ON THE TORSION DRIVEN EVOLUTION OF A UNIFORM MATERIAL BODY

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Abstract. A model of a self driven evolution law of a defective anelastic continuum is presented. Two-dimensional examples are discussed and the role of the Clausius-Duhem inequality in imposing constitutive restrictions is investigated.

1. Introduction

When a material body undergoes a process of anelastic deformation the preexisting pattern of inhomogeneities (defects, dislocations, etc.) gets modified. Such a process is often trigged by a force (stress) applied at the boundary but it may also be caused by the inhomogeneities themselves - or the internal stresses accompanying them - without any external intervention. That is, the material may prefer either to relive internal stresses by removing the inhomogeneities all together or it may just want to attain a different, more "favorable", and more stable, configuration of inhomogeneities [9]. Such spontaneous processes, and we shall call them a *self-driven evolution*, are often observed when a state is catalytically enhanced by an external field e.g., a temperature field. One may also consider such self-driven processes as an approximation of real processes triggered by a small change in boundary conditions.

In this paper, which is, in fact, a continuation and an extension of the previously published work [5], we present a linear model of a self-driven anelastic evolution of inhomogeneities law of a defective solid crystal body. Assuming that the material body is made of triclinic crystals (has no macroscopic symmetries) and that the evolution process does not alter these material symmetries, we discuss the situation when the temporal evolution of the distribution of inhomogeneities is driven primarily by the instantaneous density of inhomogeneities as represented by the torsion of the unique material connection. Some preliminary results along these lines were published already in [12].

As we deal here with dissipative processes, we feel that it is necessary to discuss also, however briefly, the thermodynamic context of these processes, in particular, the role of the Clausius-Duhem inequality. Indeed, when investigating the contorted aelotropy case [5], where only constant strain states [3] were allowed to participate, and no internal stresses were present, there was no need to look at the thermodynamic constraints. Now, as the choice of the distribution of inhomogeneities is such that residual stresses may be present such analysis seems to be necessary. Postulating, as we did in [5], that the evolution law, at least in principle, is independent of the Eshelby stress [7] and depends only on the torsion of the unique material connection, we consider the planar case only. We show subsequently that in dimension 2 the torsion tensor can be presented in terms of the Eshelby stress tensor and its derivatives. Re-writing the thermodynamic residual inequality in terms of the divergence of the Eshelby stress allows us to discuss the role of the Clausius-Duhem inequality in imposing constitutive restriction and/or selecting admissible processes.

2. Geometry of a uniform material body

We assume that the material body B is a continuum having the structure of a smooth (real) differentiable manifold. We also assume that B is orientable, simply connected and boundary-less. For the clarity and simplicity of this presentation, and without much loss of generality, we postulate that B can be covered by a global coordinate chart $\phi_0 : B \to \mathbb{R}^3$ called the *reference configuration*. Such a configuration endows the body B with the coordinate system, say, (X^1, X^2, X^3) and the induced flat matric $g = \phi_0^* h$ where ϕ_0^* denotes the pull-back of the mapping ϕ_0, and where h is the Euclidean matric of the *ambient physical space* $\mathbb{R}^3$.

The mechanical properties of a simple (elastic) material body B are fully characterized by the density of the stored energy function per unit reference volume $\mathcal{W}(F, X)$ where F denotes the gradient of the deformation $\phi \circ \phi_0^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ from the reference configuration $\phi_0(B)$ to the current configuration $\phi(B)$ and where $X \in \phi_0(B)$. Postulating that the material body is *materially uniform* [15], that is that all points are made of the same material, implies that there exist smoothly distributed throughout the body manifold B *uniformity maps* $P(X)$ [1] from a vector space V - known in the literature [6] as an *archetype* of a material point - to the tangent space of the

reference configuration, and a real-valued function $\widehat{\mathcal{W}}$ such that

$$\mathcal{W}(F, X) = J_P^{-1}\widehat{\mathcal{W}}(FP(X)) \tag{2.1}$$

for all deformation gradients F and any material point X, where J_P denotes the Jacobian of the mapping P.

