Crafting projections to analyze HDG methods

Jay Gopalakrishnan

University of Florida

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Reference

- [Cockburn, G & F.-J. Sayas, 2010]: "A projection-based error analysis of HDG methods", Math. Comp.

What is HDG?

Once upon a time, a *DG Method* dreamed of being a *Mixed Method*...
and ... vice versa ... and so the *HDG method* was born.

- "HDG" methods = Hybridizable Discontinuous Galerkin methods
- HDG methods were *discovered* in [Cockburn, G., Lazarov, '09] ("Unified hybridization of DG, mixed, and CG methods ...", SINUM).
- Many authors extended HDG to various *applications* (convection-diffusion, fluid flow, elasticity, etc.) in a short time span.
- Many authors *analyzed* HDG method and proved optimal estimates.
- Purpose of this talk: Present a *new technique* of analysis, in the spirit of (and hopefully as elegant as) mixed methods.

Why HDG?

- HDG methods have the same structural elegance as mixed methods.
- They yield matrices of the same *size* and *sparsity* as mixed methods (finally overcoming the criticism that "all DG methods have too many unknowns").
- Stability is guaranteed for *any* positive stabilization parameter. (It does *not* have to be "sufficiently large".)
- Mixed methods require carefully crafted spaces for stability, while HDG methods offer much greater *flexibility* in the choice of spaces.
- Unlike most older DG methods, HDG methods yield (provably) *optimal* error estimates for flux (and other unknowns).
- *Coupling* methods, even across non-matching mesh interfaces, is easy.

Dual hybrid methods

Dual DG methods (like Mixed Methods) for the Dirichlet problem

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega \subset \mathbb{R}^2 \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

are based on its first order reformulation: Find simultaneously the solution u and its "flux" q satisfying

$$\begin{aligned} q + \vec{\nabla} u &= 0 && \text{on } \Omega \\ \nabla \cdot q &= f && \text{on } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Derivation of DG methods

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$\vec{q} + \vec{\nabla} u = 0 \implies$ [Arnold, Brezzi, Cockburn & Marini, '01]

$$\int_K \vec{q} \cdot \vec{v} - \int_K u \ \nabla \cdot \vec{v} + \int_{\partial K} u \ (\vec{v} \cdot \vec{n}) = 0$$

$\nabla \cdot \vec{q} = f \implies$

Derivation of DG methods

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$$\int_K \vec{q}_h \cdot \vec{v} - \int_K u_h \, \nabla \cdot \vec{v} + \int_{\partial K \setminus \partial\Omega} \hat{u}_h (\vec{v} \cdot \vec{n}) = 0$$

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$\nabla \cdot \vec{q} = f \implies$

$$-\int_K \vec{\nabla} w \cdot \vec{q} \; + \int_{\partial K} w \, \vec{q} \; \cdot \vec{n} = \int_K f \, w$$

Derivation of DG methods

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Traditionally: Various DG methods are obtained by setting various expressions for the *numerical traces* $\hat{u}_h$ and $\widehat{q}_h$.

Derivation of DG methods

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Traditionally: Various DG methods are obtained by setting various expressions for the *numerical traces* $\hat{u}_h$ and $\widehat{q}_h$.

HDG methods: are obtained by letting $\hat{u}_h$ be an unknown, to be determined by adding the conservativity condition

Jump of $\widehat{q} \cdot \vec{n}$ across element interfaces $\equiv$ $[\![\widehat{q} \cdot \vec{n}]\!] = 0$.

HDG doesn't fit into the unified theory of [Arnold, Brezzi, Cockburn & Marini].

A popular HDG method

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Numerical flux: $\widehat{q}_h = \vec{q}_h + \tau(u_h - \hat{u}_h),$

(stabilization parameter $\equiv \tau > 0$).

Flux space: $\vec{q}_h|_K \in \vec{P}_k(K),$ $\forall$ mesh elements K.

Solution space: $u_h|_K \in P_k(K),$ $\forall$ mesh elements K.

Numerical trace space: $\hat{u}_h|_E \in P_k(E),$ $\forall$ mesh edges/faces E.

Equations:

$$\begin{cases} (\vec{q}_h, \vec{v})_K - (u_h, \nabla \cdot \vec{v})_K + \langle \hat{u}_h, \vec{v} \cdot \vec{n} \rangle_{\partial K} = 0, & \forall K, \\ -(\vec{q}_h, \vec{\nabla} w)_K + \langle \widehat{q}_h \cdot \vec{n}, w \rangle_{\partial K} = (f, w)_K, & \forall K, \\ [\![\widehat{q}_h \cdot \vec{n}]\!] = 0. & \end{cases}$$

Theorem (Condensed system, Cockburn, G & Lazarov, '09)

The unknown numerical trace $\hat{u}_h$ can be found by solving a sparse symmetric positive definite system. The other solution components $\vec{q}_h$ and u_h can then be locally recovered from $\hat{u}_h$.

