ANOTHER LOOK AT THE EVOLUTION OF MATERIAL STRUCTURES

ERNST BINZ AND MAREK Z. ELŻANOWSKI

ABSTRACT. The evolution of a distribution of material inhomogeneities is investigated by analyzing the evolution of the corresponding material connection. Some general relations describing how the deformation of a material G-structure modifies the associated with it material connection are derived. These relations are then analyzed for different material isotropy groups.

1. INTRODUCTION

In this article we address certain aspects of a somewhat complicated problem of material evolution laws. Laws of evolution are integral part of theories such as those of plasticity, visco-plasticity, material growth, and others. Since plasticity is often viewed as a process of re-arrangement of patterns of defects it seems natural to discuss the issue of evolution within the geometric framework of the theory of uniform material structures, *cf.*, [7], [14].

The question of the structure of the law of material evolution has already been dealt with in this realm both in the context of simple materials as well as those of the second-order, see e.g., [12], [13] and [15], [18]. In the approach presented there material evolution was modeled by a first order differential equation for the so-called uniformity maps evaluated at a material point. Using constitutive postulates of *G-covariance* and the *Principle of actual evolution*, and assuming the uniformity of evolution (to parallel the fact that the evolving structures are themselves uniform),

The work presented in this note was done in part when the second author was visiting University of Mannheim in August/September 2000. The financial support for this visit was provided by the Ministry of Education of Baden-Württemberg and the University of Mannheim. The final version of this paper was completed during the second authors visit at Nottingham University in April 2001. Travel support for both visits was provided by the Faculty Development Fund of Portland State University. The preliminary version of this paper was presented at CIMRF-2001, Berlin, *cf.*, [1].

the geometric methods were used to investigate the form and structure of such a differential equation.

In this paper we look at the evolution of material from yet another perspective by investigating how the deformation (evolution) of the underlying G-structure shows through the evolution of the corresponding material connection. Our long term objective is to find out if there may be point-wise evolutions of the uniformity maps, which although non-trivial and constitutively admissible, produce no measurable changes of the material structure in question. We believe that such evolutions my account for these non-elastic deformations which do not change "defectiveness" of the material body; as measured by the torsion of its material connection. A somewhat similar problem has been investigated by Parry (see [19] and references therein) within the context of the structurally based theory of defects following a simple model of defective crystals introduced by Davini [4][1]. Using purely kinematic considerations he was able to show that there exists a non-trivial class of inelastic deformations having the same elastic invariants. Such deformations are akin to the classical slip mechanism of the phenomenological plasticity and represent rearrangements of material points while preserving the local lattice structure. In this note, however, we shall only look at some group-theoretic obstructions related to the problem of the evolution of material connections. That is, assuming that the symmetry group of the material remains unchanged during the process of evolution we will try to show that there are nontrivial evolution processes which do not produce any measurable changes in the distribution of inhomogeneities as they do not alter the corresponding material connection.

Our presentation is divided into a number of short sections progressively leading towards the analysis of the evolution laws of material structures. After a brief review of the concepts of material uniformity, homogeneity and that of material G-structure in Section 2 the notion of a material connection is introduced in Section 3. The mathematical aspects of the evolution of G-structures are discussed in Section 4. This is followed in Section 5 by a systematic analysis of the evolution of the corresponding

[1]For the comparison between the geometric theory of the continuous distribution of defects and the structurally based theory of defective crystals we refer the reader to a very informative paper by Davini [5].

G-connections. Finally in Section 6, the last part of this note, we discuss some constitutive aspects of material evolution and the relation between the material symmetry group and the existence of particular evolution processes.

2. Uniformity

The material body B is a continuum having the structure of a (real) differentiable manifold, usually of dimension 3 or less. We assume that it is smooth, orientable, connected and boundary-less[2]. We assume also, for the clarity and simplicity of this presentation, that it can be covered by a single (global) coordinate chart. This simplifying assuption affects in no way the final analysis of the problem of evolution. Let H(B) denote the bundle of all linear frames of the body (manifold) B with the standard surjective projection $\pi : \mathrm{H(B)} \to \mathrm{B}$. A frame h at the material point $x = \pi(h)$ can be identified with a (jet) *local configuration* of the body point, *cf.*, [7]. In contrast, a *global configuration* of B is an *integrable* section of H(B), i.e., a section generated by some global chart $\chi : \mathrm{B} \to \mathbb{R}^3$.