Given a basis $\mathbf{e}_i$, $i = 1, 2, 3$, in the reference crystal V, the uniformity maps can be represented as

$$P(X)(\mathbf{e}_i) = p_i^I(X)\frac{\partial}{\partial X^I} \tag{2.2}$$

where $\frac{\partial}{\partial X^I}$ denotes a frame on the manifold B as induced by the reference configuration, and where $p_i^I(X)$ take value in the special linear group $SL(n, \mathbb{R})$. In fact, $p_i^I(X)\frac{\partial}{\partial X^I}$ can also be viewed as a section of the bundle of linear frames of B [1]. Such a section induces a global parallelism on B known as the *material parallelism*[1] [15]. The *material connection* associated with such a parallelism is given by the Lie algebra $\mathfrak{sl}(n, R)$-valued 1-form

$$\Gamma_K^I \equiv \Gamma_{JK}^I dX^J \tag{2.3}$$

the Christoffel symbols of which are

$$\Gamma_{JK}^I = -\frac{\partial p_i^I}{\partial X^J}p_K^i. \tag{2.4}$$

In general, neither the uniformity maps nor the corresponding material connection are uniquely defined due to the presence of the non-trivial material symmetry group (the isotropy of the strain energy $\mathcal{W}$) [2]. However, as we consider here only material bodies made of triclinic crystals, the symmetry group is trivial and the material connection becomes unique. It has zero curvature but its torsion

$$T_{JK}^I = \frac{1}{2}(\Gamma_{KJ}^I - \Gamma_{JK}^I) \tag{2.5}$$

does not necessarily vanish. It has been recognized [5], but only in the triclinic case, that the torsion of the material connection can be taken as the true measure of the density of the distribution of inhomogeneities. That is, if the torsion vanishes the material body is considered *homogeneous* as the distribution (2.2) is locally integrable, i.e., it is generated by a finite configuration, and the identity (2.1) is trivially satisfied by $P \equiv \mathrm{id}$.

The uniformity maps P define, in addition to the unique material connection Γ^I_K, the unique *material metric* $\widehat{h}$ (see [4]) such that

$$\widehat{h}_{IJ} = h_{ij}p^i_I p^j_J \tag{2.6}$$

in material coordinates. The material connection induced by material parallelism, although not Riemanian, is metric. Therefore, the Christoffel symbols of the Levi-Civita connection of the material metric $\widehat{h}$ take the form [13]

$$\widehat{\Gamma}^I_{JK} = \Gamma^I_{JK} - K^I_{JK} \tag{2.7}$$

where

$$K^I_{JK} = T^I_{JK} - \widehat{h}^{IL}(\widehat{h}_{JN}T^N_{KL} + \widehat{h}_{KN}T^N_{JL}). \tag{2.8}$$

We point out here that this duality of possible ways of describing geometrically the inhomogeneities proves to be particularly useful when investigating the isotropic material [15]. In this paper, we will not be investigating the modeling of the evolution of inhomogeneities in such material bodies. However, some development along these lines have already been presented in [4] when the curvature of the induced material metric was used to model the evolution of inhomogeneities.

3. Law of evolution

The mechanical properties of the uniform material body, when considered in the realm of purely elastic deformations, are adequately characterized by the density of the stored energy function $\mathcal{W}$, where the torsion of the material connection (or the curvature of the material metric [4]) measure the density of the distribution of inhomogeneities. However, once inelastic processes are allowed to participate the stored energy function alone does not fully describe the mechanical state of a material point. Indeed, inelastic processes involve mechanisms which, in general, change the mechanical properties of the material point by modifying the distribution of inhomogeneities. In mathematical terms this means that during the inelastic evolution the uniform reference (2.2) evolves causing both the material connection and its torsion to change.

We postulate here, as we proposed in previous works [5], [12], when describing the self-evolution of inhomogeneities, that regardless of the state of stress the density of inhomogeneities is the driving force behind the intrinsic (inelastic) evolution of these

inhomogeneities. In other words, assuming that internal stresses are either not present or simply negligible we postulate that the evolution of inhomogeneities is governed by the law

$$\frac{dP}{dt}(X,t) = f(T(X,t), P(X,t)) \tag{3.1}$$

where T is the torsion o the instantaneous material connection generated by the current uniformity map P, and where f is assumed as smooth as needed and independent of the material point. Invoking the *principle of covariance* [6], that is, assuming that the physical law must be independent of any particular reference configuration, one can show [5], [6] that the relation (3.1) reduces to

$$\frac{dP}{dt}(X,t) = P(X,t) f(W(X,t), I), \tag{3.2}$$

or equivalently, that it takes the form

$$L(P(X,t)) = f(W(X,t), I) \tag{3.3}$$

where

$$L \equiv P^{-1}\frac{dP}{dt} \tag{3.4}$$

is called the *inhomogeneity velocity gradient* [6] and where W denotes the pull-back, by the uniformity maps P, of the torsion tensor T to the reference crystal V. We further restrict our considerations by imposing a linear law of evolution such that

$$L(P) = CW \tag{3.5}$$

where C is a fifth order tensor of material constants. In components, the evolution law (3.5) takes the form

$$L^i_j = p^i_I \dot{p}^I_j = C^{i\ kl}_{jn} p^n_M p^N_k p^L_l T^M_{NL} = C^{i\ lm}_{jk} W^k_{lm}. \tag{3.6}$$

