Polynomial approximations of certain boundary value problems

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A typical PDE approximation

The Dirichlet problem:

$$\begin{aligned} -\operatorname{div}\big(\boldsymbol{a}(\boldsymbol{x})\,\mathbf{grad}\,u\big) &= f, && \text{on } \Omega,\\ u &= 0, && \text{on } \partial\Omega. \end{aligned}$$

Weak formulation: Find $u \in H_0^1(\Omega)$ satisfying

$$(\boldsymbol{a}\ \mathbf{grad}\,u, \mathbf{grad}\,v) = (f, v), \qquad \forall v \in H_0^1(\Omega).$$

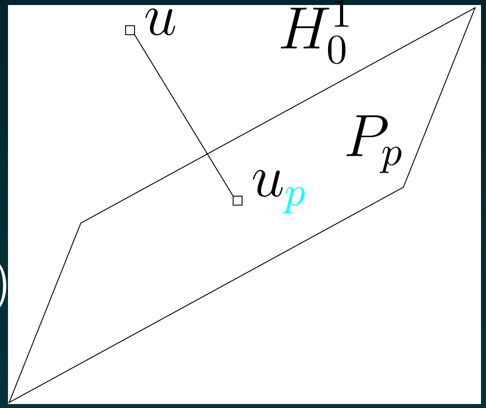
Spectral approx.: Find $u_p \in P_p \cap H_0^1(\Omega)$ satisfying

$$(\boldsymbol{a}\ \mathbf{grad}\,u_p, \mathbf{grad}\,v) = (f, v), \quad \forall v \in P_p \cap H_0^1(\Omega).$$

A typical PDE approximation

Since u_p is a projection of u, it is a quasioptimal approximation:

$$\mathcal{C}_a \|u - u_p\|_{H^1(\Omega)} \le \inf_{v \in P_p \cap H^1_0(\Omega)} \|u - v\|_{H^1(\Omega)}$$



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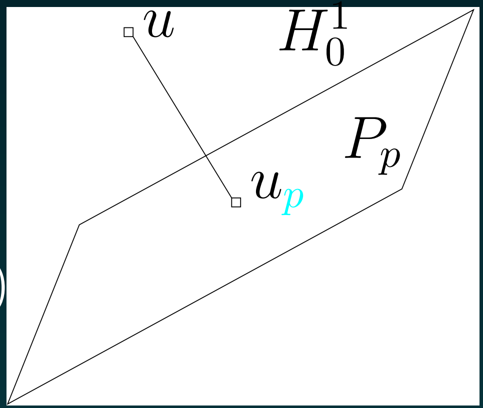

$$\mathcal{C}_a \|u - u_p\|_{H^1(\Omega)} \le \inf_{v \in P_p \cap H_0^1(\Omega)} \|u - v\|_{H^1(\Omega)}$$

Thus the error analysis of methods like the above in the variational form

$$a(u, v) = F(v)$$

with an innerproduct $a(\cdot,\cdot)$, immediately reduces to a question in approximation theory.

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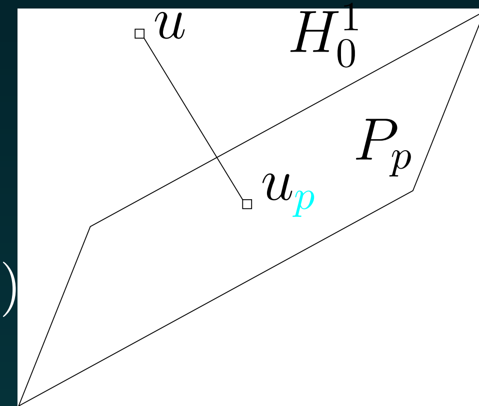

$$\mathcal{C}_a \| u - u_p \|_{H^1(\Omega)} \le \inf_{v \in P_p \cap H_0^1(\Omega)} \| u - v \|_{H^1(\Omega)}$$

What about methods in saddle point form?

$$\left.\begin{array}{l} \boldsymbol{q} \in W \\ u \in V \end{array}\right\} \qquad \begin{aligned} a(\boldsymbol{q}, \boldsymbol{r}) + b(\boldsymbol{r}, u) &= G(\boldsymbol{r}), && \forall \boldsymbol{r} \in W, \\ b(\boldsymbol{q}, v) &= F(v), && \forall v \in V. \end{aligned}$$

The spectral Raviart-Thomas method

First order reformulation of the Dirichlet problem:

$$\begin{aligned} \boldsymbol{q} + \boldsymbol{a}(\boldsymbol{x})\,\mathbf{grad}\, u &= 0, && \text{on } \Omega, \\ \operatorname{div}\, \boldsymbol{q} &= f, && \text{on } \Omega, \\ u &= g, && \text{on } \partial\Omega. \end{aligned}$$

