Crafting projections to analyze HDG methods

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Reference

What is HDG?

Once upon a time, a DG Method dreamed of being a Mixed Method... and... vice versa... and so the HDG method was born.

- “HDG” methods = Hybridizable Discontinuous Galerkin methods
- HDG methods were discovered in [Cockburn, G., Lazarov, '09] (“Unified hybridization of DG, mixed, and CG methods...”, SINUM).
- Many authors extended HDG to various applications (convection-diffusion, fluid flow, elasticity, etc.) in a short time span.
- Many authors analyzed HDG method and proved optimal estimates.
- Purpose of this talk: Present a new technique of analysis, in the spirit of (and hopefully as elegant as) mixed methods.
Why HDG?

- HDG methods have the same structural elegance as mixed methods.
- They yield matrices of the same size and sparsity as mixed methods (finally overcoming the criticism that “all DG methods have too many unknowns”).
- Stability is guaranteed for any positive stabilization parameter. (It does not have to be “sufficiently large”.)
- Mixed methods require carefully crafted spaces for stability, while HDG methods offer much greater flexibility in the choice of spaces.
- Unlike most older DG methods, HDG methods yield (provably) optimal error estimates for flux (and other unknowns).
- Coupling methods, even across non-matching mesh interfaces, is easy.
Dual hybrid methods

Dual DG methods (like Mixed Methods) for the Dirichlet problem

\[-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^2\]
\[u = 0 \quad \text{on } \partial \Omega,\]

are based on its first order reformulation: Find simultaneously the solution \(u\) and its “flux” \(q\) satisfying

\[q + \nabla u = 0 \quad \text{on } \Omega\]
\[\nabla \cdot q = f \quad \text{on } \Omega\]
\[u = 0 \quad \text{on } \partial \Omega.\]
Derivation of DG methods

\[ \vec{q} + \vec{\nabla} u = 0 \quad \Rightarrow \quad \int_{K} \vec{q} \cdot \vec{v} - \int_{K} u \ \nabla \cdot \vec{v} + \int_{\partial K} u \ (\vec{v} \cdot \vec{n}) = 0 \]

\[ \nabla \cdot \vec{q} = f \quad \Rightarrow \]

[Arnold, Brezzi, Cockburn & Marini, ’01]
Derivation of DG methods

\[ \vec{q} + \vec{\nabla} u = 0 \quad \Rightarrow \]

\[ \int_K \vec{q}_h \cdot \vec{v} - \int_K u_h \nabla \cdot \vec{v} + \int_{\partial K \setminus \partial \Omega} \hat{u}_h (\vec{v} \cdot \vec{n}) = 0 \]

\[ \nabla \cdot \vec{q} = f \quad \Rightarrow \]
Derivation of DG methods

\[ \vec{q} + \vec{\nabla} u = 0 \implies [Arnold, Brezzi, Cockburn & Marini, '01] \]

\[ \int_K \vec{q}_h \cdot \vec{v} - \int_K u_h \nabla \cdot \vec{v} + \int_{\partial K \setminus \partial \Omega} \hat{u}_h (\vec{v} \cdot \vec{n}) = 0 \]

\[ \nabla \cdot \vec{q} = f \implies \]

\[ - \int_K \vec{\nabla} w \cdot \vec{q} + \int_{\partial K} w \vec{q} \cdot \vec{n} = \int_K f w \]
Derivation of DG methods

\[
\tilde{\mathbf{q}} + \nabla \mathbf{u} = 0 \quad \Rightarrow \quad \int_K \tilde{\mathbf{q}}_h \cdot \mathbf{v} - \int_K \mathbf{u}_h \nabla \cdot \mathbf{v} + \int_{\partial K \setminus \partial \Omega} \hat{\mathbf{u}}_h (\mathbf{v} \cdot \mathbf{n}) = 0
\]

\[
\nabla \cdot \tilde{\mathbf{q}} = f \quad \Rightarrow \quad -\int_K \nabla \mathbf{w} \cdot \tilde{\mathbf{q}}_h + \int_{\partial K} \mathbf{w} \hat{\mathbf{q}}_h \cdot \mathbf{n} = \int_K f \mathbf{w}
\]

*Traditionally:* Various DG methods are obtained by setting various expressions for the *numerical traces* \(\hat{\mathbf{u}}_h\) and \(\hat{\mathbf{q}}_h\).
Derivation of DG methods

\[ \vec{q} + \vec{\nabla} u = 0 \implies \]
\[ \int_{K} \vec{q}_h \cdot \vec{v} - \int_{K} u_h \nabla \cdot \vec{v} + \int_{\partial K \setminus \partial \Omega} \hat{u}_h (\vec{v} \cdot \vec{n}) = 0 \]

\[ \nabla \cdot \vec{q} = f \implies \]
\[ -\int_{K} \vec{\nabla} w \cdot \vec{q}_h + \int_{\partial K} w \hat{q}_h \cdot \vec{n} = \int_{K} f w \]

Traditionally: Various DG methods are obtained by setting various expressions for the numerical traces \( \hat{u}_h \) and \( \hat{q}_h \).