4. 2-dimensional model

To illustrate effectively our simple model and to show the range of phenomena it can capture, we adopt the general model introduced in the previous section (3.3) to the planar evolution of an otherwise 3-dimensional uniform material body B. As we have indicated earlier, we restrict our attention to a body made of solid crystals and

such that the uniformity maps P are, and remain during the evolution, independent of, say, X^3 coordinate.

Adopting an orthonormal coordinate frame in the reference crystal V and a Cartesian coordinate system in a fixed reference configuration, we calculate the torsion of the unique material connection taking values in the Lie algebra of the special linear group $\mathfrak{sl}(2,R)$ and its realization (a pull-back) in the reference crystal V. In other words, we select the uniformity maps such that

$$p(X^1,X^2,X^3,t)=\begin{pmatrix} a(X^1,X^2,t) & b(X^1,X^2,t) & 0\\ c(X^1,X^2,t) & d(X^1,X^2,t) & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{4.1}$$

where $ab-cd=1$ at all times and at all material points of the body. The connection form of the material connection (2.4) induced by the uniformity maps (4.1) can now be calculated as

$$\begin{aligned}\Gamma^I_{JK}dX^J=&-\begin{pmatrix} da_{,X^1}-cb_{,X^1} & -ba_{,X^1}+ab_{,X^1}\\ dc_{,X^1}-cd_{,X^1} & ad_{,X^1}-bc_{,X^1}\end{pmatrix}dX^1\\ &-\begin{pmatrix} da_{,X^2}-cb_{,X^2} & -ba_{,X^2}+ab_{,X^2}\\ dc_{,X^2}-cd_{,X^2} & ad_{,X^2}-bc_{,X^2}\end{pmatrix}dX^2,\end{aligned} \tag{4.2}$$

where comma denotes partial derivative. Hence, the non-vanishing components of the torsion tensor are:

$$\begin{aligned}T^1_{12}=\frac{1}{2}(\Gamma^1_{12}-\Gamma^1_{21})=-\frac{1}{2}[-b\cdot a_{,X^1}+a\cdot b_{,X^1}-d\cdot a_{,X^2}+c\cdot b_{,X^2}],\\ T^2_{12}=\frac{1}{2}(\Gamma^2_{12}-\Gamma^2_{21})=-\frac{1}{2}[a\cdot d_{,X^1}-b\cdot c_{,X^1}-d\cdot c_{,X^2}+c\cdot d_{,X^2}]\end{aligned} \tag{4.3}$$

To facilitate our future analysis of the two dimensional evolution law, we introduce also the trace 1-form of the torsion tensor. To this end we define

$$\omega=T^I_{JI}dX^J. \tag{4.4}$$

The availability of the trace form ω allows us to decompose the torsion of a connection into its traceless and diagonal parts, respectively. Namely, we have that

$$\hat{T}=T-\tilde{T}\quad\text{where}\quad \tilde{T}^I_{JK}=\frac{1}{1-n}(\delta^I_J\omega_K-\delta^I_K\omega_J). \tag{4.5}$$

This yields

$$\tilde{T}^I_{JI} = \frac{1}{1-n}(\omega_J - n\omega_J) = \omega_J, \quad \text{and} \quad \hat{T}^I_{JI} = 0 \tag{4.6}$$

where n denotes the space dimension. In particular, when $n = 2$, $\hat{T}$ vanishes and one obtains that

$$T^I_{JK} = -(\delta^I_J\omega_K - \delta^I_K\omega_J). \tag{4.7}$$

Given the choice of the uniformity maps (4.1), the trace 1-form ω takes the form

$$\omega = T^2_{12}dX^1 - T^1_{12}dX^2 = -\frac{1}{2}(ad_{,X^1} - bc_{,X^1} - dc_{,X^2} + cd_{,X^2})dX^1 + \frac{1}{2}(-ba_{,X^1} + ab_{,X^1} - da_{,X^2} + cb_{,X^2})dX^2. \tag{4.8}$$