Weak formulation: Find $\boldsymbol{q}$ and u satisfying

$$\left.\begin{matrix} \boldsymbol{q} \in \boldsymbol{H}(\operatorname{div}) \\ u \in L^2(\Omega) \end{matrix}\right\} \qquad \begin{aligned} (\boldsymbol{a}^{-1}\boldsymbol{q}\,, \boldsymbol{r}) - (u\,, \operatorname{div} \boldsymbol{r}) &= -(g, \boldsymbol{r}\cdot\boldsymbol{n})_{\partial\Omega}, \\ (v, \operatorname{div} \boldsymbol{q}\,) &= (f, v), \end{aligned}$$

for all $\boldsymbol{r} \in \boldsymbol{H}(\operatorname{div})$ and $v \in L^2(\Omega)$.

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Spectral discretization: Find $\boldsymbol{q}_p$ and u_p satisfying

$$\left.\begin{array}{l} \boldsymbol{q}_p \in \boldsymbol{R}_p \\ u_p \in P_p \end{array}\right\} \qquad \begin{aligned} (\boldsymbol{a}^{-1}\boldsymbol{q}_p, \boldsymbol{r}) - (u_p, \operatorname{div}\boldsymbol{r}) &= -(g, \boldsymbol{r}\cdot\boldsymbol{n})_{\partial\Omega},\\ (v, \operatorname{div}\boldsymbol{q}_p) &= (f, v), \end{aligned}$$

for all $\boldsymbol{r} \in \boldsymbol{R}_p \subseteq \boldsymbol{H}(\operatorname{div})$ and $v \in P_p \subseteq L^2(\Omega)$.
($\boldsymbol{R}_p \equiv \boldsymbol{x}P_p + \boldsymbol{P}_p$.)

Quasioptimality?

Is the spectral RT method quasioptimal?

In other words, does the estimate

$$\overbrace{\|\boldsymbol{q}-\boldsymbol{q}_p\|_{\boldsymbol{H}(\mathrm{div})}+\|u-u_p\|_{L^2(\Omega)}}^{\text{Error of the method}}$$

$$\le \mathcal{C}\underbrace{\left(\inf_{\boldsymbol{r}\in\boldsymbol{R}_p}\|\boldsymbol{q}-\boldsymbol{r}\|_{\boldsymbol{H}(\mathrm{div})}+\inf_{v\in P_p}\|u-v\|_{L^2(\Omega)}\right)}_{\text{Best approximation error}}$$

hold with a constant $\mathcal{C}$ *independent of the polynomial degree* p?

Apply Babuška-Brezzi theory

The Babuška-Brezzi theory gives sufficient conditions for quasioptimality of methods in the following variational form:

$$\begin{aligned} a(\boldsymbol{q},\boldsymbol{r}) + b(\boldsymbol{r},u) &= G(\boldsymbol{r}), \\ b(\boldsymbol{q},v) &= F(v). \end{aligned}$$

or

$$\begin{aligned} A\boldsymbol{q} + B^t u &= G, \\ B\boldsymbol{q} &= F. \end{aligned}$$

- Coercivity on the kernel:

$$a(\boldsymbol{r},\boldsymbol{r}) \ge \mathcal{C}_1 \|\boldsymbol{r}\|_W^2, \qquad \forall \boldsymbol{r} \in \mathrm{Ker}(B).$$

- Inf-sup condition:

$$\|v\|_V \le \mathcal{C}_2 \|B^t v\|_W, \qquad \forall v \in V.$$

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$$\|v\|_V \le \mathcal{C}_2 \sup_{\boldsymbol{r}\in W} \frac{b(\boldsymbol{r},v)}{\|\boldsymbol{r}\|_W} \qquad \forall v \in V.$$

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$$(\boldsymbol{a}^{-1}\boldsymbol{q}\ ,\boldsymbol{r}) - (u\ ,\operatorname{div}\boldsymbol{r}) = -(g,\boldsymbol{r}\cdot\boldsymbol{n})_{\partial\Omega},$$
$$(v,\operatorname{div}\boldsymbol{q}\ ) = (f,v),$$

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- Coercivity on the kernel:

  $(\boldsymbol{a}^{-1}\boldsymbol{r},\boldsymbol{r}) \ge \mathcal{C}_1\|\boldsymbol{r}\|^2_{\boldsymbol{H}(\mathrm{div})}, \quad \forall \boldsymbol{r}\in\boldsymbol{R}_p$ with $\operatorname{div}\boldsymbol{r}=0.$
- Inf-sup condition:

$$\|v\|_{L^2(\Omega)} \le \mathcal{C}_2 \sup_{\boldsymbol{r}_p\in\boldsymbol{R}_p} \frac{(v,\operatorname{div}\boldsymbol{r}_p)}{\|\boldsymbol{r}_p\|_{\boldsymbol{H}(\mathrm{div})}}, \qquad \forall v\in P_p.$$

An exact sequence property

Let Ω be star shaped with respect to some $\boldsymbol{a} \in \overline{\Omega}$. Then the following sequences are exact:

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\textbf{grad}} \boldsymbol{H}(\textbf{curl}) \xrightarrow{\textbf{curl}} \boldsymbol{H}(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2(\Omega),$$

(This is a generalization of the classical de Rham complex to Sobolev spaces.)

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Notation:

$\boldsymbol{Q}_p =$ Nédélec space $\equiv \boldsymbol{P}_p \oplus \{$ set of homogeneous polynomials $\boldsymbol{q}$ of degree $p+1$ with $\boldsymbol{q}(\boldsymbol{x}) \cdot \boldsymbol{x} = 0\}$.

$\boldsymbol{R}_p =$ Raviart-Thomas space $= \boldsymbol{x} P_p \oplus \boldsymbol{P}_p$.

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We'll construct bounded linear maps $\boldsymbol{D}, \boldsymbol{K}, G$:

$$\begin{array}{cccccccc} & H^1(\Omega)/\mathbb{R} & \xleftarrow{G} & \boldsymbol{H}(\textbf{curl}) & \xleftarrow{\boldsymbol{K}} & \boldsymbol{H}(\mathrm{div}) & \xleftarrow{\boldsymbol{D}} & L^2(\Omega), \\ \ni & P_{p+1}/\mathbb{R} & \xleftarrow{G} & \boldsymbol{Q}_p & \xleftarrow{\boldsymbol{K}} & \boldsymbol{R}_p & \xleftarrow{\boldsymbol{D}} & P_p. \end{array}$$

The Poincaré lemma

It is well known that if $\boldsymbol{a} \in \overline{\Omega}$ and $\boldsymbol{q}$ is irrotational, the line integral

$$G\boldsymbol{q}(\boldsymbol{x}) = \int_{\boldsymbol{a}}^{\boldsymbol{x}} \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{t},$$

satisfies $\mathbf{grad}(G\boldsymbol{q}) = \boldsymbol{q}$. The Poincaré lemma is a generalization:

The Poincaré lemma

Well known fact: Let $\boldsymbol{a} \in \overline{\Omega}$. If $\boldsymbol{q}$ is irrotational, the line integral

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satisfi es $\mathbf{grad}(G\boldsymbol{q}) = \boldsymbol{q}$. The Poincaré lemma is a generalization:
For smooth $\boldsymbol{v}$ and ψ, define

$$\boldsymbol{K}\boldsymbol{v}(\boldsymbol{x}) = -(\boldsymbol{x} - \boldsymbol{a}) \times \int_0^1 t\, \boldsymbol{v}(t(\boldsymbol{x} - \boldsymbol{a}) + \boldsymbol{a})\, \mathrm{d}t,$$

$$\boldsymbol{D}\psi(\boldsymbol{x}) = \quad (\boldsymbol{x} - \boldsymbol{a}) \int_0^1 t^2\, \psi(t(\boldsymbol{x} - \boldsymbol{a}) + \boldsymbol{a})\, \mathrm{d}t.$$

Then:

- $\mathrm{div}\, \boldsymbol{D}\psi = \psi$.
- $\mathbf{curl}\, \boldsymbol{K}\boldsymbol{v} = \boldsymbol{v}$, whenever $\mathrm{div}\, \boldsymbol{v} = 0$.

Right inverses of grad, div, *and* curl

THEOREM. *The maps* $\boldsymbol{D}, \boldsymbol{K}, G$ *extend continuously to*

$$H^1(\Omega)/\mathbb{R} \xleftarrow{G} \boldsymbol{H}(\mathbf{curl}) \xleftarrow{\boldsymbol{K}} \boldsymbol{H}(\mathrm{div}) \xleftarrow{\boldsymbol{D}} L^2(\Omega).$$