**HDG methods**: are obtained by letting \( \hat{u}_h \) be an unknown, to be determined by adding the conservativity condition

Jump of \( \hat{q} \cdot \vec{n} \) across element interfaces \( \equiv [\hat{q} \cdot \vec{n}] = 0 \).

HDG doesn’t fit into the unified theory of \([Arnold, Brezzi, Cockburn & Marini]\).
A popular HDG method

Numerical flux: \[ \hat{q}_h = \bar{q}_h + \tau (u_h - \hat{u}_h), \]
(stabilization parameter \( \equiv \tau > 0 \)).

Flux space: \( \bar{q}_h |_K \in \bar{P}_k(K) \), \( \forall \) mesh elements \( K \).
Solution space: \( u_h |_K \in P_k(K) \), \( \forall \) mesh elements \( K \).
Numerical trace space: \( \hat{u}_h |_E \in P_k(E) \), \( \forall \) mesh edges/faces \( E \).

Equations:
\[
\begin{cases}
(\bar{q}_h, \bar{v})_K - (u_h, \nabla \cdot \bar{v})_K + \langle \hat{u}_h, \bar{v} \cdot \bar{n} \rangle_{\partial K} = 0, & \forall K, \\
-(\bar{q}_h, \bar{\nabla} w)_K + \langle \hat{q}_h \cdot \bar{n}, w \rangle_{\partial K} = (f, w)_K, & \forall K, \\
[\hat{q}_h \cdot \bar{n}] = 0.
\end{cases}
\]

Theorem (Condensed system, Cockburn, G & Lazarov, ’09)

The unknown numerical trace \( \hat{u}_h \) can be found by solving a sparse symmetric positive definite system. The other solution components \( \bar{q}_h \) and \( u_h \) can then be locally recovered from \( \hat{u}_h \).
Compare with mixed method

The HDG method

\[(\bar{q}_h, \bar{v})_K - (u_h, \nabla \cdot \bar{v})_K + \langle \hat{u}_h, \bar{v} \cdot \bar{n} \rangle_{\partial K} = 0 \]
\[- (\bar{q}_h, \bar{\nabla} w)_K + \langle \hat{\lambda}_h \cdot \bar{n}, w \rangle_{\partial K} = (f, w)_K \]
\[
\begin{bmatrix} \hat{\lambda}_h \cdot \bar{n} \end{bmatrix} = 0
\]

Spaces: \(\bar{q}_h|_K \in \tilde{P}_k(K)\), \quad u_h|_K \in P_k(K), \quad \hat{u}_h|_E \in P_k(E)

The Raviart-Thomas mixed method in hybridized form

\[(\bar{q}_h, \bar{v})_K - (u_h, \nabla \cdot \bar{v})_K + \langle \lambda_h, \bar{v} \cdot \bar{n} \rangle_{\partial K} = 0 \]
\[(\nabla \cdot \bar{q}_h, w)_K = (f, w)_K \]
\[
\begin{bmatrix} \bar{q}_h \cdot \bar{n} \end{bmatrix} = 0
\]

Spaces: \(\bar{q}_h|_K \in \tilde{P}_k(K) + \bar{\chi} P_k(K)\), \quad u_h|_K \in P_k(K), \quad \lambda_h|_E \in P_k(E)

Thus, the HDG method may be thought of as resulting from an attempt to stabilize the mixed method with unstable spaces.
Compare with mixed method

The HDG method

\[
(\bar{q}_h, \bar{v})_K - (u_h, \nabla \cdot \bar{v})_K + \langle \hat{u}_h, \bar{v} \cdot \vec{n} \rangle_{\partial K} = 0
\]

\[-(\bar{q}_h, \bar{\nabla} w)_K + \langle \hat{q}_h \cdot \vec{n}, w \rangle_{\partial K} = (f, w)_K\]

\[
\left[ \bar{q}_h \cdot \vec{n} + \tau (u_h - \hat{u}_h) \right] = 0
\]

Spaces: \(\bar{q}_h|_K \in \bar{P}_k(K)\), \(u_h|_K \in P_k(K)\), \(\hat{u}_h|_E \in P_k(E)\)

The Raviart-Thomas mixed method in hybridized form

\[
(\bar{q}_h, \bar{v})_K - (u_h, \nabla \cdot \bar{v})_K + \langle \lambda_h, \bar{v} \cdot \vec{n} \rangle_{\partial K} = 0
\]

\[
(\bar{\nabla} \cdot \bar{q}_h, w)_K = (f, w)_K
\]

\[
\left[ \bar{q}_h \cdot \vec{n} \right] = 0
\]

