Nédélec spaces in affine coordinates

Jay Gopalakrishnan

University of Florida

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Collaborators: Leszek F. Demkowicz

The University of Texas at Austin

Luis E. García-Castillo

Universidad de Alcalá de Henares

The Nédélec polynomial space

The k^{th} Nédélec space (in any space dimension) is

$$\boldsymbol{R}_k = \boldsymbol{P}_{k-1} \oplus \boldsymbol{S}_k.$$

Notation:

$\boldsymbol{P}_k =$ set of all vector functions whose every component is a *polynomial* of degree at most k.

$$\boldsymbol{S}_k = \{\boldsymbol{q} \in \overline{\boldsymbol{P}}_k : \quad \boldsymbol{x} \cdot \boldsymbol{q}(\boldsymbol{x}) = 0 \quad \text{for all } \boldsymbol{x}\},$$

where

$\overline{\boldsymbol{P}}_k =$ set of vector polynomials whose components are *homogeneous* of degree k.

Why Nédélec space?

- It gives $\boldsymbol{H}(\mathbf{curl})$-conforming finite elements.
- Does not produce spurious modes. Provable convergence.
- Uses only degrees of freedom needed to handle curl.
  In approximating curl, *gradients need not be included*:

$$\boldsymbol{P}_k = \boldsymbol{P}_{k-1} \oplus \underbrace{\overline{\boldsymbol{P}}_k}_{\boldsymbol{S}_k \oplus \nabla \overline{P}_{k+1}}$$

[Nédélec,1980]

$$\Longrightarrow \qquad \boldsymbol{R}_k = \boldsymbol{P}_{k-1} \oplus \boldsymbol{S}_k$$

Q: Are there other ways of removing the gradients?

A: Yes. . . (examples later).

An exactness property

Another reason for Nédélec space is that they arise canonically from an exactness property: [Hiptmair, 1999]

$$\nabla P_k = \mathrm{Ker}(\mathbf{curl}\,, \boldsymbol{R}_k),$$

where $\mathrm{Ker}(\mathbf{curl}\,, \boldsymbol{R}_k) = \{\boldsymbol{q} \in \boldsymbol{R}_k : \mathbf{curl}\,\boldsymbol{q} = 0\}$.

Consider *Nédélec-type spaces* $\boldsymbol{R}'_k = \boldsymbol{P}_{k-1} \oplus \boldsymbol{S}'_k$, where $\boldsymbol{S}'_k$ is a subspace of $\overline{\boldsymbol{P}}_k$ that is *linearly independent to* $\nabla \overline{P}_{k+1}$.

PROPOSITION. *The property*

$$\nabla P_k = \mathrm{Ker}(\mathbf{curl}\,, \boldsymbol{R}'_k),$$

holds for any Nédélec-type space $\boldsymbol{R}'_k$.

A related multilinear form

[Nédélec, 1980] shows that a smooth function $\boldsymbol{q} : \mathbb{R}^N \mapsto \mathbb{R}^N$ is in $\boldsymbol{R}_k$ if and only if

$$\varepsilon^k(\boldsymbol{q})(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_{k+1}) = 0, \qquad \forall \boldsymbol{r}_i,$$

where the multilinear form $\varepsilon^k(\boldsymbol{q})(\cdots)$ is defined by

$$\varepsilon^k(\boldsymbol{q})(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_{k+1}) = \frac{1}{(k+1)!} \sum_{\sigma} (\boldsymbol{d}^k \boldsymbol{q})(\boldsymbol{r}_{\sigma(1)}, \boldsymbol{r}_{\sigma(2)}, \ldots, \boldsymbol{r}_{\sigma(k)}) \cdot \boldsymbol{r}_{\sigma(k+1)}.$$

Here the sum runs over all permutations σ of the set $\{1, 2, \ldots, k+1\}$ and $\boldsymbol{d}^k \boldsymbol{q}$ denotes the k^{th} order Fréchet derivative of $\boldsymbol{q}$.