Moreover,

$$P_{p+1}/\mathbb{R} \xleftarrow{G} \boldsymbol{Q}_p \xleftarrow{\boldsymbol{K}} \boldsymbol{R}_p \xleftarrow{\boldsymbol{D}} P_p,$$

$$\mathrm{div}\, \boldsymbol{D}\psi = \psi, \qquad \forall \psi \in L^2(\Omega),$$

$$\mathbf{curl}\, \boldsymbol{K}\boldsymbol{v} = \boldsymbol{v}, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}) \text{ with } \mathrm{div}\, \boldsymbol{v} = 0,$$

$$\mathbf{grad}\, G\boldsymbol{q} = \boldsymbol{q}, \qquad \forall \boldsymbol{q} \in \boldsymbol{H}(\mathbf{curl}) \text{ with } \mathbf{curl}\, \boldsymbol{q} = 0.$$

(We assume that Ω is star shaped with respect to some point $\boldsymbol{a} \in \overline{\Omega}$ and $\partial\Omega$ is Lipschitz.)

Commutativity properties

A well known technique for proving the inf-sup condition is via the use of commuting projectors.

For the RT method, we need a projector $\boldsymbol{\Pi}_p^R$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\boldsymbol{H}(\mathrm{div}) & \xrightarrow{\ \mathrm{div}\ } & L^2(\Omega) \\
\Big\downarrow \boldsymbol{\Pi}_p^R & & \Big\downarrow \Pi_p \\
\boldsymbol{R}_p & \xrightarrow{\ \mathrm{div}\ } & P_p
\end{array}
$$

(Here $\Pi_p = L^2$-orthogonal projection onto P_p.)

$$\mathrm{div}\, \boldsymbol{\Pi}_p^R \boldsymbol{q} = \Pi_p \,\mathrm{div}\, \boldsymbol{q}$$

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$$

$$
\|v\|_{L^2(\Omega)} \le \mathcal{C} \sup_{\boldsymbol{r} \in \boldsymbol{H}(\mathrm{div})} \frac{(v, \mathrm{div}\, \boldsymbol{r})}{\|\boldsymbol{r}\|_{\boldsymbol{H}(\mathrm{div})}}
$$

$$
\downarrow
$$

$$
\|v_p\|_{L^2(\Omega)} \le \mathcal{C} \sup_{\boldsymbol{r}_p \in \boldsymbol{R}_p} \frac{(v_p, \mathrm{div}\, \boldsymbol{r}_p)}{\|\boldsymbol{r}_p\|_{\boldsymbol{H}(\mathrm{div})}}
$$

It allows one to use inf-sup conditions at the top level to prove inf-sup conditions at the bottom level.

A de Rham diagram

More generally, we will construct a sequence of projectors such that the following diagram commutes:

$$
\begin{array}{ccccccc}
H^1(\Omega)/\mathbb{R} & \xrightarrow{\textbf{grad}} & \boldsymbol{H}(\textbf{curl}\,) & \xrightarrow{\textbf{curl}} & \boldsymbol{H}(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\big\downarrow \Pi_p^W & & \big\downarrow \boldsymbol{\Pi}_p^Q & & \big\downarrow \boldsymbol{\Pi}_p^R & & \big\downarrow \Pi_p \\
P_{p+1}/\mathbb{R} & \xrightarrow{\textbf{grad}} & \boldsymbol{Q}_p & \xrightarrow{\textbf{curl}} & \boldsymbol{R}_p & \xrightarrow{\mathrm{div}} & P_p
\end{array}
$$

Projectors

$$\boldsymbol{\Pi}_p^R \boldsymbol{v} = \boldsymbol{\Pi}_p^{R0} \; \boldsymbol{v} + (\boldsymbol{I} - \boldsymbol{\Pi}_p^{R0}) \boldsymbol{D}(\; \Pi_p \; \operatorname{div} \boldsymbol{v})$$

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$\boldsymbol{\Pi}_p^{R0}$ is the L^2-orthogonal projection onto $\boldsymbol{R}_p^0 = \{\boldsymbol{r} \in \boldsymbol{R}_p : \operatorname{div} \boldsymbol{r} = 0\}$.

$\Pi_p = L^2$-orthogonal projection onto P_p

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Then,

$$\begin{aligned} \operatorname{div} \boldsymbol{\Pi}_p^R \boldsymbol{v} &= \operatorname{div}(\boldsymbol{I} - \boldsymbol{\Pi}_p^{R0} \boldsymbol{v}) \boldsymbol{D}(\Pi_p \operatorname{div} \boldsymbol{v}) \\ &= \operatorname{div} \boldsymbol{D}(\Pi_p \operatorname{div} \boldsymbol{v}) \\ &= \Pi_p \operatorname{div} \boldsymbol{v}. \end{aligned}$$

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$$\boldsymbol{\Pi}_p^Q \boldsymbol{q} = \boldsymbol{\Pi}_p^{Q0} \boldsymbol{q} + (\boldsymbol{I} - \boldsymbol{\Pi}_p^{Q0}) \boldsymbol{K}(\, \boldsymbol{\Pi}_p^{R0}\, \mathbf{curl}\, \boldsymbol{q})$$

Here $\boldsymbol{\Pi}_p^{Q0}$ is the L^2-orthogonal projection onto $\boldsymbol{Q}_p^0 = \{\boldsymbol{q} \in \boldsymbol{Q}_p : \mathbf{curl}\, \boldsymbol{q} = 0\}$.

