An HDG method for the velocity-vorticity formulation

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What is HDG?

- Hybridizable DG methods (HDG) were discovered in
  - [Cockburn, G., Lazarov, 2009] "Unified hybridization of DG, mixed, and CG methods for second order elliptic problems", SINUM.
  - Many authors analyzed HDG, and extended to various applications.
- This talk is on an HDG method for Stokes flow:
  - [Cockburn, & G., 2009] "The derivation of hybridizable discontinuous Galerkin methods for Stokes flow", SINUM.

Stokes system

$$\begin{aligned} -\boldsymbol{\Delta}\mathbf{u} + \mathbf{grad}\, p &= \mathbf{f}, && \text{on } \Omega,\\ \operatorname{div}\mathbf{u} &= 0, && \text{on } \Omega,\\ \mathbf{u} &= \mathbf{0}, && \text{on } \partial\Omega. \end{aligned}$$

Since $-\boldsymbol{\Delta}\mathbf{u} = \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \mathbf{grad}\operatorname{div}\mathbf{u}$, the Stokes equations can be rewritten using vorticity $\boldsymbol{\omega}$:

$$\begin{aligned} \boldsymbol{\omega} - \mathbf{curl}\,\mathbf{u} &= \mathbf{0}, && \text{on } \Omega,\\ \mathbf{curl}\,\boldsymbol{\omega} + \mathbf{grad}\, p &= \mathbf{f}, && \text{on } \Omega,\\ \operatorname{div}\mathbf{u} &= 0, && \text{on } \Omega. \end{aligned}$$

Velocity-vorticity formulation

$$\begin{aligned} \boldsymbol{\omega} - \mathbf{curl}\, \mathbf{u} = 0 \implies & (\boldsymbol{\omega}, \boldsymbol{\tau})_\Omega - (\mathbf{u}, \mathbf{curl}\, \boldsymbol{\tau})_\Omega &= 0 \\ \mathbf{curl}\, \omega + \mathbf{grad}\, p = \mathbf{f} \implies & (\mathbf{v}, \mathbf{curl}\, \omega)_\Omega &= (\mathbf{v}, \mathbf{f})_\Omega \\ \operatorname{div} \mathbf{u} = 0 \implies & \text{(imposed in the space).} & \end{aligned}$$

$$\begin{aligned} \boldsymbol{\omega}, \boldsymbol{\tau} \in \quad & H(\mathbf{curl}) \\ \mathbf{u}, \mathbf{v} \in \quad & \{\mathbf{v} \in H(\operatorname{div}) : \operatorname{div} \mathbf{v} = 0, \quad \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega = 0}\}. \end{aligned}$$

Known approaches:

- Use stream function [Girault & Raviart, 1986]
- Use a double hybridization [Cockburn & G., 2000]
- Use DG [Carrero, Cockburn, Schötzau, 2006]

This talk's approach: hybrid DG

$$\boldsymbol{\omega} - \mathbf{curl}\,\mathbf{u} = \mathbf{0} \quad \Longrightarrow$$

$$(\boldsymbol{\omega}_h, \boldsymbol{\tau})_K - (\mathbf{u}_h, \mathbf{curl}\,\boldsymbol{\tau})_K + \langle \widehat{\mathbf{u}}_h, \mathbf{n} \times \boldsymbol{\tau} \rangle_{\partial K} = 0,$$

$$\mathbf{curl}\,\omega + \mathbf{grad}\,p = \mathbf{f} \quad \Longrightarrow$$

$$(\boldsymbol{\omega}_h, \mathbf{curl}\,\mathbf{v})_K + \langle \widehat{\boldsymbol{\omega}}_h, \mathbf{v} \times \mathbf{n} \rangle_{\partial K} - (p_h, \mathrm{div}\,\mathbf{v})_K + \langle \widehat{p}_h, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial K} = (\mathbf{f}, \mathbf{v})_K,$$

$$\mathrm{div}\,\mathbf{u} = 0 \quad \Longrightarrow$$

$$-(\mathbf{u}_h, \mathbf{grad}\,q)_K + \langle \widehat{\mathbf{u}}_h \cdot \mathbf{n}, q \rangle_{\partial K} = 0,$$

- Numerical traces: $\widehat{\mathbf{u}}_h \times \mathbf{n}$, $\widehat{\boldsymbol{\omega}}_h \times \mathbf{n}$, $\widehat{p}_h$, $\widehat{\mathbf{u}}_h \times \mathbf{n}$.
- Element spaces: $\boldsymbol{\omega}_h, \boldsymbol{\tau} \in \mathbf{W}(K)$, $\mathbf{u}_h, \mathbf{v} \in \mathbf{V}(K)$, $p_h, q \in P(K)$.