Spaces: \(\bar{q}_h|_K \in \bar{P}_k(K) + \bar{\chi}P_k(K)\), \(u_h|_K \in P_k(K)\), \(\lambda_h|_E \in P_k(E)\)

Thus, the HDG method may be thought of as resulting from an attempt to stabilize the mixed method with unstable spaces.
Compare with mixed method

The HDG method

\[(\bar{q}_h, \bar{v})_K - (u_h, \nabla \cdot \bar{v})_K + \langle \hat{u}_h, \bar{v} \cdot \vec{n} \rangle_{\partial K} = 0\]

\[(\nabla \cdot \bar{q}_h, w) + \langle \tau(u_h - \hat{u}_h), w \rangle_{\partial K} = (f, w)_K\]

\[\left[\bar{q}_h \cdot \vec{n} + \tau(u_h - \hat{u}_h)\right] = 0\]

Spaces: \(\bar{q}_h|_K \in \bar{P}_k(K), \ u_h|_K \in P_k(K), \ \hat{u}_h|_E \in P_k(E)\)

The Raviart-Thomas mixed method in hybridized form

\[(\bar{q}_h, \bar{v})_K - (u_h, \nabla \cdot \bar{v})_K + \langle \lambda_h, \bar{v} \cdot \vec{n} \rangle_{\partial K} = 0\]

\[(\nabla \cdot \bar{q}_h, w)_K = (f, w)_K\]

\[\left[\bar{q}_h \cdot \vec{n}\right] = 0\]

Spaces: \(\bar{q}_h|_K \in \bar{P}_k(K) + \bar{x}P_k(K), \ u_h|_K \in P_k(K), \ \lambda_h|_E \in P_k(E)\)

Thus, the HDG method may be thought of as resulting from an attempt to stabilize the mixed method with unstable spaces.
Projections in error analysis

Method for analyzing Raviart-Thomas mixed method

A flux projection $\Pi$ that commutes with the $L^2$-orthogonal projection $P$

$$\nabla \cdot \Pi \vec{q} = P \nabla \cdot \vec{q}$$

gives us an analogue of Galerkin orthogonality for mixed methods:

$$(\vec{q} - \vec{q}_h, \Pi \vec{q} - \vec{q}_h) = 0 \implies \text{simple analysis.}$$

Can we mimic this for HDG methods?

Is there a projection $\Pi$ into the HDG flux space satisfying

$$\nabla \cdot \Pi \vec{q} = P \nabla \cdot \vec{q}$$

Perhaps

$$\nabla \cdot \Pi \vec{q} = P \nabla \cdot \vec{q} + \cdots?$$
The HDG projection

The Raviart-Thomas projection: \( \Pi^\text{RT}_h(\vec{q}) \)

\[ \Pi^\text{RT}_h \vec{q} \in \tilde{P}_k(K) + \vec{x}P_k(K) \] satisfies

\[ (\Pi^\text{RT}_h \vec{q}, \vec{v})_K = (\vec{q}, \vec{v})_K \quad \text{for all } \vec{v} \in \tilde{P}_{k-1}(K), \]

\[ \langle \Pi^\text{RT}_h \vec{q} \cdot \vec{n}, \mu \rangle_F = \langle \vec{q} \cdot \vec{n}, \mu \rangle_F \quad \text{for all } \mu \in P_k(F). \]

Key ideas to extend this to the HDG method:

- Couple both \( \vec{q} \) and \( u \) into a projection. This gives enough degrees of freedom.

- Use the form of the numerical flux (with \( \tau \)) in projector’s definition. This simplifies error analysis.
The HDG projection

The Raviart-Thomas projection: $\Pi^\text{RT}_h(\vec{q})$

$\Pi^\text{RT}_h \vec{q} \in \tilde{P}_k(K) + \vec{x}P_k(K)$ satisfies

\[
\begin{align*}
(\Pi^\text{RT}_h \vec{q}, \vec{v})_K &= (\vec{q}, \vec{v})_K \\
\langle \Pi^\text{RT}_h \vec{q} \cdot \vec{n}, \mu \rangle_F &= \langle \vec{q} \cdot \vec{n}, \mu \rangle_F
\end{align*}
\]

for all $\vec{v} \in \tilde{P}_{k-1}(K)$,

for all $\mu \in P_k(F)$.

The new HDG projection: $\Pi_h(\vec{q}, u)$

The (flux) $q$-component of $\Pi_h(\vec{q}, u)$ is $\Pi^q_h \vec{q}$. It depends on both $\vec{q}$ and $u$!