The components of the pullback of the torsion tensor to the reference crystal W in the standard euclidian basis of V and the material frame induced by the reference configuration are

$$W^i_{jk} = p^i_M p^N_j p^L_k T^M_{NL} = p^i_M(p^1_j p^2_k - p^2_j p^1_k)T^M_{12}. \tag{4.9}$$

As both the pullback W and the torsion T have the same symmetries, and as $p^1_1p^2_2 - p^2_1p^1_2 = \det p = 1$, the only non-vanishing components of W are:

$$W^i_{12} = p^i_M(p^1_1p^2_2 - p^2_1p^1_2)T^M_{12} = p^i_M T^M_{12} \tag{4.10}$$

for $i = 1, 2$. Combining relations (4.5) and using the definition of the 1-form ω (4.4), we obtain that

$$W^i_{jk} = p^i_I p^J_j p^K_k T^I_{JK} = -\delta^i_j p^K_k \omega_K + \delta^i_k p^K_j \omega_K. \tag{4.11}$$

Consequently, the only non-vanishing components of W are

$$W^1_{12} = -p^K_2\omega_K = -b\omega_1 - d\omega_2 \quad \text{and} \quad W^2_{12} = p^K_1\omega_K = a\omega_1 + c\omega_2. \tag{4.12}$$

More specifically, either using the original definition of W (4.10) or the relation (4.12), we obtain that in our special two dimensional case (4.1)

$$W^1_{12} = dT^1_{12} - bT^2_{12} = -\frac{1}{2}[b_x + d_y] \quad \text{and} \quad W^2_{12} = -cT^1_{12} + aT^2_{12} = \frac{1}{2}[a_x + c_y]. \tag{4.13}$$

where we have benefited from the fact that $ad - bc = 1$.

5. Linear evolution

As we have indicated earlier, the material velocity of evolution of the uniformity maps P is given by the $\mathfrak{sl}(2,R)$-valued inhomogeneity velocity gradient L (3.4). Its coordinate representation in the planar case (4.1) takes the form

$$L = \begin{pmatrix} d\dot{a} - b\dot{c} & d\dot{b} - b\dot{d} & 0 \\ -c\dot{a} + a\dot{c} & -c\dot{b} + a\dot{d} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.1}$$

Thus, when the linear evolution law (3.6) is adopted, and taking into consideration the fact that the tensor W has only two non-vanishing components, the four non-vanishing components of the tensorial evolution law (3.6) can be written as

$$L^i_j =_{(k)} C^i_j W^k_{12} =_{(2)} C^i_j p^K_1 \omega_K -_{(1)} C^i_j p^K_2 \omega_K, \tag{5.2}$$

where

$${}_{(1)}C = \begin{pmatrix} A & E \\ F & -A \end{pmatrix}, \quad {}_{(2)}C = \begin{pmatrix} B & D \\ G & -B \end{pmatrix} \tag{5.3}$$

are constant, volume preserving, matrices of constitutive parameter such that

$${}_{(k)}C^j_l \equiv C^{j\ 12}_{kl} - C^{j\ 21}_{kl}. \tag{5.4}$$

Taking into account the form of the pull-back of the torsion tensor W (4.13) the evolution law (5.2) reduces to a system of four first-order quasi-linear partial differential equations

$$\begin{aligned} \dot{a} &= (Aa + Fb)[b_{,X^1} - d_{,X^2}] + (Ba + Gb)[a_{,X^1} + c_{,X^2}], \\ \dot{b} &= (Ea - Ab)[b_{,X^1} - d_{,X^2}] + (Da - Bb)[a_{,X^1} + c_{,X^2}], \\ \dot{c} &= (Ac + Fd)[b_{,X^1} - d_{,X^2}] + (Bc + Gd)[a_{,X^1} + c_{,X^2}], \\ \dot{d} &= (Ec - Ad)[b_{,X^1} - d_{,X^2}] + (Dc - Bd)[a_{,X^1} + c_{,X^2}]. \end{aligned} \tag{5.5}$$

It is worth pointing out that the equations (5.5) are not independent as $\dot{a}d + a\dot{d} - \dot{b}c - b\dot{c} = 0$ due to the fact that the inhomogeneity velocity gradient L takes values in the algebra of traceless matrices. Moreover, the system (5.5) is not strictly hyperbolic for all choice of material constants. In fact, it is often degenerate as at least one eigenvalue of its characteristic matrix always vanishes.