The mechanical response of a *simple material body*, measured at a material point, is completely determined by the present (and possibly past history) values of the local configuration at that point measured relative to a reference crystal - a prototype of a material point. The density (per unit volume of the reference crystal) of its *stored energy function* W is assumed to be a real-valued function on the space of local configurations H(B)[3]. Namely, if the material point, say $x \in \mathrm{B}$, is placed at the local configuration $h \in \mathrm{H(B)}$ the density of the stored energy at that point is given by $W(h)$. We say that the body is *materially uniform* if there exists a smooth section $p : \mathrm{B} \to \mathrm{H(B)}$ such that

$$W(p(x)) = W(p(y)) \tag{2.1}$$

for any pair of material points $x, y \in \mathrm{B}$. Such a section is called the *uniform configuration.* It represents a hypothetical re-arrangement of local configurations of material

[2]As far as the Differential Geometry background of this presentation, and the mathematical theory of uniform material structures is concerned we refer a reader to [16] and [20].

[3]As pointed in [12] this assumption does not necessarily imply that the material is elastic.

points so that the relative mechanical response becomes point independent[4]. Being materially uniform is the mathematical way of saying that the body B is made of the same material at all points. Thus, for the remainder of this exposition we will deal with materially uniform bodies only.

The uniform configuration does not necessarily represent any true physical state of the body B as a whole as it may not be integrable. Moreover, the material body, as defined by the stored energy function W, may have more than one uniform reference. This is the case if a materially uniform body is such that the stored energy W has a nontrivial isotropy group, i.e., if there exists a subgroup G_{B} of $\mathrm{GL}_3(\mathbb{R})$ - the structure group of H(B) - such that

$$W(hg) = W(h) \tag{2.2}$$

for every local configuration $h \in \mathrm{H(B)}$ and every $g \in G_{\mathrm{B}}$. The product $hg \in \mathrm{H(B)}$ denotes here the right multiplication of the frame h by the element $g \in \mathrm{GL}_3(\mathbb{R})$. The (maximal) isotropy group G_{B} of W is called the *material symmetry group* of the body B. Given the uniform configuration $p : \mathrm{B} \to \mathrm{H(B)}$ and a smooth map $g : \mathrm{B} \to \mathrm{G_B}$ the section $pg : \mathrm{B} \to \mathrm{H(B)}$, where $pg(x) := p(x)g(x)$ for every $x \in \mathrm{B}$, represents yet another uniform reference, as evident from equations (2.1) and (2.2). Conversely, any two uniform configurations vary, as it is elementary to show [9], by a point-wise smooth action (*gauging*) by the symmetry group G_{B}. Note, that if the symmetry group G_{B} is trivial or is a discreet subgroup of $\mathrm{GL}_3(\mathbb{R})$ the (smooth) uniform configuration is uniquely defined.

A uniform configuration may or may not be a global configuration of the body B. However, if it is, the material body B is said to be *homogeneous*. In other words, the materially uniform body B is homogeneous if among all its uniform configurations there exists at least one which is also a global configuration.

[4]Given the energy density $W : \mathrm{H(B)} \to \mathbb{R}$ and utilizing some trivialization of H(B) there exist a function $\hat{W} : \mathrm{B} \times \mathrm{GL}_3(\mathbb{R}) \to \mathbb{R}$ such that $W(h) = \hat{W}(x, g)$ for some $g \in \mathrm{GL}_3(\mathbb{R})$ corresponding (via the trivialization used) to h. Having the uniform configuration p available, and using the induced by p trivialization, one can show that there exists a function $\tilde{W} : \mathrm{GL}_3(\mathbb{R}) \to \mathbb{R}$ such that $W(h) = \tilde{W}(g)$, where g is the element of the group $\mathrm{GL}_3(\mathbb{R})$ for which $h = p(x)g$, *cf.*, [10].

3. Material Connection

Consider the materially uniform body B with the density of the stored energy function W. Let us select one of its uniform configurations, say p, and suppose that G_{B} is its material symmetry group. It was shown in [9] that this pair induces in a natural way a G_{B}-structure, i.e., a reduction of the bundle of linear frames H(B) to the subgroup G_{B} of its structure group $\mathrm{GL}_3(\mathbb{R})$, *cf.*, [20]. This reduced bundle, called the *material structure* (or G_{B}-*material structure*), is the central mathematical construct in the analysis of the homogeneity of the body B. Indeed, as shown in [9] (see also [10] and [14]), the materially uniform body is homogeneous if and only if the corresponding material structure is flat.

The uniform configuration $p : \mathrm{B} \to \mathrm{H(B)}$ introduces on H(B) a parallelism - known as the *material parallelism* - by lifting the tangent space of the body manifold B onto the submanifold $p(\mathrm{H(B)}) \subset \mathrm{H(B)}$ and extending it by the natural right action of $\mathrm{GL}_3(\mathbb{R})$ to the (equivariant) horizontal distribution on the whole bundle H(B) . In fact, every uniform configuration does induce a material parallelism although two different uniform configurations generate two, in general different parallelisms.