Characterizations of S_k

THEOREM. *A C^k-function $\boldsymbol{q} \equiv (q_\ell)$ is in $\boldsymbol{R}_k$ if and only if*

$$\sum_{\ell=1}^{N} \beta_\ell \, \partial^{\boldsymbol{\beta}-\boldsymbol{e}_\ell} q_\ell = 0, \qquad \textit{for all } |\boldsymbol{\beta}| = k+1.$$

Notation:

- $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_N)$ are multi-indices.
- $|\boldsymbol{\beta}| = \beta_1 + \beta_2 + \cdots + \beta_N$ and $\boldsymbol{e}_\ell = (0, \ldots 1, \ldots 0)$. ($\ell^{\text{th}}$ entry)
- For any multi-index $\boldsymbol{\alpha}$, $\quad \partial^{\boldsymbol{\alpha}} = \dfrac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}}$.
- All terms involving $\boldsymbol{\beta} - \boldsymbol{e}_\ell$ are considered to be zero if a component of $\boldsymbol{\beta} - \boldsymbol{e}_\ell$ is negative.

Characterizations of S_k

THEOREM. *A C^k-function $\boldsymbol{q} \equiv (q_\ell)$ is in $\boldsymbol{R}_k$ if and only if*

$$\sum_{\ell=1}^{N} \beta_\ell \, \partial^{\boldsymbol{\beta}-\boldsymbol{e}_\ell} q_\ell = 0, \qquad \text{for all } |\boldsymbol{\beta}| = k+1.$$

COROLLARY. *The polynomial* $\boldsymbol{q}(\boldsymbol{x}) = \begin{bmatrix} \displaystyle\sum_{|\boldsymbol{\alpha}|=k} c_{\boldsymbol{\alpha},1} \, \boldsymbol{x}^{\boldsymbol{\alpha}} \\ \vdots \\ \displaystyle\sum_{|\boldsymbol{\alpha}|=k} c_{\boldsymbol{\alpha},N} \, \boldsymbol{x}^{\boldsymbol{\alpha}} \end{bmatrix}$

is in $\boldsymbol{S}_k$ if and only if

$$\sum_{\ell=1}^{N} c_{\boldsymbol{\beta}-\boldsymbol{e}_\ell,\ell} = 0, \qquad \text{for all } |\boldsymbol{\beta}| = k+1.$$

An example

$$\sum_{\ell=1}^{3} c_{\boldsymbol{\beta}-\boldsymbol{e}_\ell,\ell} = 0, \quad \text{for all } |\boldsymbol{\beta}| = 2+1$$

Case of
$N = 3, \ k = 2$

$$\boldsymbol{q}(\boldsymbol{x}) \equiv \Big(\sum_{|\boldsymbol{\alpha}|=2} c_{\boldsymbol{\alpha},1}\, \boldsymbol{x}^{\boldsymbol{\alpha}}\Big)\boldsymbol{e}_1 + \Big(\sum_{|\boldsymbol{\alpha}|=2} c_{\boldsymbol{\alpha},2}\, \boldsymbol{x}^{\boldsymbol{\alpha}}\Big)\boldsymbol{e}_2 + \Big(\sum_{|\boldsymbol{\alpha}|=2} c_{\boldsymbol{\alpha},3}\, \boldsymbol{x}^{\boldsymbol{\alpha}}\Big)\boldsymbol{e}_3$$

is in $\boldsymbol{S}_2$ if and only if its coeffi cients $\{c_{\boldsymbol{\alpha},\ell}\}$ satisfy:

$$\begin{aligned}
\boldsymbol{\beta} &= (2,1,0): & c_{(1,1,0),1} + c_{(2,0,0),2} &= 0,\\
\boldsymbol{\beta} &= (2,0,1): & c_{(1,0,1),1} + c_{(2,0,0),3} &= 0,\\
\boldsymbol{\beta} &= (1,2,0): & c_{(0,2,0),1} + c_{(1,1,0),2} &= 0,\\
\boldsymbol{\beta} &= (1,1,1): & c_{(0,1,1),1} + c_{(1,0,1),2} + c_{(1,1,0),3} &= 0,\\
\boldsymbol{\beta} &= (1,0,2): & c_{(0,0,2),1} + c_{(1,0,1),3} &= 0,\\
&\vdots & \text{(10 equations)} \qquad &\vdots
\end{aligned}$$