The (scalar) $u$-component of $\Pi_h(\vec{q}, u)$ is $\Pi^u_h u$. They satisfy:

\[
\begin{align*}
(\Pi^q_h \vec{q}, \vec{v})_K &= (\vec{q}, \vec{v})_K \\
(\Pi^u_h u, w)_K &= (u, w)_K \\
\langle \Pi^q_h \vec{q} \cdot \vec{n} + \tau \Pi^u_h u, \mu \rangle_F &= \langle \vec{q} \cdot \vec{n} + \tau u, \mu \rangle_F
\end{align*}
\]

for all $\vec{v} \in \tilde{P}_{k-1}(K)$,

for all $w \in P_{k-1}(K)$,

for all $\mu \in P_k(F)$. 

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Commutativity property of the Raviart-Thomas projection

For all $w \in P_k(K)$,

$$(w, \nabla \cdot \bar{q})_K = (w, \nabla \cdot \Pi_h^{RT} \bar{q})_K.$$ 

Lemma (Weak commutativity property for the HDG projection)

For all $w \in P_k(K)$,

$$(w, \nabla \cdot \bar{q})_K = (w, \nabla \cdot \Pi_h^q \bar{q})_K + \langle w, \tau(\Pi_h^u u - u) \rangle_{\partial K}.$$
Approximation

Suppose $\tau|_{\partial K}$ is nonnegative and $\tau_K^{\max} := \max \tau|_{\partial K} > 0$. Let $F^*$ be a face of $K$ at which the maximum of $\tau$ is attained. Put $\tau_K^* := \max \tau|_{\partial K \setminus F^*}$.

**Theorem (Dependence on approximation on $\tau$ and $h$)**

Let $k \geq 0$ and $s_u, s_q \in (1/2, k + 1]$. There is a constant $C$ independent of element diameter $h_K$ and stabilization parameter $\tau$ such that

\[
\| \Pi_h^q \tilde{q} - \tilde{q} \|_K \leq C \, h_K^{s_q} \| \tilde{q} \|_{H^{s_q}(K)} + C \, h_K^{s_u} \tau_K^* \| u \|_{H^{s_u}(K)}
\]

\[
\| \Pi_h^u u - u \|_K \leq C \, h_K^{s_u} \| u \|_{H^{s_u}(K)} + C \, \frac{h_K^{s_q}}{\tau_K^{\max}} \| \tilde{q} \|_{H^{s_q}(K)}.
\]
Unlike many DG methods, HDG methods have optimally convergent fluxes:

**Theorem (Flux error estimate)**

For any $k \geq 0$,

$$
\| \Pi_h^q \vec{q} - \vec{q}_h \| \leq \| \Pi_h^q \vec{q} - \vec{q} \|.
$$

- Thus, combining with the approximation property of the projection,
  $$
  \| \vec{q} - \vec{q}_h \| \leq C h^{k+1} \left[ \| \vec{q} \|_{H^{k+1}} + \max_K (\tau^*_K) \| u \|_{H^{k+1}} \right].
  $$

- If $\tau$ is such that it is nonzero only on one edge of every mesh triangle, then $\tau^*_K = 0$ and flux error is independent of $\tau$. 
Numerical convergence of flux

Degree $k = 0$ case: 1st order convergence observed.
Numerical convergence of flux

Degree \( k = 1 \) case: 2nd order convergence observed.
Convergence of $u$

**Theorem (Optimal convergence of $u$)**

For any $k \geq 0$,

$$
\| u - u_h \| \leq C \| u - \Pi_h^u u \| + b_\tau C \| \vec{q} - \Pi_h^q \vec{q} \|,
$$

where $b_\tau = \max\{1 + h_K \tau_K^* + h_K / \tau_K^{\max} : K \in \mathcal{T}_h\}$.

**Theorem (Superconvergence of projected error by duality)**

Under the full regularity assumption

$$
\| \Pi_h^u u - u_h \| \leq c_\tau C h^{\min\{k,1\}} \| \Pi_h^q \vec{q} - \vec{q} \| \quad \text{for } k \geq 0,
$$

where $h = \max\{h_K : K \in \mathcal{T}_h\}$ and $c_\tau = \max\{1, h_K \tau_K^* : K \in \mathcal{T}_h\}$.
Numerical convergence of $u$

Degree $k = 0$ case: 1st order convergence observed.
Numerical convergence of $u$

Degree $k = 1$ case: 2nd order convergence observed.
Conclusion

- There is a weakly commuting projector that renders the analysis of HDG methods simple and concise.

- The local approximation properties of the projector can be precisely characterized in terms of $h$ and $\tau$.

- The global HDG errors and their $\tau$-dependence can be understood using the local properties of the projector.

- All variables converge at optimal order when $\tau$ is of unit size.

- Standard postprocessing techniques can be applied to obtain enhanced accuracy (using the superconvergence of projected error).

- Similar projectors can be constructed to analyze HDG methods for Stokes flow.