We do not attempt to present here the complete analysis of this system. Rather, for the sake of specificity and clarity of presentation as well as to be able to illustrate effectively different types of evolution, we shall concentrate on a number of particular cases restricting the choice of the uniformity maps P to specific subgroups of the special linear group. To be able to present our analysis effectively and efficiently we point out first that the coordinate representation of the uniformity map depends, naturally, on the choice of bases in the reference crystal V. On the other hand, a change of bases, does not change the coordinate representation of the corresponding material connection. It, however, does change by conjugation the inhomogeneity velocity gradient L and the pull-back of the torsion tensor W. Consequently, the material coefficients C transform in V like a tensor. It would now be easy to show that the components of matrices ${}_{(k)}C$ transform by conjugation too. In other words, viewing the change of bases as the action of the special linear group $SL(2,\mathbb{R})$ on the pair of 2×2 traceless matrices of material constants ${}_{(1)}C$ and ${}_{(2)}C$ we realize that the choice of such parameters is invariant within orbits of such an action. Indeed, selecting two different sets of material coefficients from within the same orbit will produce the same tensorial evolution law (5.2). Thus, it appears that when analyzing different uniform structures, we may benefit from selecting canonical representations of the corresponding $SL(2,\mathbb{R})$-orbits. In particular, we are interested in the Borel subalgebra of upper-triangular traceless matrices

$$\begin{pmatrix} A & E \\ 0 & -A \end{pmatrix} \tag{5.6}$$

as well as such special subalgebras as the nilpotent subalgebra and the diagonal subalgebras

$$\begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -A \\ A & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}, \tag{5.7}$$

respectively.

6. Examples

In order to illustrate better different types of evolution, we consider in this section a number of specific cases selecting the uniformity maps P from different subgroups of the special linear group, such as, orthogonal, diagonal and triangular matrices,

respectively. Our objective is show how different distributions of inhomogeneities evolve.

6.1. **Contorted aelotropy (orthogonal) case.** Assuming that the uniform reference represents the state of constant strain [14] and that the evolution proceeds only within the realm of such states, implies [3] that the uniformity maps (4.1) can be represented as planar rotations

$$p(X^1, X^2, X^3, t) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.1}$$

where $\theta = \theta(X^1, X^2, t)$ measures the counterclockwise rotation. The essential Christoffel symbols of the corresponding material connection (2.4) can now be easily evaluated as

$$\Gamma^I_{JK} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \theta_{,X^I} \tag{6.2}$$

where $I, J, K = 1, 2$ only. This yields the following non-vanishing components of the torsion tensor

$$T^1_{12} = \frac{1}{2}\theta_{,X^1}\,, \quad T^2_{12} = \frac{1}{2}\theta_{,X^2}\,, \tag{6.3}$$

while the torsion trace 1-form ω reduces to

$$\omega = \frac{1}{2}\theta_{,X^2}\, dX^1 - \frac{1}{2}\theta_{X^1} dX^2. \tag{6.4}$$

Substituting these formulas into (4.13) shows that

$$W^1_{12} = \frac{\cos\theta}{2}\theta_{,X^1} + \frac{\sin\theta}{2}\theta_{,X^2}\,, \quad W^2_{12} = \frac{\cos\theta}{2}\theta_{,X^2} - \frac{\sin\theta}{2}\theta_{,X^1}\,. \tag{6.5}$$

Skew symmetry of the tensor W renders a number of material constants redundant reducing the evolution law to a single hyperbolic quasi-linear differential equation for the angle of rotation θ

$$2\theta_{,t} = (D\sin\theta - E\cos\theta)\theta_{,X^1} - (E\sin\theta + D\cos\theta)\theta_{,X^2}\,. \tag{6.6}$$

The above equation models, depending on the choice of initial conditions and the value of material parameters E and D, such phenomena as dissipation of dislocations (rarefaction waves) and dislocation pileups (shock waves), as it was shown in [5].