An example

$$\sum_{\ell=1}^{3} c_{\boldsymbol{\beta}-\boldsymbol{e}_\ell,\ell} = 0, \quad \text{for all } |\boldsymbol{\beta}| = 2+1$$

Case of $N = 3,\ k = 2$

$$\boldsymbol{q}(\boldsymbol{x}) \equiv \Big(\sum_{|\boldsymbol{\alpha}|=2} c_{\boldsymbol{\alpha},1}\, \boldsymbol{x}^{\boldsymbol{\alpha}}\Big)\boldsymbol{e}_1 + \Big(\sum_{|\boldsymbol{\alpha}|=2} c_{\boldsymbol{\alpha},2}\, \boldsymbol{x}^{\boldsymbol{\alpha}}\Big)\boldsymbol{e}_2 + \Big(\sum_{|\boldsymbol{\alpha}|=2} c_{\boldsymbol{\alpha},3}\, \boldsymbol{x}^{\boldsymbol{\alpha}}\Big)\boldsymbol{e}_3$$

is in $\boldsymbol{S}_2$ if and only if its coefficients $\{c_{\boldsymbol{\alpha},\ell}\}$ satisfy:

$$\boldsymbol{\beta} = (2,1,0): \qquad c_{(1,1,0),1} + c_{(2,0,0),2} = 0,$$

$$\vdots$$

Eg., the first equation tells us that $x_1 x_2 \boldsymbol{e}_1 - x_1^2 \boldsymbol{e}_2 \equiv \begin{bmatrix} x_1 x_2 \\ -x_1^2 \\ 0 \end{bmatrix}$ is in $\boldsymbol{S}_2$.

More generally, observe that $\boldsymbol{x}^{\boldsymbol{\beta}-\boldsymbol{e}_1}\boldsymbol{e}_1 - \boldsymbol{x}^{\boldsymbol{\beta}-\boldsymbol{e}_2}\boldsymbol{e}_2$ *is in* $\boldsymbol{S}_2$ *for all* $|\boldsymbol{\beta}| = 3$ *with positive* β_1 *and* β_2.

A basis for S_k

Consider all such linearly independent two-term expressions. E.g., for a $\boldsymbol{\beta}$ with three nonzero entries and $|\boldsymbol{\beta}| = k+1$,

$$\boldsymbol{\beta} = (0 \cdots \underset{}{\overset{l^{\text{th}}\text{entry}}{\times}} \cdots \overset{m^{\text{th}}\text{entry}}{\times} \cdots \overset{n^{\text{th}}\text{entry}}{\times} \cdots 0),$$

we have two expressions:

$$\boldsymbol{x}^{\boldsymbol{\beta}-\boldsymbol{e}_l}\boldsymbol{e}_l - \boldsymbol{x}^{\boldsymbol{\beta}-\boldsymbol{e}_m}\boldsymbol{e}_m, \qquad \boldsymbol{x}^{\boldsymbol{\beta}-\boldsymbol{e}_m}\boldsymbol{e}_m - \boldsymbol{x}^{\boldsymbol{\beta}-\boldsymbol{e}_n}\boldsymbol{e}_n.$$

THEOREM. *The collection of all such expressions (for all multi-indices $\boldsymbol{\beta}$ with $|\boldsymbol{\beta}| = k+1$) forms a basis for $\boldsymbol{S}_k$ (for any order k and any dimension N).*

