathbf{F}^p \circ \alpha_*. \tag{21}$$

Once the uniformity tensor $\mathbf{D}$ is available the uniform material metric $\mathbf{g}$, (13), can be represented as

$$g_{KM} = D^I_K D^J_M h_{IJ}. \tag{22}$$

The metric $\mathbf{g}$ is, in general, not flat. Hence, the corresponding material Levi-Civita connection $\omega_{\mathbf{g}}$ has non-vanishing curvature. On the other hand, the vanishing of the curvature tensor, say $\mathbf{R}_{\mathbf{g}}$, of the connection $\omega_{\mathbf{g}}$, or equivalently its Ricci tensor $\mathbf{Rc}_{\mathbf{g}}$, implies the flatness of the metric $\mathbf{g}$, Kobayashi and Nomizu [5]. The flatness of the material metric should be viewed as the indication of the local material homogeneity. In fact, when the metric $\mathbf{g}$ is flat, one is allowed to select the uniformity tensor $\mathbf{D}$ as the identity $\mathbf{I}$ rendering the choice of the intermediate configuration $\widehat{\phi}$ arbitrary and the BKL-decomposition integrable.

Remark 1. The commonly used measure of the deformation of a material body, particularly well suited for the theory of the small deformations, is the **strain tensor**

$$\mathbf{E} = \frac{1}{2} \ln(\mathbf{h}^{-1}\mathbf{C}) \cong \frac{1}{2}\mathbf{h}^{-1}(\mathbf{C} - \mathbf{h}) \tag{23}$$

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[6] See also Ciancio *et al.* [3].

[7] See Epstein and Elżanowski [4] for the discussion of this point.

comparing the metric of the deformed state ϕ and the metric of the undeformed (reference) state, Marsden and Hughes [11]. Utilizing this measure of the deformation the **inelastic strain** of the BKL-decomposition takes the form

$$\mathbf{E}^p = \frac{1}{2}\ln(\mathbf{h}^{-1}\mathbf{D}\widehat{\mathbf{C}}\mathbf{D}^T) \cong \frac{1}{2}\mathbf{h}^{-1}(\mathbf{D}\widehat{\mathbf{C}}\mathbf{D}^T - \mathbf{h}) \tag{24}$$

where

$$\mathbf{E}^p_{in} = \frac{1}{2}\ln(\mathbf{h}^{-1}\widehat{\mathbf{C}}) \cong \frac{1}{2}\mathbf{h}^{-1}(\widehat{\mathbf{C}} - \mathbf{h}) \tag{25}$$

is its integrable part, $\mathbf{D}^T$ denotes the transpose, and the coordinate representation of the Cauchy-Green tensor of the intermediate configuration $\widehat{\mathbf{C}} \equiv \widehat{\phi}^*\mathbf{h}$ is given by

$$\widehat{C}_{IJ} = h_{ij}\widehat{\phi}^i_{,I}\widehat{\phi}^j_{,J}. \tag{26}$$

In this framework the **material strain**

$$\mathbf{E}^m = \frac{1}{2}\ln(\mathbf{h}^{-1}\mathbf{D}\mathbf{h}\mathbf{D}^T) \cong \frac{1}{2}\mathbf{h}^{-1}(\mathbf{D}\mathbf{h}\mathbf{D}^T - \mathbf{h}) \tag{27}$$

may be viewed as a measure of inhomogeneity of a material.

4 The stress space

We are now ready to present the construction of the stress space of Stojanovic [15] and Kröner [7] modified to encompass inhomogeneous materials. First, let us assume that the linear isomorphism

$$\mathfrak{F} : T^*M \to TM, \tag{28}$$

relating the covariant and contravariant tensor fields on the body manifold M, is given[8]. We shall call the isomorphism $\mathfrak{F}$ the **residual stress function** and use it to pull back the kinematic objects, such as deformation or strain, from the tangent space TM to the cotangent bundle T^*M, establishing this way the **stress space**. In particular, let