6.2. **Diagonal case.** Let us suppose now that we deal with the inhomogeneous material body such that the only distributions of inhomogeneity allowed are those represented by the "diagonal" uniformity maps. In other words, the state of inhomogeneities is modeled at all points and all times by

$$p(X^1, X^2, X^3, t) = \begin{pmatrix} a(X^1, X^2, t) & 0 & 0 \\ 0 & 1/a(X^1, X^2, t) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.7}$$

The material connection takes the form

$$\Gamma^I_J = -\frac{1}{a}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}[a_{,X^1}\, dX^1 + a_{,X^2}\, dX^2], \tag{6.8}$$

$I, J = 1, 2$, inducing the torsion tensor such that

$$T^1_{12} = \frac{1}{2}a^{-1}a_{,X^2}\,, \quad T^2_{12} = \frac{1}{2}a^{-1}a_{,X^1}\,, \tag{6.9}$$

and the torsion trace 1-form

$$\omega = \frac{1}{2}a^{-1}a_{,X^1}\, dX^1 - \frac{1}{2}a^{-1}a_{,X^2}\, dX^2. \tag{6.10}$$

The pull-back of the torsion tensor has only two non-vanishing components

$$W^1_{12} = -\frac{1}{2}a^{-2}a_{,X^2}\,, \quad W^2_{12} = \frac{1}{2}a_{,X^1}\,. \tag{6.11}$$

One can easily calculate the inhomogeneity velocity gradient as

$$L^i_j = \frac{1}{a}\begin{pmatrix} a_{,t} & 0 & 0 \\ 0 & -a_{,t} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.12}$$

Comparing this inhomogeneity velocity gradient L with the corresponding *plastic distortion rate tensor*

$$L_P \equiv P^{-1}LP \tag{6.13}$$

of finite strain plasticity [10] shows that it represents a plastic slip system on two perpendicular planes with the identical shear rates [12].

The form of the inhomogeneity velocity gradient (6.12), and that of the torsion tensor, forces most material constants to vanish thus reducing the law of evolution to

a single equation for the function a, namely

$$a_{,t} = \frac{B}{2} a a_{,X^1} - \frac{A}{2} a^{-1} a_{,X^2} = \left(\frac{Ba}{2}, -\frac{A}{2a} \right) \nabla a \tag{6.14}$$

where the pair $\left(\frac{Ba}{2}, -\frac{A}{2a}\right)$ shall be called the *evolution vector*. The equation is hyperbolic and it models the same class of phenomena as the evolution equation of the contorted aelotropy (6.6). In particular, its generic continuous solutions can be generated by the following procedure. First, by re-scaling the spatial variables we can reduce the equation (6.14) to

$$a_t = a a_{,X^1} - a^{-1} a_{,X^2} \, . \tag{6.15}$$

Assuming invertibility of solutions, we re-write our differential equation in terms of the new unknown function $t(X^1, X^2, a)$

$$a t_{,X^1} - a^{-1} t_{,X^2} = 1. \tag{6.16}$$

This is a linear, inhomogeneous, first order differential equation where X^1, X^2 and a are considered independent variables. Its particular solution is, for example,

$$t_p = \frac{X^1}{a}. \tag{6.17}$$

The general solution of the homogeneous counterpart of the equation (6.16) takes the form

$$t_h = f(a, x + a^2 y) \tag{6.18}$$

where f is an arbitrary function of two variables. Superposing both solutions one obtains the implicit form of the general solution of the evolution law (6.14)

$$t = \frac{x}{a} + f(a, x + a^2 y). \tag{6.19}$$

6.3. **Upper triangular case.** Consider now a situation in which the evolution of inhomogeneities is such that the uniformity maps are at all times and at all material points represented by the upper triangular matrices

$$p = \begin{pmatrix} a & b & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.20}$$

where a and b are functions of X^1, X^2 and t only. The corresponding inhomogeneity gradients L and the matrices of material constants ${}_{(i)}C$, $i = 1, 2$, belong to the subalgebra of traceless upper triangular matrices. We also note that

$$W^1_{12} = -\frac{1}{2}[b_{,X^1} - a^{-2}a_{,X^2}], \quad W^2_{12} = \frac{1}{2}a_{,X^1}\,. \tag{6.21}$$

As a result, the evolution equations take the form

$$\begin{aligned} a_{,t} &= -\frac{Aa}{2}[b_{,X^1} - a^{-2}a_{,X^2}] + \frac{Ba}{2}a_{,X^1}\,, \\ b_{,t} &= -\frac{1}{2}(Ea - Ab)[b_{,X^1} - a^{-2}a_{,X^2}] + \frac{Da - Bb}{2}a_{,X^1}\,. \end{aligned} \tag{6.22}$$

The characteristic matrix of the system (6.22) is

$$M = \frac{1}{2}\begin{pmatrix} aB & -aA \\ Da - Bb & Ab - Ea \end{pmatrix}\sigma_1 + \frac{1}{2}\begin{pmatrix} a^{-1}A & 0 \\ a^{-2}(Ea - Ab) & 0 \end{pmatrix}\sigma_2, \tag{6.23}$$

where $\sigma = \sigma_1 dx + \sigma_2 dy$ is the co-vector of characteristic co-direction. It should be easy to see that depending on the choice of material parameters the type of the system changes. In particular, it is hyperbolic only for some, very particular, choice of the parameters A, B, D, E. To show how different this case can be from the previously investigating evolutions we shall consider the nilpotent case.