Some Nédélec-type spaces

Let $\boldsymbol{S}_k^{(a)}$ denote the set of all homogeneous polynomials

$$\boldsymbol{q}(\boldsymbol{x}) = \sum_{\ell=1}^{N} \sum_{|\boldsymbol{\alpha}|=k} c_{\boldsymbol{\alpha},\ell}\, \boldsymbol{x}^{\boldsymbol{\alpha}}\ \boldsymbol{e}_\ell$$

whose coefficients $\{c_{\boldsymbol{\alpha},\ell}\}$ satisfy

$$\sum_{\ell=1}^{N} a_{\boldsymbol{\beta},\ell}\ c_{\boldsymbol{\beta}-\boldsymbol{e}_\ell,\ell} = 0, \quad \text{for all } |\boldsymbol{\beta}| = k+1,$$

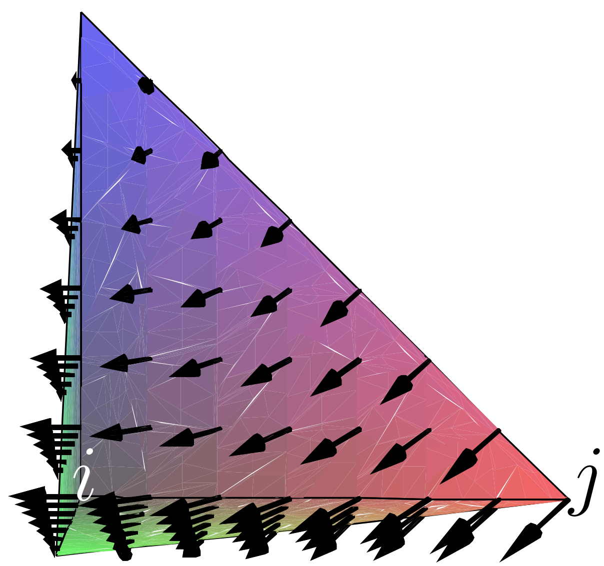
for some numbers $\{a_{\boldsymbol{\beta},\ell}\}$ such that $\sum_{\ell=1}^{N} a_{\boldsymbol{\beta},\ell}\beta_\ell \neq 0$.

PROPOSITION. *Then,* $\overline{\boldsymbol{P}}_k = \boldsymbol{S}_k^{(a)} \oplus \boldsymbol{\nabla}\overline{P}_{k+1}$, *so* $\boldsymbol{R}_k^{(a)} = \boldsymbol{P}_{k-1} \oplus \boldsymbol{S}_k^{(a)}$ *is a Nédélec-type space.*

Barycentric coordinates

In the lowest order case, Whitney forms give expressions in barycentric (or *affine*) coordinates that form a basis for the Nédélec space:

$$\{\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i\}.$$



Q: Can we systematically generalize such expressions to obtain bases for higher order Nédélec spaces?

A: Yes, as we shall now show . . .

Background

Many papers have given basis expressions in affine coordinates and investigated their utility in electromagnetics.

Papers from the engineering literature in the 90's:

- [Lee, Sun & Csendes, 1991]
- [Ahagon & Kashimoto, 1995]
- [Savage & Peterson, 1996]
- [Yioultsis & Tsiboukis, 1996]
- [Graglia, Wilton & Peterson, 1997]
- [Webb, 1999]

However, some of them do *not* span the Nédélec space (e.g. the first and the last – more remarks on this *later*).

Some isomorphisms

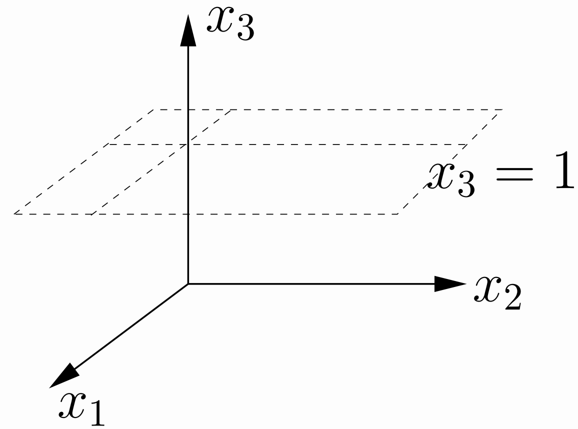
$$Ax_1^2 + Bx_2^2 + Cx_3^2 + Dx_1x_2 + Ex_1x_3 + Fx_2x_3$$

$$\downarrow$$

$$(x_3 = 1)$$

$$\downarrow$$

$$Ax_1^2 + Bx_2^2 + C + Dx_1x_2 + Ex_1 + Fx_2$$



PROPOSITION. *The restriction of homogeneous polynomials of degree k in $N+1$ variables $x_1, x_2, \ldots x_{N+1}$ to the hyperplane $x_{N+1} = 1$ is an isomorphism onto the space of all polynomials of degree at most k in the first N variables.*