Having the material parallelism induced on H(B) we get to represent it by the *linear connection*, i.e, the Lie algebra $\mathfrak{gl}_3(\mathbb{R})$-valued equivariant 1-form ω on the frame bundle H(B) vanishing on the horizontal distribution induced by the section (uniform configuration) p. This linear connection ω is said to be "integrable" as the corresponding horizontal distribution is a locally integrable distribution on H(B). Hence, its curvature vanishes identically. Its torsion, however, is not necessarily zero. Such a connection is known as "pure gauge" and, with some abuse of notation, can be represented as $\omega := p^{-1}dp$[5]. Moreover, it can be shown, *cf.*, [7], that our linear connection ω can be reduced to the corresponding G_{B}-material structure. We call such a reduced connection ω the *material connection*, and denote it by the same letter. Thus, G_{B}-reduction of the linear connection induced on H(B) by some uniform configuration

[5]The section p is viewed here - via the canonically induced by it trivialization of the frame bundle H(B) - as a $\mathrm{GL}_3(\mathbb{R})$-valued mapping on H(B), *cf.*, [6]. The pre-multiplication by p^{-1} denotes the induced left action of the Lie group $\mathrm{GL}_3(\mathbb{R})$ on its own tangent space.

is a material connection. Conversely, any integrable connection on the G_{B}-material structure of H(B) is a material connection.

Given the uniform body B and its G_{B}-material structure one should be able to determine if the body is homogeneous. This is equivalent, as we have indicated earlier, to establishing if the given G_{B}-reduction of H(B) is flat. The material connection comes handy here. Indeed, it was shown in [8] (see also [22]) that the G_{B}-material structure is flat if and only if there exists a material connection ω (a Lie algebra $\mathfrak{g}_{\mathrm{B}}$-valued integrable connection on the reduced G_{B}-bundle) with the vanishing torsion. It is worth pointing out that there are other ways of identifying a flat G-structure. One such method is the method of the characteristic object [10], see also [3]. Although looking for a torsion free connection may be not the most effective method it is certainly the most universal one. The material body B represented by the stored energy function W is homogeneous if there exists a material connection such that its torsion vanishes. In fact, in case the symmetry group is discreet, the material connection is unique, and the non-vanishing torsion of the material connection becomes the measure of the inhomogeneity of the uniform material body. When the symmetry group is continuous such a unique measure does not exist, however.

4. Evolution of Structures

In this and the next section we will investigate how the material connection evolves under gauging by the elements of $\mathrm{GL}_3(\mathbb{R})$. As the material connection is uniquely defined by a section of H(B) we look first at the gauging of such sections.

Hence, let us consider the G_{B}-material structure uniquely defined by the uniform configuration, say p_0, and the material symmetry group G_{B}. Suppose $p_t : \mathrm{B} \to \mathrm{H(B)}$ is a one parameter family of smooth sections of the frame bundle. Such a family of sections can always be obtained from the section p_0 by the right action of some one-parameter (smooth) family of transformations $g_t : \mathrm{B} \to \mathrm{GL}_3(\mathbb{R})$ where $g_0(\mathrm{B}) \equiv$ I, the identity of the group $\mathrm{GL}_3(\mathbb{R})$. That is, $p_t(x) := p_0(x)g_t(x)$. The sections $p_t : \mathrm{B} \to \mathrm{H(B)}$ do not necessarily represent uniform configurations unless, although not only, the gauge transformations g_t take values in the symmetry group G_{B}. If this is the case we shall call such an evolution of uniform configurations *trivial* as each and every section p_t is just yet another representation of the same G_{B}-structure. However, if at some moment in time and some material point an element of the family g_t takes value in the complement of the symmetry group G_{B} the induced evolution of configurations is *non-trivial.* For example, if the material structure is "rigid", that is the symmetry group $G_{\mathrm{B}} = \mathrm{GL}_3(\mathbb{R})$, every evolution is trivial. On the other hand, every evolution, except constant, of the uniform configurations of the triclinic crystal ,where $G_{\mathrm{B}} = \{I\}$, is non-trivial.