Projectors

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$$\Pi_p^W w = \Pi^{W0}\ w + (\boldsymbol{I} - \Pi^{W0})G(\ \boldsymbol{\Pi}_p^{Q0}\ \mathbf{grad}\, w)$$

Properties of the projectors

THEOREM. *The following diagram commutes:*

$$\begin{array}{ccccccc}
H^1(\Omega)/\mathbb{R} & \xrightarrow{\textbf{grad}} & \boldsymbol{H}(\textbf{curl}) & \xrightarrow{\textbf{curl}} & \boldsymbol{H}(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\Big\downarrow \Pi_p^W & & \Big\downarrow \boldsymbol{\Pi}_p^Q & & \Big\downarrow \boldsymbol{\Pi}_p^R & & \Big\downarrow \Pi_p \\
P_{p+1}/\mathbb{R} & \xrightarrow{\textbf{grad}} & \boldsymbol{Q}_p & \xrightarrow{\textbf{curl}} & \boldsymbol{R}_p & \xrightarrow{\mathrm{div}} & P_p\,.
\end{array}$$

We have norm bounds independent of degree p:

$$\begin{aligned}
\|\boldsymbol{\Pi}_p^R \boldsymbol{v}\|^2_{\boldsymbol{H}(\mathrm{div})} &\le (1+\mathcal{C}_D^2)\|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathrm{div})} && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}),\\
\|\boldsymbol{\Pi}_p^Q \boldsymbol{q}\|^2_{\boldsymbol{H}(\textbf{curl})} &\le (1+\mathcal{C}_K^2)\|\boldsymbol{q}\|^2_{\boldsymbol{H}(\textbf{curl})} && \forall \boldsymbol{q} \in \boldsymbol{H}(\textbf{curl}),\\
\|\Pi_p^W w\|^2_{H^1(\Omega)} &\le (1+\mathcal{C}_G^2)\|w\|^2_{H^1(\Omega)} && \forall w \in H^1(\Omega).
\end{aligned}$$

Return to the RT method

THEOREM. *There is a positive constant $\mathcal{C}$ independent of polynomial degree p such that*

$$\|v\|_{L^2(\Omega)} \le \mathcal{C} \sup_{\boldsymbol{r}_p \in \boldsymbol{R}_p} \frac{(v, \operatorname{div} \boldsymbol{r}_p)_\Omega}{\|\boldsymbol{r}_p\|_{\boldsymbol{H}(\mathrm{div})}}, \qquad \forall v \in P_p.$$

COROLLARY. *The spectral RT method is quasioptimal.*

Other applications

Application to proving Poincaré-Friedrichs type inequalities:

Let $\boldsymbol{Q}_p^{\perp} = \{\boldsymbol{q} \in \boldsymbol{Q}_p : (\boldsymbol{q}, \mathbf{grad}\, w) = 0$ for all $w \in P_{p+1}\}$.

THEOREM. *For all* $\boldsymbol{q} \in \boldsymbol{Q}_p^{\perp}$

$$\|\boldsymbol{q}\|_{L^2(\Omega)} \le \mathcal{C} \|\mathbf{curl}\, \boldsymbol{q}\|_{L^2(\Omega)}.$$

PROOF:

$$\begin{aligned}
\|\boldsymbol{q}\|_{L^2(\Omega)} &= \inf_{w \in P_{p+1}} \|\boldsymbol{q} - \mathbf{grad}\, w\|_{L^2(\Omega)} \\
&\le \|\boldsymbol{q} - (\boldsymbol{q} - \boldsymbol{K}\, \mathbf{curl}\, \boldsymbol{q})\|_{L^2(\Omega)} \\
&\le \mathcal{C}_K \|\mathbf{curl}\, \boldsymbol{q}\|_{L^2(\Omega)}.
\end{aligned}$$