Various DG methods are obtained by prescribing various numerical traces and element spaces.

HDG method

Q: *Are there choices of numerical traces* $\widehat{\mathbf{u}}_h \times \mathbf{n}, \quad \widehat{\boldsymbol{\omega}}_h \times \mathbf{n}, \quad \widehat{p}_h, \quad \widehat{\mathbf{u}}_h \times \mathbf{n}$ *that yield a hybridizable method?* A: (our main result) Yes!

HDG method

Q: *Are there choices of numerical traces* $\widehat{\mathbf{u}}_h \times \mathbf{n}$, $\widehat{\omega}_h \times \mathbf{n}$, $\widehat{p}_h$, $\widehat{\mathbf{u}}_h \times \mathbf{n}$ *that yield a hybridizable method?* A: (our main result) Yes!

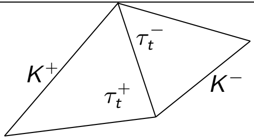
$$(\widehat{\omega}_h)_t = \left(\frac{\tau_t^- (\omega_h^+)_t + \tau_t^+ (\omega_h^-)_t}{\tau_t^- + \tau_t^+} \right) + \left(\frac{\tau_t^+ \tau_t^-}{\tau_t^- + \tau_t^+} \right) [\![\mathbf{u}_h \times \mathbf{n}]\!],$$

$$(\widehat{\mathbf{u}}_h)_t = \left(\frac{\tau_t^+ (\mathbf{u}_h^+)_t + \tau_t^- (\mathbf{u}_h^-)_t}{\tau_t^- + \tau_t^+} \right) + \left(\frac{1}{\tau_t^- + \tau_t^+} \right) [\![\mathbf{n} \times \omega_h]\!],$$

$$(\widehat{\mathbf{u}}_h)_n = \left(\frac{\tau_n^+ (\mathbf{u}_h^+)_n + \tau_n^- (\mathbf{u}_h^-)_n}{\tau_n^- + \tau_n^+} \right) + \left(\frac{1}{\tau_n^- + \tau_n^+} \right) [\![p_h \, \mathbf{n}]\!],$$

$$\widehat{p}_h = \left(\frac{\tau_n^- p_h^+ + \tau_n^+ p_h^-}{\tau_n^- + \tau_n^+} \right) + \left(\frac{\tau_n^+ \tau_n^-}{\tau_n^- + \tau_n^+} \right) [\![\mathbf{u}_h \cdot \mathbf{n}]\!],$$

- $[\![\cdots]\!] \to$ jump, $(\cdot)_t \to$ tangential, $(\cdot)_n \to$ normal
- $\tau_t, \tau_n \to$ two stabilization parameters
- $\pm$ indicate values from adjacent elements $K^{\pm}$.



Solvability

Theorem

Assume that τ_t and τ_n are positive everywhere. Assume also that

$$\mathbf{curl}\,\mathbf{V}(K) \subset \mathbf{W}(K),$$
$$\mathbf{grad}\,P(K) \subset \mathbf{V}(K),$$
$$\operatorname{div}\mathbf{V}(K) \subset P(K),$$

for every element $K \in \Omega_h$. Then there is one and only one $(\boldsymbol{\omega}_h, \mathbf{u}_h, p_h)$ satisfying the equations of the method (including the numerical trace expressions and boundary conditions).

If $\mathbf{W}(K)$, $\mathbf{V}(K)$, $P(K)$ are set to polynomials of degree d_W, d_V, d_P, resp., then for any $k \geq 1$, we may choose (d_W, d_V, d_P) to

$$\begin{array}{ccc} \begin{array}{l} (k-1,\quad k-1,\quad k), \\ (k-1,\quad k,\quad k), \end{array} & \begin{array}{l} (k,\quad k-1,\quad k), \\ (k,\quad k,\quad k), \\ (k,\quad k+1,\quad k), \end{array} & \begin{array}{l} (k+1,\quad k-1,\quad k), \\ (k+1,\quad k,\quad k), \\ (k+1,\quad k+1,\quad k). \end{array} \end{array}$$

There are 4 transmission conditions for Stokes flow:

$$[\![\boldsymbol{\omega} \times \mathbf{n}]\!] = 0, \qquad [\![\mathbf{u} \times \mathbf{n}]\!] = 0, \qquad [\![\mathbf{u} \cdot \mathbf{n}]\!] = 0, \qquad [\![p\, \mathbf{n}]\!] = 0.$$

Hybridization strategy:

- Pick two as unknowns, and find equations by the remaining two!