Consider now two different evolutions of uniform configurations, say p_t and h_t. We say that these two evolutions are *parallel* if there exists time independent gauge transformation $g : \mathrm{B} \to \mathrm{GL}_3(\mathbb{R})$ such that

$$h_t(x) = p_t(x)g(x), \quad h_0(x) \neq p_0(x) \tag{4.1}$$

at every $(t, x) \in [0, T) \times \mathrm{B}$. According to the intuitive idea of evolution as a time dependent phenomenon it seems natural to expect that parallel evolutions are somewhat "equivalent". Comparing the "time" derivatives one notices immediately that they not only differ by the gauge g but also, if not because of this, they take values, even for the same parameter t, at different vector (tangent) spaces. However, an elementary calculation shows that the mapping $L_p(t) := \dot{p}_t p_t^{-1} : \mathrm{B} \to \mathfrak{gl}_3(\mathbb{R})$ is such that

$$L_p(t)(x) = L_h(t)(x) \tag{4.2}$$

as long as the relation (4.1) holds. The converse is obviously true as

$$L_h(t)(x) = L_p(t)(x) + p_t(x)L_g(t)(x)p_t^{-1}(x) \tag{4.3}$$

for any two time compatible evolutions of sections, and (4.2) holds only if $L_g(t)$ vanishes identically.

Proposition 1. *Two time compatible evolutions of sections of* H(B)*, say p_t and h_t, are parallel if and only if the corresponding mappings $L_p(t)$ and $L_h(t)$ are identical.*

We shall call the mapping $L_p : [0, T) \times \mathrm{B} \to \mathfrak{gl}_3(\mathbb{R})$ the *objective velocity* mapping of the evolution $p_t : \mathrm{B} \to \mathrm{H(B)}$. In the context of the theory of evolution of material structures $L_p(t)(x)$ is called the *inhomogeneity velocity gradient* and it was first introduced by Epstein and Maugin, [15].

5. Evolution of Connections

A smooth section of the bundle of frames of the body manifold B induces, as we have reviewed briefly in Section 3, a connection on H(B). When such a section evolves the induced connection may evolve with it. Suppose than that the section p_0 defines the connection $\omega_0 := p_0^{-1}dp_0$. The family p_t of sections representing the evolution of p_0 generates the one-parameter family of connections $\omega_p(t) := p_t^{-1}dp_t$. Let h_t represent yet another smooth evolution of sections where $\omega_h(t)$ is the corresponding one-parameter family of connection forms. As any two sections differ by the point-wise action of the structure group $\mathrm{GL}_3(\mathbb{R})$,

$$\omega_h(t) = [p_t g_t]^{-1} d[p_t g_t] = g_t^{-1}\omega_p(t) g_t + g_t^{-1} dg_t \tag{5.1}$$

where $g : [0,T) \times \mathrm{B} \to \mathrm{GL}_3(\mathbb{R})$ represents the gauge transformation while $g_t := g(t,\cdot)$ for every $t \in [0,T)$. If however the sections p_0 and h_0 evolve in parallel, i.e., the gauge transformation g_t is time independent, the corresponding time derivatives are related by

$$\dot{\omega}_h(t) = g^{-1}\dot{\omega}_p(t) g. \tag{5.2}$$

This implies that

$$ad(h_t)\dot{\omega}_h(t) = ad(p_t)\dot{\omega}_p(t) \tag{5.3}$$

where $ad : \mathrm{GL}_3(\mathbb{R}) \to \mathfrak{gl}_3(\mathbb{R})$ denotes the adjoint representation of the group $\mathrm{GL}_3(\mathbb{R})$ in its algebra $\mathfrak{gl}_3(\mathbb{R})$, *cf.*, [16]. When considering the evolution of integrable connections $\omega_p(t) = p_t^{-1}dp_t$ the *induced connection velocity* $\pounds_p(t) := ad(p_t)\dot{\omega}_p(t)$ becomes the connection counterpart of the inhomogeneity velocity gradient $L_p(t)$ in the sense that

Proposition 2. *For two parallel evolutions of sections, say p_t and h_t, the corresponding induced velocities of connections $\pounds_h(t)$ and $\pounds_p(t)$ are identical. In particular, if the families of connections $\omega_h(t)$ and $\omega_p(t)$ are such that at same point in time, say τ, $\dot{\omega}_h(\tau) = g_\tau^{-1}\dot{\omega}_p(\tau)g_\tau$, for $g_\tau = p_\tau^{-1}h_\tau : \mathrm{B} \to \mathrm{GL}_3(\mathbb{R})$, then*

$$\pounds_h(\tau) = \pounds_p(\tau).$$

In contrast to Proposition 1 the converse to Proposition 2 is not obvious at all. Indeed, given two evolutions of sections such that the corresponding induced velocities of connections are identical - we shall call such evolutions of connections *parallel* - it is not clear that the evolution of sections must be parallel. To illustrate this fact let us consider again two evolutions of sections h_t and p_t. Therefore, $h_t = p_t(p_t^{-1}h_t) = p_t g_t$, and

$$\dot{\omega}_h = -g_t^{-1}\dot{g}_t g_t^{-1}\omega_p g_t + g_t^{-1}\dot{\omega}_p g_t + g_t^{-1}\omega_p\dot{g}_t - g_t^{-1}\dot{g}_t g_t^{-1}dg_t + g_t^{-1}d\dot{g}_t. \tag{5.4}$$