$$\theta \equiv \mathfrak{F}^*\mathbf{h} \tag{29}$$

be the **residual stress metric** on T^*M corresponding to the intrinsic (flat) Euclidean metric $\mathbf{h}$ of the body manifold M. Although the metric $\mathbf{h}$ is flat, the stress metric θ is, in general, not flat, unless the isomorphism $\mathfrak{F}$ is holonomic. Denote by ω_θ the Levi-Civita connection of the metric θ and let $\mathbf{R}_\theta$ be its Riemannian curvature tensor. Finally, let $\mathbf{Rc}_\theta$ be the corresponding Ricci tensor and $\mathfrak{R}_\theta$ its scalar curvature. Following Kröner's lead, let us postulate that the **residual stress** measured at the

[8] This should be viewed as an additional constitutive postulate we will try to reconcile later with the previously made assumptions leading to the introduction of the uniformity tensor $\mathbf{D}$.

intermediate (unloaded) configuration $\widehat{\phi}$ is represented by the Ricci tensor $\mathbf{Rc}_\theta$ of the Levi-Civita connection ω_θ. If, in addition, we assume that the isomorphism $\mathfrak{F}$ is such that the Levi-Civita connection ω_θ has constant scalar curvature $\mathfrak{R}_\theta$, then the first Bianchi identity ($\nabla_\theta \mathbf{R}_\theta = 0$, where ∇_θ denotes the covaraint derivative of the connection θ) implies, as often postulated in the literature (see Kröner [7], Minogawa [13], Stojanovic [15]), that the residual stresses are self-equilibrated, that is, that

$$\mathrm{div}_\theta \mathbf{Rc}_\theta = 0 \tag{30}$$

where div_θ denotes the covariant divergence.

Remark 2. The Einstein tensor

$$\mathbf{E}_\theta \equiv \mathbf{Rc}_\theta - \frac{\mathfrak{R}_\theta}{2}\theta, \tag{31}$$

rather than the Ricci tensor $\mathbf{Rc}_\theta$, is the geometric object which in dimension 3 is always the covariant divergence free, Besse [1]. However, in dimension 2, as the Einstein tensor is identically zero, the Ricci tensor becomes its natural substitute. It is not self-equilibrated but the vanishing of its covariant divergence is equivalent, as we mentioned earlier, to postulating that the scalar curvature $\mathfrak{R}_\theta$ is constant.

Having the residual stress defined by the Ricci tensor $\mathbf{Rc}_\theta$, we are now in the position to look again at the constitutive assumption (28) that there exists a linear transformation relating the material tangent and cotangent spaces. But first, viewing the material strain as the natural counter part of the residual stress, let us postulate that

$$\mathbf{E}^m = \frac{1}{2}\mathfrak{F}_* \mathbf{Rc}_\theta. \tag{32}$$

For this definition to be consistent with the earlier definition of the material strain tensor (27) the material metric $\mathbf{g}$, as given by the equation (22), must obey the following relation:

$$\mathbf{g} = \mathbf{h}\exp(\mathbf{E}^m) = \mathbf{h}\exp(\mathfrak{F}_* \mathbf{Rc}_\theta). \tag{33}$$

In other words, postulating the above relation (32), between the residual stress and the material strain, we de facto assume that given the intermediate configuration $\widehat{\phi}$ and the uniformity tensor $\mathbf{D}$ there exists an isomorphism $\mathfrak{F}$ such that

$$\ln(\mathbf{D}\mathbf{D}^T\mathbf{h}^{-1}) = \mathfrak{F}_* \mathbf{Rc}_\theta \tag{34}$$

where $\theta = \mathfrak{F}^* \mathbf{h}$. Conversely, given the constitutive isomorphism $\mathfrak{F}$ the stress metric θ defines the stress space and the uniformity tensor $\mathbf{D}$ is given (up to an isometry of the metric $\mathbf{h}$) by the equation (34). The relation (32), between the Ricci (residual stress) tensor $\mathbf{Rc}$ and the material strain tensor $\mathbf{E}^m$, plays the role of the Hooke's law of the linear elasticity. When presented in coordinates, it takes the form