□

A magnetostatics problem

Case of zero magnetic boundary condition:

$$
\begin{aligned}
\mathbf{curl}\, \mu^{-1} \mathbf{curl}\, \boldsymbol{E} &= \boldsymbol{J} && \text{on } \Omega, \\
\operatorname{div} \boldsymbol{E} &= 0 && \text{on } \Omega, \\
\boldsymbol{n} \times \mu^{-1} \mathbf{curl}\, \boldsymbol{E} &= 0 && \text{on } \partial\Omega.
\end{aligned}
$$

Case of zero electric boundary condition:

$$
\begin{aligned}
\mathbf{curl}\, \mu^{-1} \mathbf{curl}\, \boldsymbol{E} &= \boldsymbol{J} && \text{on } \Omega, \\
\operatorname{div} \boldsymbol{E} &= 0 && \text{on } \Omega, \\
\boldsymbol{n} \times \boldsymbol{E} &= 0 && \text{on } \partial\Omega.
\end{aligned}
$$

A magnetostatics problem

Case of zero magnetic boundary condition:

$$\begin{aligned} \mathbf{curl}\,\mu^{-1}\mathbf{curl}\,\boldsymbol{E} &= \boldsymbol{J} && \text{on } \Omega, \\ \operatorname{div} \boldsymbol{E} &= 0 && \text{on } \Omega, \\ \boldsymbol{n} \times \mu^{-1}\mathbf{curl}\,\boldsymbol{E} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Spectral method: Find $(\boldsymbol{E}_p, \psi_p) \in \boldsymbol{Q}_p \times P_{p+1}/\mathbb{R} \quad \ni$

$$\begin{aligned} (\mu^{-1}\mathbf{curl}\,\boldsymbol{E}_p, \mathbf{curl}\,\boldsymbol{q}_p) - (\mathbf{grad}\,\psi_p, \boldsymbol{q}_p) &= (\boldsymbol{J}, \boldsymbol{q}_p), \\ (\mathbf{grad}\,w_p, \boldsymbol{E}_p) &= 0, \end{aligned}$$

for all $\boldsymbol{q}_p \in \boldsymbol{Q}_p$ and $w_p \in P_{p+1}/\mathbb{R}$.

Quasioptimality

- Inf-sup condition:

$$\|w_p\|_{H^1(\Omega)} \le \mathcal{C} \sup_{\boldsymbol{q}_p \in \boldsymbol{Q}_p} \frac{(\mathbf{grad}\, w_p, \boldsymbol{q}_p)}{\|\boldsymbol{q}_p\|_{\boldsymbol{H}(\mathbf{curl}\,)}}, \qquad \forall w_p \in P_{p+1}/\mathbb{R}.$$

  This follows from the imbedding $P_{p+1} \xrightarrow{\mathbf{grad}} \boldsymbol{Q}_p$.

- Coercivity on the kernel:

$$\|\boldsymbol{q}\|_{L^2(\Omega)} \le \mathcal{C}\|\mathbf{curl}\,\boldsymbol{q}\|_{L^2(\Omega)}, \qquad \forall \boldsymbol{q} \in \boldsymbol{Q}_p^{\perp}.$$

  This follows from Poincaré-Friedrichs estimate we proved earlier.

Hence Babuška-Brezzi theory $\Longrightarrow$ quasioptimality.

Case of electric boundary conditions

Spectral method: Find $(\boldsymbol{E}_p, \psi_p) \in \mathring{\boldsymbol{Q}}_p \times \mathring{P}_{p+1}$ $(p \geq 3)$:

$$(\mu^{-1}\mathbf{curl}\,\boldsymbol{E}_p, \mathbf{curl}\,\boldsymbol{q}_p) - (\mathbf{grad}\,\psi_p, \boldsymbol{q}_p) = (\boldsymbol{J}, \boldsymbol{q}_p), \;\forall \boldsymbol{q}_p \in \mathring{\boldsymbol{Q}}_p$$
$$(\mathbf{grad}\,w_p, \boldsymbol{E}_p) = 0, \qquad \forall w_p \in \mathring{P}_{p+1}$$

where $\mathring{\boldsymbol{Q}}_p = \boldsymbol{Q}_p \cap \boldsymbol{H}_0(\mathbf{curl})$ and $\mathring{P}_{p+1} = P_{p+1} \cap H^1_0(\Omega)$.

- Inf-sup condition is trivial again.
- But we need to show that

$$\|\mathring{\boldsymbol{q}}\|_{L^2(\Omega)} \leq \mathcal{C}\|\mathbf{curl}\,\mathring{\boldsymbol{q}}\|_{L^2(\Omega)}$$

$\forall \mathring{\boldsymbol{q}} \in \mathring{\boldsymbol{Q}}_p^{\perp} = \{\boldsymbol{r} \in \mathring{\boldsymbol{Q}}_p : (\boldsymbol{r}, \mathbf{grad}\,w) = 0$ for all $w \in \mathring{P}_{p+1}\}$.