This in turn implies that

$$\begin{aligned}\pounds_h(t) = h_t\dot{\omega}_h h_t^{-1} = \pounds_p(t) - p_t^{-1}\dot{g}_t g_t^{-1}\omega_p p_t^{-1} + p_t\omega_p\dot{g}_t g_t^{-1}p_t^{-1} \\ - p_t\dot{g}_t g_t^{-1}dg_t g_t^{-1}p_t^{-1} + p_t d\dot{g}_t g_t^{-1}p_t^{-1}.\end{aligned} \tag{5.5}$$

Realizing that

$$dL_g(t) := d(\dot{g}_t g_t^{-1}) = d\dot{g}_t g_t^{-1} - \dot{g}_t g_t^{-1}dg_t g_t^{-1} \tag{5.6}$$

the equation (5.5) can be rewritten as

$$\pounds_h(t) - \pounds_p(t) = ad(p_t)[\omega_p(t), L_g(t)] + ad(p_t)dL_g(t) \tag{5.7}$$

where $[\cdot,\cdot]$ is the Lie algebra commutator, and d denotes spatial differentiation.

Proposition 3. *Two evolutions of connections, $\omega_h(t)$ and $\omega_p(t)$, are parallel if and only if*

$$[\omega_p(t), L_g(t)] + dL_g(t) = 0 \tag{5.8}$$

where $g_t = h_t p_t^{-1}$.

In other words, given the one-parameter family of connections $\omega_p(t)$ and the family of gauge transformations g_t satisfying equation (5.8) the induced evolution of connections $\omega_h(t)$, where $h_t = p_t g_t$, is parallel to $\omega_p(t)$. Note, that if the gauge transformations g_t of the Proposition 3 are material point independent the equation (5.8) reduces to

$$[\omega_p(t), L_g(t)] = 0. \tag{5.9}$$

The existence of particular gauge transformations satisfying these equations for different material symmetry groups will be discussed in the following section.

It seems now natural to inquire about the relation between the induced velocity $\pounds_p$ and the corresponding objective velocity (the inhomogeneity velocity gradient) L_p. To this end let us consider the family of sections p_t in a neighborhood of the material point $x \in \mathrm{B}$. The evolution of the corresponding connection forms $\omega_p(t)$ at x is such that

$$\dot{\omega}_p(t) = \frac{d}{dt}[p_t^{-1} dp_t] = -p_t^{-1}\dot{p}_t p_t^{-1} dp_t + p_t^{-1} d\dot{p}_t. \tag{5.10}$$

Thus, the induced velocity at the point x

$$\pounds_p(t) = p_t \dot{\omega}_p(t) p_t^{-1} = -\dot{p}_t p_t^{-1} dp_t p_t^{-1} + d\dot{p}_t p_t^{-1}. \tag{5.11}$$

Comparing this with the equation (5.6), one gets that

$$\pounds_p(t) = dL_p(t). \tag{5.12}$$

We finish this section by deriving the general evolution relation for the pure gauge connection under the time dependent gauge transformation. Henceforth, let us assume that the connection $\omega_0 = p_0^{-1} dp_0$ is being deformed by the family of gauge transformations $g_t : \mathrm{B} \to \mathrm{GL}_3(\mathbb{R})$. Therefore, $p_t = p_0 g_t$ and $\omega_p(t) = g_t^{-1}\omega_0 g_t + g_t^{-1} dg_t$. Modifying equation (5.4) we obtain that

$$\dot{\omega}_p(t) = -g_t^{-1}\dot{g}_t g_t^{-1}\omega_0 g_t + g_t^{-1}\omega_0 \dot{g}_t - g_t^{-1}\dot{g}_t g_t^{-1} dg_t + g_t^{-1} d\dot{g}_t. \tag{5.13}$$