Compare with mixed method

The HDG method

$$
\begin{aligned}
(\vec{q}_h, \vec{v})_K - (u_h, \nabla \cdot \vec{v})_K + \langle \hat{u}_h, \vec{v} \cdot \vec{n} \rangle_{\partial K} &= 0 \\
-(\vec{q}_h, \vec{\nabla} w)_K + \langle \widehat{q}_h \cdot \vec{n}, w \rangle_{\partial K} &= (f, w)_K \\
[\![\widehat{q}_h \cdot \vec{n}]\!] &= 0
\end{aligned}
$$

Spaces: $\vec{q}_h|_K \in \vec{P}_k(K)$, $\quad u_h|_K \in P_k(K)$, $\quad \hat{u}_h|_E \in P_k(E)$

The Raviart-Thomas mixed method in hybridized form

$$
\begin{aligned}
(\vec{q}_h, \vec{v})_K - (u_h, \nabla \cdot \vec{v})_K + \langle \lambda_h, \vec{v} \cdot \vec{n} \rangle_{\partial K} &= 0 \\
(\nabla \cdot \vec{q}_h, w)_K &= (f, w)_K \\
[\![\vec{q}_h \cdot \vec{n}]\!] &= 0
\end{aligned}
$$

Spaces: $\vec{q}_h|_K \in \vec{P}_k(K) + \vec{x} P_k(K)$, $\quad u_h|_K \in P_k(K)$, $\quad \lambda_h|_E \in P_k(E)$

Thus, the HDG method may be thought of as resulting from an attempt to stabilize the mixed method with unstable spaces.

Compare with mixed method

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-(\vec{q}_h, \vec{\nabla} w)_K + \langle \widehat{q}_h\cdot\vec{n}, w\rangle_{\partial K} &= (f, w)_K \\
[\![\vec{q}_h\cdot\vec{n} + \tau(u_h - \hat{u}_h)]\!] &= 0
\end{aligned}
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(\nabla\cdot\vec{q}_h, w) + \langle \tau(u_h - \hat{u}_h), w\rangle_{\partial K} &= (f, w)_K\\
[\![\vec{q}_h\cdot\vec{n} + \tau(u_h - \hat{u}_h)]\!] &= 0
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Thus, the HDG method may be thought of as resulting from an attempt to stabilize the mixed method with unstable spaces.

Projections in error analysis

Method for analyzing Raviart-Thomas mixed method

A flux projection $\mathit{\Pi}$ that commutes with the L^2-orthogonal projection P

$$\nabla \cdot \mathit{\Pi} \vec{q} = P \, \nabla \cdot \vec{q}$$

gives us an analogue of Galerkin orthogonality for mixed methods:

$$(\vec{q} - \vec{q}_h, \mathit{\Pi} \vec{q} - \vec{q}_h) = 0 \qquad \Longrightarrow \text{ simple analysis.}$$

Can we mimic this for HDG methods?

Is there a projection $\mathit{\Pi}$ into the HDG flux space satisfying

$$\nabla \cdot \mathit{\Pi} \vec{q} = P \, \nabla \cdot \vec{q} \;?$$

Perhaps

$$\nabla \cdot \mathit{\Pi} \vec{q} = P \, \nabla \cdot \vec{q} + \cdots ?$$

The HDG projection

The Raviart-Thomas projection: $\mathit{\Pi}_h^{RT}(\vec{q})$

$\mathit{\Pi}_h^{RT}\vec{q} \in \vec{P}_k(K) + \vec{x}P_k(K)$ satisfies

$$(\mathit{\Pi}_h^{RT}\vec{q}, \vec{v})_K = (\vec{q}, \vec{v})_K \qquad \text{for all } \vec{v} \in \vec{P}_{k-1}(K),$$
$$\langle \mathit{\Pi}_h^{RT}\vec{q}\cdot\vec{n}, \mu\rangle_F = \langle \vec{q}\cdot\vec{n}, \mu\rangle_F \qquad \text{for all } \mu \in P_k(F).$$

Key ideas to extend this to the HDG method:

- Couple both $\vec{q}$ and u into a projection. This gives enough degrees of freedom.
- Use the form of the numerical flux (with τ) in projector's definition. This simplifies error analysis.