Some isomorphisms

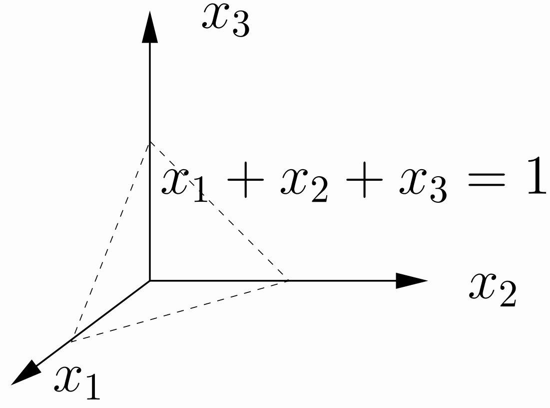
$$Ax_1^2 + Bx_2^2 + Cx_3^2 + Dx_1x_2 + Ex_1x_3 + Fx_2x_3$$

$$\downarrow$$

$$(x_1 + x_2 + x_3 = 1)$$

$$\downarrow X$$

$$A\lambda_1^2 + B\lambda_2^2 + C\lambda_3^2 + D\lambda_1\lambda_2 + E\lambda_1\lambda_3 + F\lambda_2\lambda_3$$



PROPOSITION. *If λ_i's are the barycentric coordinates of an N-simplex, the above indicated map X is an isomorphism from the space of homogeneous polynomials of degree k in $N+1$ variables onto the space of all polynomials of degree at most k on the simplex.*

Some isomorphisms

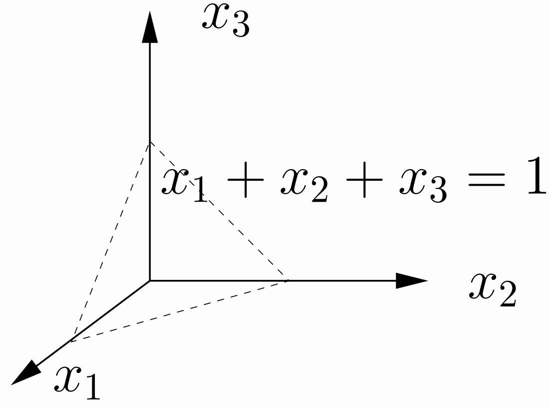
$$Ax_1^2 + Bx_2^2 + Cx_3^2 + Dx_1x_2 + Ex_1x_3 + Fx_2x_3$$

$$\downarrow$$

$$(x_1 + x_2 + x_3 = 1)$$

$$\downarrow X$$

$$A\lambda_1^2 + B\lambda_2^2 + C\lambda_3^2 + D\lambda_1\lambda_2 + E\lambda_1\lambda_3 + F\lambda_2\lambda_3$$



For vector polynomials, we consider $\boldsymbol{Y}$ so that the following diagram commutes:

$$\begin{array}{ccc} \overline{P}_{k+1}^{(N+1)} & \xrightarrow{\boldsymbol{\nabla}^{(N+1)}} & \overline{\boldsymbol{P}}_k^{(N+1)} \\ \downarrow X & & \downarrow \boldsymbol{Y} \\ P_{k+1}^{(N)} & \xrightarrow{\boldsymbol{\nabla}^{(N)}} & \boldsymbol{P}_k^{(N)} \end{array}$$