Comparing this relation with equations (5.5) and (5.6) and adopting them to our particular situation we get that

$$\dot{\omega}_p(t) = ad(g_t^{-1})[\omega_0, L_g(t)] + ad(g_t^{-1}) dL_g(t), \tag{5.14}$$

or equivalently

$$ad(p_0^{-1})\pounds_p(t) = [\omega_0, L_g(t)] + \pounds_g(t). \tag{5.15}$$

This gives us the following analog of Proposition 3

Proposition 4. *Given the pure gauge connection ω_0 and the family of gauge transformations g_t the family of connection forms $\omega_p(t)$, where $p_t = p_0 g_t$, will not evolve if and only if*

$$\pounds_g(t) = [L_g(t), \omega_0]. \tag{5.16}$$

When the gauge transformations g_t are material point independent the relation (5.16) reduces to

$$[L_g(t), \omega_0] = 0. \tag{5.17}$$

6. Material Evolution

The material structure of an elastic body serves, as pointed out earlier, as a geometric representation of the distribution of inhomogeneities. As long as a (uniform) body remains elastic its material structure, as determined by the density of its stored energy function W, remains unchanged. However, if we allow the body to experience other than elastic deformations, like for example in the case of visco-plasticity, while assuming that the strain energy is still measurable, the underlying geometric structure may change. Indeed, it is traditionally assumed for example that plasticity involves a mechanism which modifies the distribution of inhomogeneities, defects in particular. Mathematically, such a re-arrangement of defects patterns can only be observed if the underlying material structure evolves.

The exact form of the law of evolution of any particular material structure can only be determine through constitutive modeling. This aspect of the theory will not be dealt with in this paper. There are, however, some general principles we would like any "reasonable" law of evolution of structures to satisfy. In particular, we postulate that any such law satisfies the following two fundamental principles:

- **Principle of covariance:** *a law of evolution must be independent of the particular reference configuration chosen.*
- **Principle of actual evolution:** *a law of evolution must at all times select the inhomogeneity velocity gradient $L_p(t)$ outside of the algebra of the instantaneous symmetry group G_{B}.*

These principles were originally postulated by Epstein and Maugin in [15] (see also [12]) where it was also suggested that the evolution of a material is governed by a first order differential equation for the uniform configurations with the Eshelby tensor $\mathbf{b}$ as the driving force, *cf.*, [15]. In this work, consistent with our view on the evolution of structures, it seems natural to assume an evolution law of the form:

$$\dot{\omega} = \mathfrak{f}(\omega, p, \cdot) \tag{6.1}$$

where $\omega = p^{-1}dp$ is the instantaneous material connection as generated by the uniform configuration p, and where the law of evolution is assumed material point independent due to the uniformity of the body. As indicated, the functional $\mathfrak{f}$ may still depend on other objects like for example the Eshelby tensor or the deformation gradient. According to the *Principle of covariance* this evolution law must be invariant under the change of the global reference configuration. In mathematical terms this means that whenever two uniform material evolutions, say $p_1(t)$, $p_2(t)$, differ by a global (integrable) configuration κ, the evolutions of the corresponding material connections remain equivalent. To this end, and viewing the image of the uniform configurations p as linear isomorphisms between tangent spaces, let us note first that

$$\omega_2 = p_2^{-1} dp_2 = (\nabla\kappa\, p_1)^{-1} d(\nabla\kappa\, p_1) = \omega_1 \tag{6.2}$$

whenever there exists a global configuration κ such that

$$p_2(t) p_1^{-1}(t) = \nabla\kappa. \tag{6.3}$$

This, in turn, implies that

$$\dot{\omega}_2 = \dot{\omega}_1. \tag{6.4}$$

Hence

$$\mathfrak{f}(\omega_2, p_2, \cdot) = \mathfrak{f}(\omega_1, p_1, \cdot) \tag{6.5}$$

for any two uniform references satisfying (6.3). As the relation (6.5) is local in nature it leads to the identity

$$\mathfrak{f}(\omega, \mathbf{K}\, p, \cdot) = \mathfrak{f}(\omega, p, \cdot) \tag{6.6}$$

to be satisfied for all non-singular tensors (invertible linear transformations) $\mathbf{K}$. Thus, the functional $\mathfrak{f}$ is proved to be independent, in any direct way, of the underlying uniform reference. It may still depends on p indirectly via the material connection ω as well as possibly through other objects[6].

[6] If we postulate, as it was done in [12], that the functional $\mathfrak{f}$ depends on the Eshelby tensor $\mathbf{b}$ then the covariance of the law of evolution leads to the identity

$$\mathfrak{f}(\omega, \mathbf{K}\, p, J_{\mathbf{K}}^{-1} \mathbf{K}^{-T} \mathbf{b} \mathbf{K}^{T}) = \mathfrak{f}(\omega, p, \mathbf{b}).$$

In other words, the law of evolution (6.1) may be rewritten as:

$$\dot{\omega} = \mathfrak{f}(\omega, \mathbf{b}_0)$$

where $\mathbf{b}_0 = J_{\mathbf{P}} \mathbf{P}^T \mathbf{b} \mathbf{P}^{-T}$. The subscript T denotes here the transpose, the tensor $\mathbf{P}$ is a point-wise representation the uniform configuration p while $J_{\mathbf{P}}$ is its Jacobian.