The HDG projection

The Raviart-Thomas projection: $\Pi_h^{RT}(\vec{q})$

$\Pi_h^{RT}\vec{q} \in \vec{P}_k(K) + \vec{x}P_k(K)$ satisfies

$$(\Pi_h^{RT}\vec{q}, \vec{v})_K = (\vec{q}, \vec{v})_K \qquad \text{for all } \vec{v} \in \vec{P}_{k-1}(K),$$
$$\langle \Pi_h^{RT}\vec{q}\cdot\vec{n}, \mu\rangle_F = \langle \vec{q}\cdot\vec{n}, \mu\rangle_F \qquad \text{for all } \mu \in P_k(F).$$

The new HDG projection: $\Pi_h(\vec{q}, u)$

The (flux) q-component of $\Pi_h(\vec{q}, u)$ is $\Pi_h^q\vec{q}$. It depends on both $\vec{q}$ and u!
The (scalar) u-component of $\Pi_h(\vec{q}, u)$ is $\Pi_h^u u$. They satisfy:

$$(\Pi_h^q\vec{q}, \vec{v})_K = (\vec{q}, \vec{v})_K \qquad \text{for all } \vec{v} \in \vec{P}_{k-1}(K),$$
$$(\Pi_h^u u, w)_K = (u, w)_K \qquad \text{for all } w \in P_{k-1}(K),$$
$$\langle \Pi_h^q\vec{q}\cdot\vec{n} + \tau\Pi_h^u u, \mu\rangle_F = \langle \vec{q}\cdot\vec{n} + \tau u, \mu\rangle_F \qquad \text{for all } \mu \in P_k(F).$$

Commutativity

Commutativity property of the Raviart-Thomas projection

For all $w \in P_k(K)$,

$$(w, \nabla \cdot \vec{q})_K = (w, \nabla \cdot \Pi_h^{RT} \vec{q})_K.$$

Lemma (Weak commutativity property for the HDG projection)

For all $w \in P_k(K)$,

$$(w, \nabla \cdot \vec{q})_K = (w, \nabla \cdot \Pi_h^{q} \vec{q})_K + \langle w, \tau(\Pi_h^u u - u)\rangle_{\partial K}.$$

Approximation

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Suppose $\tau|_{\partial K}$ is nonnegative and $\tau_K^{\max} := \max \tau|_{\partial K} > 0$. Let F^* be a face of K at which the maximum of τ is attained. Put $\tau_K^* := \max \tau|_{\partial K \setminus F^*}$.

Theorem (Dependence on approximation on τ and h)

Let $k \geq 0$ and $s_u, s_q \in (1/2, k+1]$. There is a constant C independent of element diameter h_K <u>and stabilization parameter τ</u> such that

$$\|\Pi_h^q \vec{q} - \vec{q}\,\|_K \leq C\, h_K^{s_q}\, |\vec{q}|_{H^{s_q}(K)} + C\, h_K^{s_u}\, \tau_K^*\, |u|_{H^{s_u}(K)}$$

$$\|\Pi_h^u u - u\|_K \leq C\, h_K^{s_u}\, |u|_{H^{s_u}(K)} + C\, \frac{h_K^{s_q}}{\tau_K^{\max}}\, |\vec{q}|_{H^{s_q}(K)}.$$

HDG flux error estimates

Unlike many DG methods, HDG methods have optimally convergent fluxes:

Theorem (Flux error estimate)

For any $k \geq 0$,

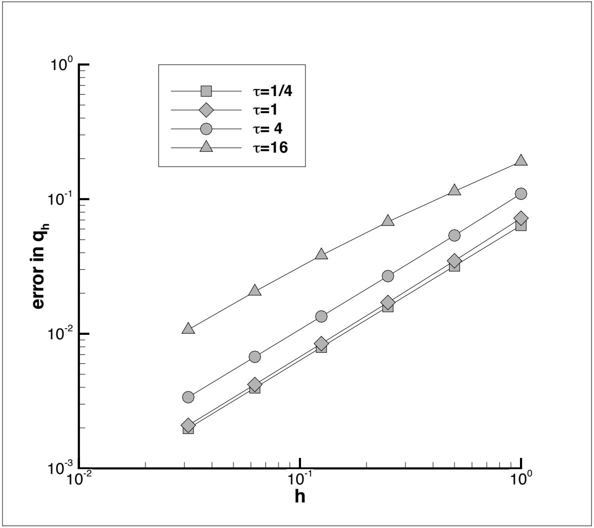
$$\|\Pi_h^q \vec{q} - \vec{q}_h\| \leq \|\Pi_h^q \vec{q} - \vec{q}\|.$$

- Thus, combining with the approximation property of the projection,

$$\|\vec{q} - \vec{q}_h\| \leq C\, h^{k+1} \left[|\vec{q}|_{H^{k+1}} + \max_K(\tau_K^*)\, |u|_{H^{k+1}} \right].$$