Preserving boundary conditions

We need new right inverses and projectors.

Exact sequences with zero boundary conditions:

$$H_0^1(\Omega) \xrightarrow{\textbf{grad}} \boldsymbol{H}_0(\textbf{curl}) \xrightarrow{\textbf{curl}} \boldsymbol{H}_0(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2(\Omega)/\mathbb{R}$$

$$\mathring{P}_{p+1} \xrightarrow{\textbf{grad}} \mathring{\boldsymbol{Q}}_p \xrightarrow{\textbf{curl}} \mathring{\boldsymbol{R}}_p \xrightarrow{\mathrm{div}} P_p/\mathbb{R}$$

Preserving boundary conditions

We need new right inverses and projectors.

Q1: Are there projectors satisfying

$$
\begin{array}{ccccccc}
H^1_0(\Omega) & \xrightarrow{\textbf{grad}} & \boldsymbol{H}_0(\textbf{curl}) & \xrightarrow{\textbf{curl}} & \boldsymbol{H}_0(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2(\Omega)/\mathbb{R} \\
\downarrow \mathring{\Pi}^W_p & & \downarrow \mathring{\boldsymbol{\Pi}}^Q_p & & \downarrow \mathring{\boldsymbol{\Pi}}^R_p & & \downarrow \Pi_p \\
\mathring{P}_{p+1} & \xrightarrow{\textbf{grad}} & \mathring{\boldsymbol{Q}}_p & \xrightarrow{\textbf{curl}} & \mathring{\boldsymbol{R}}_p & \xrightarrow{\mathrm{div}} & P_p/\mathbb{R}\ ?
\end{array}
$$

Preserving boundary conditions

We need new right inverses and projectors.

Q2: Are there right inverses

$$H_0^1(\Omega) \xleftarrow{\mathring{G}} \boldsymbol{H}_0(\mathbf{curl}) \xleftarrow{\mathring{\boldsymbol{K}}} \boldsymbol{H}_0(\mathrm{div}) \xleftarrow{\mathring{\boldsymbol{D}}} L^2(\Omega)/\mathbb{R}$$

$$\mathring{P}_{p+1} \xleftarrow{\mathring{G}} \mathring{\boldsymbol{Q}}_p \xleftarrow{\mathring{\boldsymbol{K}}} \mathring{\boldsymbol{R}}_p \xleftarrow{\mathring{\boldsymbol{D}}} P_p/\mathbb{R} \;?$$

(We need $\mathring{\boldsymbol{K}}$ to analyze the spectral method with electric boundary conditions.)

Right inverse with zero b.c.

THEOREM. *Let Ω be a tetrahedron. Then there exists an operator $\mathring{\boldsymbol{K}}$ on*

$$\boldsymbol{H}_0(\operatorname{div} 0, \Omega) \equiv \{\boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{div}) : \operatorname{div} \boldsymbol{v} = 0\}$$

with the following properties:

- $\mathbf{curl}\, \mathring{\boldsymbol{K}} \boldsymbol{v} = \boldsymbol{v}$, *for all* $\boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{div} 0, \Omega)$.
- $\boldsymbol{n} \times \mathring{\boldsymbol{K}} \boldsymbol{v} = 0$, *on* $\partial\Omega$ *for all* $\boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{div} 0, \Omega)$.
- $\|\mathring{\boldsymbol{K}} \boldsymbol{v}\|_{L^2(\Omega)} \le \mathcal{C} \|\boldsymbol{v}\|_{L^2(\Omega)}, \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{div} 0, \Omega)$.
- *Whenever* $\boldsymbol{v}$ *is in* $\mathring{\boldsymbol{R}}_p$, *the function* $\mathring{\boldsymbol{K}} \boldsymbol{v}$ *is in* $\mathring{\boldsymbol{Q}}_p$.

Right inverse with zero b.c.