$$E^m{}_{KJ} = \mathfrak{F}^M_K \mathfrak{F}^N_J Rc_{MN} \tag{35}$$

where the tensor $\mathfrak{F}_K^M \mathfrak{F}_J^N$ may be viewed as an (inelastic) analog of the material modula. Following this line of thought, we may want to replace the constitutive isomorphism $\mathfrak{F}$ by a more general linear isomorphism Υ between the bundles of covariant $(0,2)$-tensors on the tangent and cotangent spaces of the body manifold M. Indeed, given the isomorphism

$$\Upsilon : S_2(T^*M) \to S_2(TM), \tag{36}$$

where $S_2(\cdot)$ denotes the bundle of covariant symmetric $(0,2)$-tensors, the stress metric $\theta = \Upsilon^{-1}\mathbf{h}$ and the material strain

$$E^m{}_{KL} = \Upsilon_{KL}^{IJ} Rc_{IJ}, \tag{37}$$

while the uniformity tensor $\mathbf{D}$ is such that

$$\ln(\mathbf{D}\mathbf{D}^T \mathbf{h}^{-1}) = \Upsilon \mathbf{Rc}_\theta. \tag{38}$$

Note that the existence of the isomorphism Υ implies the existence of the isomorphism $\mathfrak{F}$ as its base map.

5 Examples

We present here two simple examples of the stress space and the objects associated with it.

Example 1. Einstein metric
Consider a test case when the stress metric θ is the Einstein metric in dimension 3, that is, when

$$\mathbf{Rc}_\theta = \frac{\mathfrak{R}_\theta}{3}\theta. \tag{39}$$

In such a stress space the uniform material metric $\mathbf{g}$, the uniformity tensor $\mathbf{D}$, and the material strain $\mathbf{E}^m$ are given by:

$$\begin{aligned} \mathbf{g} &= e^{\frac{\mathfrak{R}_\theta}{3}}\mathbf{h}, \\ \mathbf{D} &= e^{\frac{\mathfrak{R}_\theta}{6}}\mathbf{h}, \\ \mathbf{E}^m &= e^{\frac{\mathfrak{R}_\theta}{6}}\mathbf{h}. \end{aligned} \tag{40}$$

Example 2. Isotropic material
Consider the material isomorphism Υ such that the tensor

$$\Upsilon_{IJ}^{KL} = -\frac{\lambda}{2\mu(3\lambda+2\mu)} h_{IJ} h^{KL} + \frac{1}{2\mu}\delta_I^K \delta_J^L. \tag{41}$$

It can be interpreted as the inverse elasticity tensor of an isotropic material with the inhomogeneous Lame constants λ and μ, Landau and Lifshitz [8]. The corresponding residual stress metric θ is conformally equivalent to the metric $\mathbf{h}$, that is, given a material point X

$$\theta(X) = \Upsilon^{-1}\mathbf{h}(X) = \frac{1}{3K(X)}\mathbf{h}, \tag{42}$$

where $K(X) = \mu(X) + \frac{2}{3}\lambda(X)$ is the inhomogeneous bulk module. The Levi-Civita connection of θ has the Christoffel symbols given by

$$\Gamma^i_{JL} = \delta^I_J N_{,L} + \delta^I_L N_{,J} - h^{IS} h_{JL} N_{,S} \tag{43}$$

where $N(X) \equiv -\frac{1}{2}\ln(3K(X))$. Hence, the Ricci tensor $\mathbf{Rc}_\theta$ is represented by a matrix with coordinates:

$$Rc_{KL} = N_{,KL} - 3N_{,K}N_{,L} + [||dN||^2 - \triangle N]h_{KL}, \tag{44}$$

Besse [1]. This, in turn, implies that the material strain (37) has the following representation:

$$E^m{}_{IJ} = -\frac{\lambda}{18K^2\mu}\mathfrak{R}_\theta h_{IJ} + \frac{1}{6K\mu}Rc_{IJ}. \tag{45}$$

One may view this relation as the residual stress analog of the (linear) Hooke's law for an isotropic material.

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