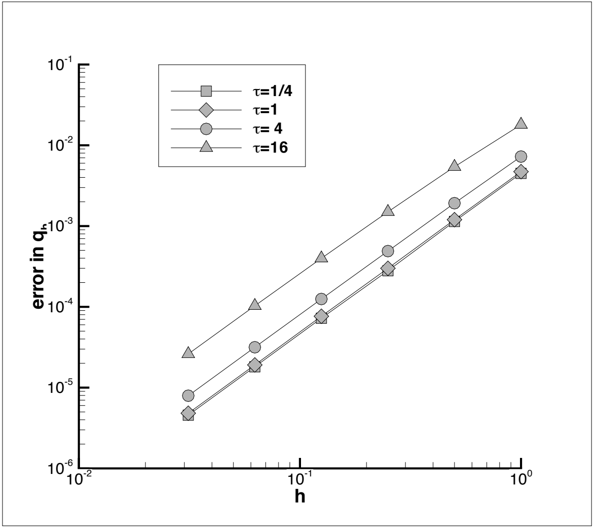
- If τ is such that it is nonzero only on one edge of every mesh triangle, then $\tau_K^* = 0$ and flux error is independent of τ.

Numerical convergence of flux



Degree $k = 0$ case: 1st order convergence observed.

Numerical convergence of flux



Degree $k = 1$ case: 2nd order convergence observed.

Convergence of u

Theorem (Optimal convergence of u)

For any $k \geq 0$,

$$\|u - u_h\| \leq C\|u - \Pi_h^u u\| + b_\tau C\|\vec{q} - \Pi_h^q \vec{q}\,\|,$$

where $b_\tau = \max\{1 + h_K \tau_K^* + h_K/\tau_K^{\max} : K \in \mathcal{T}_h\}$.

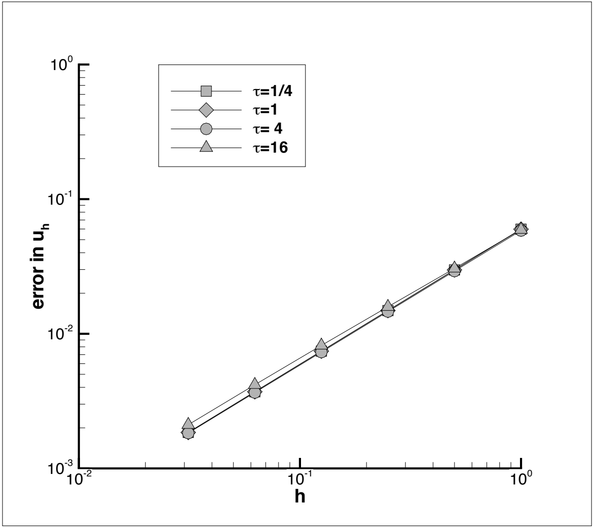
Theorem (Superconvergence of projected error by duality)

Under the full regularity assumption

$$\|\Pi_h^u u - u_h\| \leq c_\tau C\, h^{\min\{k,1\}}\, \|\Pi_h^q \vec{q} - \vec{q}\,\| \qquad \text{for } k \geq 0,$$

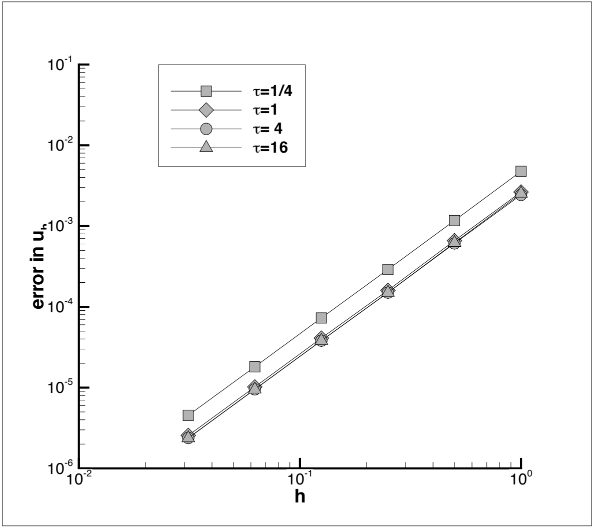
where $h = \max\{h_K : K \in \mathcal{T}_h\}$ *and* $c_\tau = \max\{1, h_K \tau_K^* : K \in \mathcal{T}_h\}$.

Numerical convergence of u



Degree $k = 0$ case: 1st order convergence observed.

Numerical convergence of u



Degree $k = 1$ case: 2nd order convergence observed.

Conclusion

- There is a weakly commuting projector that renders the analysis of HDG methods simple and concise.
- The local approximation properties of the projector can be precisely characterized in terms of h and τ.
- The global HDG errors and their τ-dependence can be understood using the local properties of the projector.
- All variables converge at optimal order when τ is of unit size.
- Standard postprocessing techniques can be applied to obtain enhanced accuracy (using the superconvergence of projected error).
- Similar projectors can be constructed to analyze HDG methods for Stokes flow.