6.4. **Nilpotent case.** We assume that the state of inhomogeneity is given by the upper-triangular matrices and remains during the planar evolution as

$$p(X^1, X^2, t) = \begin{pmatrix} 1 & b(X^1, X^2, t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.24}$$

As a result

$$W^1_{12} = -\frac{1}{2}b_{,X^1} \quad \text{and} \quad W^2_{12} = 0. \tag{6.25}$$

The only equation left from the system of evolution equations (5.5) is a simple transport equation

$$\dot{b} = EW^1_{12} = -\frac{E}{2}b_{,X^1}\,. \tag{6.26}$$

Its general solution is a traveling wave

$$b = f(X^1 - \frac{E}{2}t), \tag{6.27}$$

where $f(\cdot)$ is an arbitrary function defining the initial profile of the distribution.

7. Thermodynamic constraints

In this final section, considering the dissipative nature of the processes we have been investigating, we the role of the Clausius-Duhem [10], [6] in imposing thermodynamic constraints on the evolution law.

As we have pointed out in [12], the main obstacle in dealing effectively with the issue of thermodynamic constrains, within the framework adopted here, is the very fact that our evolution law (3.1) seems to be independent of the Eshelby stress. Yet, it is commonly accepted that that the Eshelby stress is the driving force behind the evolution of inhomogeneities [8], [11]. As we show in the next section, this apparent independence of the torsion driven evolution from the Eshelby tensor may not necessarily be true. Indeed, after all, both objects describe the same physical concept of a distribution of inhomogeneities. Hence, our first task is to express, when possible, the torsion of the unique material connection in terms of the Eshelby stress.

7.1. **Eshelby stress tensor and torsion.** The Eshelby stress tensor, known also as the Eshelby energy-momentum tensor, being a generalization of the Piola-Kirchoff stress, is defined for uniform material [6] as

$$b \equiv -\frac{\partial \mathcal{W}}{\partial P} = -\frac{\partial \widehat{\mathcal{W}}}{P} P^T \tag{7.1}$$

where P^T denotes the transpose of the representation of the uniformity map P and where P's are assumed volume preserving. Enforcing the balance of linear momentum law implies, as shown in [6], the following relation between the torsion of the material connection and the Eshelby tensor:

$$b^I_{J;I} = -b^I_K T^K_{JI} + b^I_J T^K_{IK} = -b^I_K T^K_{JI} + b^I_J \omega_I, \tag{7.2}$$

where ; denotes covariant differentiation with respect to the material connection, and where the definition of the torsion trace 1-from (4.4) was utilized. The balance equation (7.2) is satisfied identically be the Eshelby stress tensor associated with a solution to a particular boundary-value problem.

As we are able to represent (in two dimensions) the torsion tensor via the trace form ω (4.5), the balance equation (7.2) reduces to

$$b^I_{J;I} = \omega_J - b^I_J \omega_J. \tag{7.3}$$

In other words, in the planar case, and the planar case only, one is able to represent the torsion of the material connection as a function of the Eshelby stress tensor and its covariant derivatives. This rather simple fact has far reaching consequences for the analysis of cases we consider in this presentation. Indeed, having the relation between the torsion tensor and the Eshelby stress available allows us to present the residual inequality of the Clausius-Duhem inequality in terms of the components of the Eshelby stress without compromising our original assumption about the torsion being the impetus for the evolution of inhomogeneities.