Type I:	Unknowns:	$\widehat{\mathbf{u}}_h \times \mathbf{n}$,	$\widehat{p}_h \mathbf{n}$
	Equations:	$[\![\widehat{\boldsymbol{\omega}}_h \times \mathbf{n}]\!] = 0$,	$[\![\widehat{\mathbf{u}}_h \cdot \mathbf{n}]\!] = 0$
Type II:			
Type III:			
Type IV:			

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Discard the remaining two choices.

Results on Type I hybridization

Type I:	Unknowns:	$\widehat{\mathbf{u}}_h \times \mathbf{n}$,	$\widehat{p}_h \mathbf{n}$
	Equations:	$[\![\widehat{\boldsymbol{\omega}}_h \times \mathbf{n}]\!] = 0$,	$[\![\widehat{\mathbf{u}}_h \cdot \mathbf{n}]\!] = 0$

Results on Type I hybridization

Type I:	Unknowns:	$\boldsymbol{\lambda}_t$	ρ
	Equations:	$[\![\widehat{\boldsymbol{\omega}}_h \times \mathbf{n}]\!] = 0,$	$[\![\widehat{\mathbf{u}}_h \cdot \mathbf{n}]\!] = 0$

- The "Equations" give a uniquely solvable condensed system

$$\begin{pmatrix} A & B^t \\ B & C \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda}_t \\ \rho \end{pmatrix} = \mathrm{r.h.s}, \qquad \text{(plus one eq. for } \mathrm{mean}(\rho)\text{)}$$

  with locally computable operators A, B, C, on appropriate piecewise polynomial spaces on element interfaces.

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  with locally computable operators A, B, C, on appropriate piecewise polynomial spaces on element interfaces.
- The solution $(\boldsymbol{\lambda}_t, \rho)$ gives the same numerical traces stated earlier.
- On any element K, all variables are recovered locally from $(\boldsymbol{\lambda}_t, \rho)$ by:

Solve for $\boldsymbol{\omega}_h$, $\mathbf{u}_h$, p_h using the HDG discretization of

$$\begin{cases} \boldsymbol{\omega} - \mathbf{curl}\, \mathbf{u} = 0 & \text{in } K, \\ \mathbf{curl}\, \boldsymbol{\omega} + \mathbf{grad}\, p = \mathbf{f} & \text{in } K, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } K, \\ (\mathbf{u})_t = \boldsymbol{\lambda}_t & \text{on } \partial K, \\ p = \rho & \text{on } \partial K. \end{cases}$$

Results on Type II hybridization

Type II:	Unknowns:	$\widehat{\mathbf{u}}_h \times \mathbf{n}$,	$\widehat{\mathbf{u}}_h \cdot \mathbf{n}$
	Equations:	$[\![\widehat{\boldsymbol{\omega}}_h \times \mathbf{n}]\!] = 0$,	$[\![\widehat{p}_h \mathbf{n}]\!] = 0$

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Type II:	Unknowns:	$\boldsymbol{\lambda}_t$	$\boldsymbol{\lambda}_n$
	Equations:	$[\![\widehat{\omega}_h \times \mathbf{n}]\!] = 0,$	$[\![\widehat{p}_h \mathbf{n}]\!] = 0$

- Local recovery from $\boldsymbol{\lambda} \equiv (\boldsymbol{\lambda}_t, \boldsymbol{\lambda}_n)$?

Solving for $\boldsymbol{\omega}_h$, $\mathbf{u}_h$, p_h using the HDG discretization of

$$\begin{cases} \omega - \mathbf{curl}\,\mathbf{u} = 0 & \text{in } K, \\ \mathbf{curl}\,\omega + \mathbf{grad}\, p = \mathbf{f} & \text{in } K, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } K, \\ (\mathbf{u})_t = \boldsymbol{\lambda}_t & \text{on } \partial K, \\ (\mathbf{u})_n = \boldsymbol{\lambda}_n & \text{on } \partial K. \end{cases}$$

is possible only if

$$\int_{\partial K} \boldsymbol{\lambda} \cdot \mathbf{n} = 0 \qquad \ldots!$$

Results on Type II hybridization

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- Revised local solver:

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after solving a condensed system for $(\boldsymbol{\lambda}, \bar{\rho})$:

$$\begin{pmatrix} A_2 & B_2^t \\ B_2 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \bar{\rho} \end{pmatrix} = \text{r.h.s}$$

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- The solution $\boldsymbol{\lambda}$ gives the same numerical traces stated earlier.

Conclusion

- There is an HDG method for velocity-vorticity formulation of Stokes flow.

- While it may appear that there are four ways to hybridize, all four ways give the same global HDG method.

- Proof of error estimates is an open question, as of now.