Proof proceeds by finding a map Φ such that

$$\mathring{\boldsymbol{K}}\boldsymbol{v} = \boldsymbol{K}\boldsymbol{v} - \mathbf{grad}\,\Phi(\boldsymbol{v})$$

satisfies the required properties.
Ingredients in the proof:

- A p-optimal extension operator [Muñoz–Sola, 1997]

$$H^{1/2}(\partial\Omega) \xrightarrow{\mathcal{E}} H^1(\Omega).$$

- Hodge decomposition on $\partial\Omega$ [Buffa & Ciarlet, 2002]

$$0 \to H^{1/2}(\partial\Omega)/\mathbb{R} \xrightarrow{\mathbf{grad}_\tau} \boldsymbol{H}_\perp^{-1/2}(\partial\Omega) \xrightarrow{\mathbf{curl}_\tau} H^{-3/2}(\partial\Omega) \to 0$$

A quasioptimality result

Existence of $\mathring{\boldsymbol{K}}$ implies that

$$\|\mathring{\boldsymbol{q}}\|_{L^2(\Omega)} \le \mathcal{C}\|\mathbf{curl}\,\mathring{\boldsymbol{q}}\|_{L^2(\Omega)}$$

$\forall \mathring{\boldsymbol{q}} \in \mathring{\boldsymbol{Q}}_p^{\perp} = \{\boldsymbol{r} \in \mathring{\boldsymbol{Q}}_p : (\boldsymbol{r}, \mathbf{grad}\, w) = 0$ for all $w \in \mathring{P}_{p+1}\}$.

Hence quasioptimality of the spectral method with electric boundary condition follows on tetrahedra.

Conclusion

1. We began with these exact sequences:

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\textbf{grad}} \boldsymbol{H}(\textbf{curl}\,) \xrightarrow{\textbf{curl}} \boldsymbol{H}(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2(\Omega)$$

$$P_{p+1}/\mathbb{R} \xrightarrow{\textbf{grad}} \boldsymbol{Q}_p \xrightarrow{\textbf{curl}} \boldsymbol{R}_p \xrightarrow{\mathrm{div}} P_p$$

Conclusion

2. We gave right inverses of $\mathbf{grad}$, div, and $\mathbf{curl}$.

$$H^1(\Omega)/\mathbb{R} \xleftarrow{\;G\;} \boldsymbol{H}(\mathbf{curl}) \xleftarrow{\;\boldsymbol{K}\;} \boldsymbol{H}(\mathrm{div}) \xleftarrow{\;\boldsymbol{D}\;} L^2(\Omega)$$

$$P_{p+1}/\mathbb{R} \xleftarrow{\;G\;} \boldsymbol{Q}_p \xleftarrow{\;\boldsymbol{K}\;} \boldsymbol{R}_p \xleftarrow{\;\boldsymbol{D}\;} P_p$$

Conclusion

3. We constructed commuting projectors.

$$
\begin{array}{ccccccc}
H^1(\Omega)/\mathbb{R} & \xrightarrow{\textbf{grad}} & \boldsymbol{H}(\textbf{curl}) & \xrightarrow{\textbf{curl}} & \boldsymbol{H}(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\downarrow \mathit{\Pi}_p^W & & \downarrow \boldsymbol{\mathit{\Pi}}_p^Q & & \downarrow \boldsymbol{\mathit{\Pi}}_p^R & & \downarrow \mathit{\Pi}_p \\
P_{p+1}/\mathbb{R} & \xrightarrow{\textbf{grad}} & \boldsymbol{Q}_p & \xrightarrow{\textbf{curl}} & \boldsymbol{R}_p & \xrightarrow{\mathrm{div}} & P_p
\end{array}
$$

We showed how to apply these constructions to various spectral methods.

Conclusion

4. We tried to extend the results to zero bc.

$$H_0^1(\Omega) \xrightarrow{\textbf{grad}} \boldsymbol{H}_0(\textbf{curl}) \xrightarrow{\textbf{curl}} \boldsymbol{H}_0(\text{div}) \xrightarrow{\text{div}} L^2(\Omega)/\mathbb{R}$$

$$\mathring{P}_{p+1} \xrightarrow{\textbf{grad}} \mathring{\boldsymbol{Q}}_p \xrightarrow{\textbf{curl}} \mathring{\boldsymbol{R}}_p \xrightarrow{\text{div}} P_p/\mathbb{R}$$

Conclusion

5. One right inverse is missing in the zero bc case.

$$
\begin{array}{ccccccc}
H_0^1(\Omega) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}_0(\mathbf{curl}) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}_0(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2(\Omega)/\mathbb{R} \\
\Big\downarrow \mathring{\Pi}_p^W & & \Big\downarrow \mathring{\boldsymbol{\Pi}}_p^Q & & \Big\downarrow ? & & \Big\downarrow \Pi_p \\
\mathring{P}_{p+1} & \xrightarrow{\mathbf{grad}} & \mathring{\boldsymbol{Q}}_p & \xrightarrow{\mathbf{curl}} & \mathring{\boldsymbol{R}}_p & \xrightarrow{\mathrm{div}} & P_p/\mathbb{R}
\end{array}
$$

6. Further questions: How can one modify and use such projectors to prove hp-optimality of hp-mixed methods?