We proceed now to investigate the role of the isotropy (symmetry) group and the implications of imposing the *Principle of actual evolution.* However, in contrast to what was done in previous works like [13, 15] we shall not investigate the form of the evolution law. Especially, we will not analyze how the functional $\mathfrak{f}$ need to be specified in order to conform with the principle. Rather, we shall focus on finding the group-theoretic obstructions to the existence of solutions to the evolution relation (5.15). That is, looking at different isotropy groups and accepting the *Principle of actual evolution* we shall try to determine the confines within which one could find a proper - as we see it - evolution, that is the evolution changing the essential characteristics of a distribution of material inhomogeneities like its material connection. This way we should get a better understanding what the "reasonable" evolutions are. In other words, aided by Proposition 4 we shall try to determine the sets of solutions to the relations (5.16) and (5.17).

We assume that any allowable gauge transformation is unimodular and that we only consider unimodular symmetry transformations, that is that $G_{\mathrm{B}} \subset \mathrm{SL}_3(\mathbb{R})$ and that we gauge within $\mathrm{SL}_3(\mathbb{R})$ only. We also assume that during the evolution the symmetry group remains unchanged. The much more difficult case of the evolution process in which not only the uniform configuration but also the structure group of the corresponding material structure may change is left for future presentation.

To start with our analysis let us look closer at $\mathfrak{sl}_3(\mathbb{R})$, the Lie algebra of the special linear group $\mathrm{SL}_3(\mathbb{R})$, that is the space of all trace-less 3×3 matrices. By $\mathfrak{so}_3$ we denote the algebra of the special orthogonal group SO_3, namely the set of all skew-symmetric 3×3 matrices representing the symmetry group of the isotropic material. Furthermore let $\mathfrak{sym}_3$ be the space of all trace-less symmetric 3×3 matrices while $\mathfrak{so}_{2,3}$ stands for the Lie algebra of the group of all rotations about a fixed axis. This is a one-dimensional subalgebra of $\mathfrak{so}_3$, the symmetry group of the transversely isotropic material. Thus we have

$$\mathfrak{sl}_3(\mathbb{R}) = \mathfrak{so}_3 \oplus \mathfrak{sym}_3. \qquad (6.7)$$

In addition, both components may be decomposed further as follows:

$$\mathfrak{so}_3 = \mathfrak{so}_{2,3} \oplus \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -a & -b & 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 & c & 0 \\ -c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -a & -b & 0 \end{pmatrix} \right\} \tag{6.8}$$

and

$$\mathfrak{sym}_3 = \left\{ \begin{pmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & d \\ c & d & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & -2e \end{pmatrix} \right\}. \tag{6.9}$$

It is elementary to observe that $\mathfrak{so}_3$, $\mathfrak{so}_{2,3}$, and the set of all trace-less diagonal 3×3 matrices $\mathfrak{d}\{a, b, -(a+b)\}$, are all abelian subalgebras of $\mathfrak{sl}_3(\mathbb{R})$ while $\mathfrak{sym}_3$ is only a vector subspace.

First, let us look at the relation (5.17) where the gauge transformations g_t are assumed material point independent. Supposing that the connection form ω_0 takes value in a non-trivial subalgebra $\mathfrak{h} \subset \mathfrak{sl}_3(\mathbb{R})$ and accepting the *Principle of actual evolution* we look for the deformations $g_t \in \mathrm{SL}_3(\mathbb{R})$ such that $L_g(t) \notin \mathfrak{h}$ and $[L_g(t), X] = 0$ for every $X \in \mathfrak{h}$. In other words, given the subalgebra $\mathfrak{h} \subset \mathfrak{sl}_3(\mathbb{R})$, we look for the set

$$\mathfrak{c}(\mathfrak{h}) := \{Y \in \mathfrak{sl}_3(\mathbb{R})/\mathfrak{h} : [Y, X] = 0 \quad \text{for all} \quad X \in \mathfrak{h}\} \tag{6.10}$$

where $\mathfrak{sl}_3(\mathbb{R})/\mathfrak{h}$ denotes the complement of $\mathfrak{h}$ in $\mathfrak{sl}_3(\mathbb{R})$. Note that in general $\mathfrak{sl}_3(\mathbb{R})/\mathfrak{h}$ is not a Lie algebra. Consequently, $\mathfrak{c}(\mathfrak{h})$ is not a Lie algebra either, despite the fact that the Jacobi identity is always satisfied, *cf.*, [2]. It is now a matter of simple calculations to show that:

- *full isotropy:*

$$\mathfrak{c}(\mathfrak{so}_3) = \{0\}. \tag{6.11}$$

- *transversal isotropy:*

$$\mathfrak{c}(\mathfrak{so}_{2,3}) = \left\{ \begin{pmatrix} y & z & 0 \\ -z & y & 0 \\ 0 & 0 & -2y \end{pmatrix} \right\}_{y \neq 0} \oplus \mathfrak{o}, \tag{6.12}$$

- *volumetric isotropy:*

$$\mathfrak{c}(\mathfrak{d}\{a,a,-2a\}) = \left\{ \begin{pmatrix} 0 & x & y \\ v & 0 & z \\ u & t & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} r & s & 0 \\ s & -r & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}_{rs\neq 0}. \tag{6.13}$$

Let us now turn our attention to the relation (5.16) where the gauge transformations g_t may be material point dependent. Assuming once again that the material connection ω_0 takes values in a non-trivial subalgebra $\mathfrak{h} \subset \mathfrak{sl}_3(\mathbb{R})$ and invoking the *Principle of actual evolution* our objective is to identify these deformations $g_t \in \mathrm{SL}_3(\mathbb{R})$ that $L_g(t) \notin \mathfrak{h}$ and $[L_g(t), X] \in \mathfrak{sl}_3(\mathbb{R})/\mathfrak{h}$ for all $X \in \mathfrak{h}$. Therefore, given the subalgebra $\mathfrak{h}$, we define the set

$$\mathfrak{i}(\mathfrak{h}) := \{Y \in \mathfrak{sl}_3(\mathbb{R})/\mathfrak{h} : [Y,X] \in \mathfrak{sl}_3(\mathbb{R})/\mathfrak{h} \quad \text{for all} \quad X \in \mathfrak{h}\}. \tag{6.14}$$

Note that the set $\mathfrak{i}(\mathfrak{h})$ is a complement of the idealizer of $\mathfrak{h}$ in $\mathfrak{sl}_3(\mathbb{R})$. Thus $\mathfrak{i}(\mathfrak{h}) = \mathfrak{sl}_3(\mathbb{R})/\mathfrak{h}$ whenever $\mathfrak{h}$ is a Cartan subalgebra[7] of $\mathfrak{sl}_3(\mathbb{R})$. Notice also that $\mathfrak{i}(\mathfrak{h}) \supseteqq \mathfrak{c}(\mathfrak{h})$. It is now easy to see that:

- *full isotropy:*

$$\mathfrak{i}(\mathfrak{so}_3) = \mathfrak{sym}_3, \tag{6.15}$$

- *transversal isotropy:*

$$\mathfrak{i}(\mathfrak{so}_{2,3}) = \mathfrak{sym}_3 \oplus \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ -x & -y & 0 \end{pmatrix} \right\}, \tag{6.16}$$

- *volumetric isotropy:*

$$\mathfrak{i}(\mathfrak{d}\{a,a,-2a\}) = \mathfrak{c}(\mathfrak{d}\{a,a,-2a\}). \tag{6.17}$$

We have shown that as far as the material point independent deformations of material structures are concerned every proper deformation (i.e., obeying the principle

[7]The idealizer of the subalgebra $\mathfrak{h} \subset \mathfrak{sl}_3(\mathbb{R})$ is the set $I(\mathfrak{h}) := \{Y \in \mathfrak{sl}_3(\mathbb{R}) : [Y,X] \in \mathfrak{h} \quad \text{for all} \quad X \in \mathfrak{h}\}$. $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{sl}_3(\mathbb{R})$ if $I(\mathfrak{h}) = \mathfrak{h}$, *cf.*, [2].

of actual evolution) of the isotropic material structure yields a change in the material connection, see (6.11). However, there are some nontrivial proper evolutions of transversaly isotropic and volumetric structures which while deforming the structure will not alter the corresponding material connections, (6.12), (6.13). If, on the other hand, we allow all proper deformations to participate in the process of evolution there may even be fewer deformations of material structure producing any measurable changes in them as confirmed by (6.15) and (6.16). In the case of volumetric isotropy it appears - interestingly enough - that there is hardly any (material point dependent or not) evolution of the corresponding material structure which produces measurable changes, see (6.17). We would like to add that the above results cannot be compared with those of the structurally based theory of Parry as the latter is purely kinematic and, as it stands, material symmetry groups have no role to play there.

Acknowledgments: Both authors would like to thank Serge Preston for stimulating discussions concerning the group-theoretic aspects of this research. Thanks are also due to Gareth Parry for taking time out to explain the structurally based theory of defective crystals.

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Department of Mathematics, University of Mannheim, Mannheim, Germany.
E-mail address: `binz@euler.math.uni-mannheim.de`

Department of Mathematical Sciences, Portland State University, Portland, Oregon, U.S.A.
E-mail address: `marek@mth.pdx.edu`