Some isomorphisms

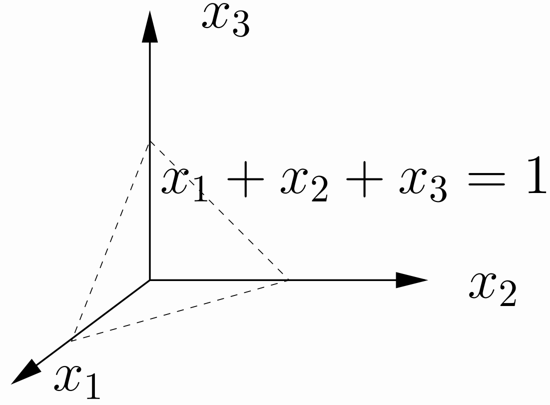
Polynomial $\sum_{\ell=1}^{N+1} \sum_{|\boldsymbol{\alpha}|=k} c_{\boldsymbol{\alpha},\ell}\, \boldsymbol{x}^{\boldsymbol{\alpha}}\, \boldsymbol{e}_\ell$ in $N+1$ variables

$$\downarrow$$

$$(x_i \to \lambda_i, \quad \boldsymbol{e}_\ell \to \boldsymbol{\nabla}\lambda_\ell)$$

$$\downarrow \boldsymbol{Y}$$

Polynomial $\sum_{\ell=1}^{N+1} \sum_{|\boldsymbol{\alpha}|=k} c_{\boldsymbol{\alpha},\ell}\, \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\, \boldsymbol{\nabla}\lambda_\ell$ in N variables



THEOREM. *The map $\boldsymbol{Y}$ considered as a map from*

$$\boldsymbol{S}_k^{(N+1)} \longrightarrow \boldsymbol{R}_k^{(N)}$$

(the highest degree part of the $N+1$ dimensional Nédélec space)

(N-dimensional Nédélec space)

is an isomorphism.

Basis in affine coordinates

Therefore, to get a basis for the Nédélec space $\boldsymbol{R}_k$ in 3 dimensions, we simply apply $\boldsymbol{Y}$ to the previously constructed basis for $\boldsymbol{S}_k$ in 4 dimensions.

$$\boldsymbol{x}^{\boldsymbol{\beta}-\boldsymbol{e}_l}\boldsymbol{e}_l - \boldsymbol{x}^{\boldsymbol{\beta}-\boldsymbol{e}_m}\boldsymbol{e}_m$$

$$\downarrow \boldsymbol{Y}$$

$$\boldsymbol{\lambda}^{\boldsymbol{\beta}-\boldsymbol{e}_l}\boldsymbol{\nabla}\lambda_l - \boldsymbol{\lambda}^{\boldsymbol{\beta}-\boldsymbol{e}_m}\boldsymbol{\nabla}\lambda_m$$

Collecting such expressions for all admissible $\boldsymbol{\beta}$, we can categorize them as edge, face, and interior basis functions:

Basis in affine coordinates

Basis expressions categorized: Edge, face, and interior basis functions for any order (k) follows. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

$\lvert\boldsymbol{\alpha}\rvert = k - 1$ with two nonzero entries α_l, α_m	$\boldsymbol{\lambda}^{\boldsymbol{\alpha}}(\lambda_l \boldsymbol{\nabla}\lambda_m - \lambda_m \boldsymbol{\nabla}\lambda_l)$. *(Edge basis functions)*
$\lvert\boldsymbol{\alpha}\rvert = k - 2$ with three nonzero entries $\alpha_l, \alpha_m, \alpha_n$	$\boldsymbol{\lambda}^{\boldsymbol{\alpha}}(\lambda_l\lambda_m \boldsymbol{\nabla}\lambda_n - \lambda_m\lambda_n \boldsymbol{\nabla}\lambda_l)$, $\boldsymbol{\lambda}^{\boldsymbol{\alpha}}(\lambda_m\lambda_n \boldsymbol{\nabla}\lambda_l - \lambda_n\lambda_l \boldsymbol{\nabla}\lambda_m)$. *(Face basis functions)*
$\lvert\boldsymbol{\alpha}\rvert = k - 3$ with all four entries nonzero	$\boldsymbol{\lambda}^{\boldsymbol{\alpha}}(\lambda_1\lambda_2\lambda_3 \boldsymbol{\nabla}\lambda_4 - \lambda_2\lambda_3\lambda_4 \boldsymbol{\nabla}\lambda_1)$, $\boldsymbol{\lambda}^{\boldsymbol{\alpha}}(\lambda_2\lambda_3\lambda_4 \boldsymbol{\nabla}\lambda_1 - \lambda_3\lambda_4\lambda_1 \boldsymbol{\nabla}\lambda_2)$, $\boldsymbol{\lambda}^{\boldsymbol{\alpha}}(\lambda_3\lambda_4\lambda_1 \boldsymbol{\nabla}\lambda_2 - \lambda_4\lambda_3\lambda_2 \boldsymbol{\nabla}\lambda_1)$. *(Interior basis functions)*