7.2. **Entropy production.** The residual part of the Clausius-Duhem inequality under the isothermal conditions takes the form [6], [10]

$$tr(b^T L_P) \leq 0 \tag{7.4}$$

where b^T denotes the transpose of the Eshelby stress tensor (7.1) and L_P is the plastic distortion rate tensor (6.13). In the planar case, and for the linear evolution law (5.2), the inequality reduces to

$$b^I_J p^i_I p^J_j ({}_{(2)}C^j_i p^K_1 \omega_K - {}_{(1)}\, C^j_i p^K_2 \omega_K) \leq 0 \tag{7.5}$$

where $i, j, I, J = 1, 2$. Although, in general, it does not seem to be possible to replace in (7.5) the trace torsion form ω by the Eshelby stress and/or its derivatives via (7.3), in all particular cases investigated in this work the Clausius-Duhem inequality (7.5) gets reduced to a rather simple form . Indeed, assuming that the evolution of inhomogeneities is given the diagonal uniformity maps (6.7), the inhomogeneity velocity gradient becomes

$$L^i_j = \begin{pmatrix} -A & 0 \\ 0 & A \end{pmatrix} a^{-1}\omega_2 + \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix} a\omega_1. \tag{7.6}$$

On the other hand, using (7.3), elementary calculations show that

$$\omega_I = \frac{b^J_{I,J}}{b^2_2 - b^1_1} \tag{7.7}$$

provided the Eshelby stress tensor is such that $b_1^1 \neq b_2^2$. Note that this assumption does not restrict in any way the choice of the Eshelby stress as the condition $b_1^1 = b_2^2$ implies the vanishing of the divergence $b^J_{I,J}$ implying, in turn, homogeneity [3]. Combining the expression for the inhomogeneity velocity gradient (7.6) and the form of the torsion trace 1-form (7.7) we finally obtain the following corresponding plastic distortion rate tensor

$$L^I_{P\ J} = \frac{1}{a(b_2^2 - b_1^1)} \begin{pmatrix} -[Ab_{2,\ K}^K - Ba^2 b_{1,\ K}^K] & 0 \\ 0 & [Ab_{2,\ K}^K - Ba^2 b_{1,\ K}^K] \end{pmatrix}, \tag{7.8}$$

where $I, J = 1, 2$ and the summation convention of repeated indices is enforced. This yields the following form of the Clausius-Duhem inequality (7.5):

$$A\frac{b_{2,\ K}^K}{a} - Bab_{1,\ K}^K \leq 0. \tag{7.9}$$

Not that the left hand side of the residual inequality represents the scalar product of the divergence vector of the Eshelby stress tensor and the evolution vector (6.14). Given the stored energy function $\mathcal{W}(F, X)$ of a uniform material body admitting the particular class of uniformity maps and the corresponding Eshelby stress tensor, in particular, its divergence, and assuming that the material parameters A and B of the evolution law are arbitrary, the residual inequality (7.9) restricts the choice of the evolution processes.

In the contorted aelotropy case the entropy production $tr(b^T L_P)$ can again be represented as the product of the divergence vector of the Eshelby stress and the corresponding evolution vector

$$-\left(D\sin\theta - E\cos\theta, E\sin\theta + D\cos\theta\right). \tag{7.10}$$

However, due to the fact that the plastic distortion rate tensor of the orthogonal evolution

$$L^i_j = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[\left(D\sin\theta - E\cos\theta\right)\theta_{X^1} + \left(E\sin\theta + D\cos\theta\right)\theta_{X^2}\right] \tag{7.11}$$

is skew-symmetric while the corresponding Eshelby stress tensor is symmetric, in fact, diagonal [6], the residual inequality is trivially satisfied.

We would like to conclude our investigations by looking again at the nilpotent case (6.24). Given the non-vanishing components of the torsion pullback (6.25) we

know that the trace torsion 1-form is

$$\omega = \frac{E}{2} b_{,X^1}\, dX^2, \tag{7.12}$$

and that the only non-vanishing component of the inhomogeneity velocity gradient, which in this case is identical to the plastic distortion rate tensor, is

$$L^2_1 = -\frac{E}{2} b_{,X^1}\,. \tag{7.13}$$

The plastic distortion rate tensor can now be represented as

$$L^2_{P1} = -\frac{E}{2b^2_1} b^J_{1,J} \tag{7.14}$$

which yields the following residual Clausius-Duhem inequality

$$-\frac{E b^1_2}{2b^2_1} b^J_{1,J} \leq 0. \tag{7.15}$$

Taking $(-E/2, 0)$ as the evolution vector, the entropy production is again proportional to the scalar product of the divergence of the Eshelby stress tensor and the corresponding evolution vector. Moreover, as the parameters of the evolution process are not explicitly present, and assuming that the Eshelby stress tensor can be selected arbitrarily[1] , it implies that $E = 0$ and no processes of this kind are allowed.

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