Two previous works

Different expressions for the quadratic case were suggested:

- [Lee, Sun & Csendes, 1991]:

Edge(i,j): $\lambda_i \boldsymbol{\nabla} \lambda_j, \ \ \lambda_j \boldsymbol{\nabla} \lambda_i$ (spans $\boldsymbol{P}_1$)

Face(i,j,k): $\lambda_i \lambda_j \boldsymbol{\nabla} \lambda_k, \ \lambda_k \lambda_i \boldsymbol{\nabla} \lambda_j$ (adds 8 quadratics)

- [Savage & Peterson, 1996]:

Edge(i,j): $\lambda_i \boldsymbol{\nabla} \lambda_j, \ \ \lambda_j \boldsymbol{\nabla} \lambda_i$ (spans $\boldsymbol{P}_1$)

Face(i,j,k): $\lambda_i \lambda_j \boldsymbol{\nabla} \lambda_k - \lambda_j \lambda_k \boldsymbol{\nabla} \lambda_i,$

$\lambda_j \lambda_k \boldsymbol{\nabla} \lambda_i - \lambda_k \lambda_i \boldsymbol{\nabla} \lambda_j$ (adds 8 quadratics)

The latter is a hierarchical rearrangement of our previously established expressions, so it spans the Nédélec space $\boldsymbol{R}_2$.

But the former does not span the Nédélec space . . .

Two previous works

Different expressions for the quadratic case were suggested:

- [Lee, Sun & Csendes, 1991]:

Edge(i,j):	$\lambda_i \boldsymbol{\nabla}\lambda_j, \;\; \lambda_j \boldsymbol{\nabla}\lambda_i$	(spans $\boldsymbol{P}_1$)
Face(i,j,k):	$\lambda_i\lambda_j \boldsymbol{\nabla}\lambda_k, \;\; \lambda_k\lambda_i \boldsymbol{\nabla}\lambda_j$	(adds 8 quadratics)

Using our techniques, it is easy to see why an expression like $\lambda_i\lambda_j\boldsymbol{\nabla}\lambda_k$ cannot be in the Nédélec space:

$$x_i x_j \boldsymbol{e}_k \;\longrightarrow\; \text{this is not in } \boldsymbol{S}_2^{(4)}, \text{ because } \sum_{\ell=1}^{4} c_{\boldsymbol{\beta}-\boldsymbol{e}_\ell,\ell} \neq 0,$$

$$\updownarrow \boldsymbol{Y}$$

$$\lambda_i\lambda_j\boldsymbol{\nabla}\lambda_k \;\longrightarrow\; \text{so this cannot be in } \boldsymbol{R}_2^{(3)}.$$

One question remains: Why does it work so well?

Two previous works

Different expressions for the quadratic case were suggested:

- [Lee, Sun & Csendes, 1991]:

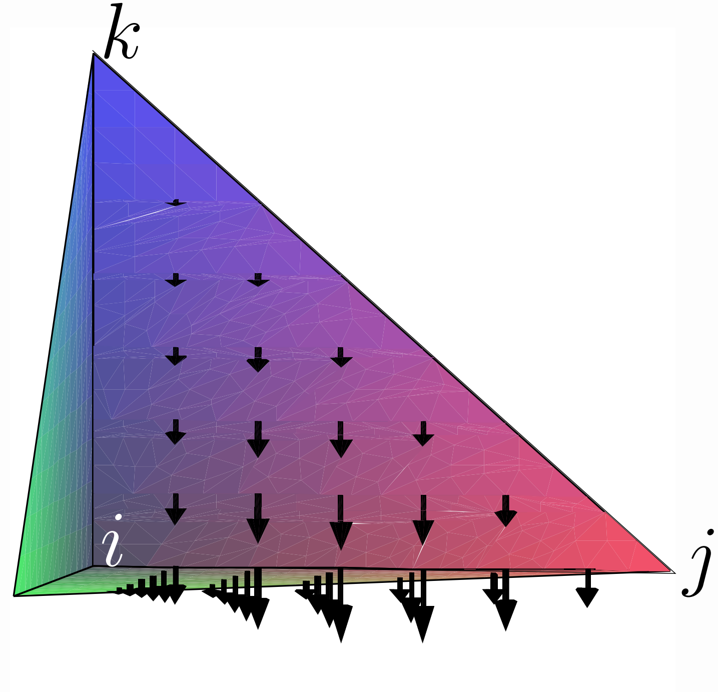
Edge(i,j): $\lambda_i \boldsymbol{\nabla} \lambda_j, \;\; \lambda_j \boldsymbol{\nabla} \lambda_i$ (spans $\boldsymbol{P}_1$)

Face(i,j,k): $\lambda_i \lambda_j \boldsymbol{\nabla} \lambda_k, \;\; \lambda_k \lambda_i \boldsymbol{\nabla} \lambda_j$ (adds 8 quadratics)

– It spans a *Nédélec-type space*, so standard analysis using discrete Helmholtz decomposition etc. holds.

– Although asymmetric, gets global tangential continuity by assigning two basis functions per face globally.

– Same approximation order as the Nédélec space $\boldsymbol{R}_2$.



$\lambda_i \lambda_j \boldsymbol{\nabla} \lambda_k$

Hierarchical shape functions

We prefer shape functions that are hierarchical in k.
But our expressions were not written out hierarchically, eg.:

$\lvert\boldsymbol{\alpha}\rvert = k-1$ with two nonzero entries α_l, α_m	$\boldsymbol{\lambda}^{\boldsymbol{\alpha}}(\lambda_l \boldsymbol{\nabla}\lambda_m - \lambda_m \boldsymbol{\nabla}\lambda_l)$. *(Edge basis functions)*

But we can use any hierarchical basis for polynomials of degree at most $k-1$ on the edge in place of $\boldsymbol{\lambda}^{\boldsymbol{\alpha}}$ above.
Doing this for each category of basis functions, we find:

Hierarchical shape functions for standard polynomial spaces can be used to build hierarchical shape functions for the Nédélec space.

The hierarchy

Eg., $\boxed{k = 1}$

Edge(l, m) : $(\lambda_l \nabla \lambda_m - \lambda_m \nabla \lambda_l)$

Face(l, m, n) :

Interior :

The hierarchy

Eg., $\boxed{k = 2}$

Edge(l, m) : $(\lambda_l \nabla \lambda_m - \lambda_m \nabla \lambda_l)$ $\cdot \langle \lambda_l, \lambda_m \rangle$

Face(l, m, n) : $(\lambda_m \lambda_n \nabla \lambda_l - \lambda_n \lambda_l \nabla \lambda_m)$

$(\lambda_n \lambda_l \nabla \lambda_m - \lambda_l \lambda_m \nabla \lambda_n)$

Interior :

The hierarchy

Eg., $\boxed{k = 3}$

Edge(l, m) : $(\lambda_l \boldsymbol{\nabla} \lambda_m - \lambda_m \boldsymbol{\nabla} \lambda_l) \cdot \langle \lambda_l, \lambda_m \rangle \cdot \langle$edge bubbles$\rangle$

Face(l, m, n) : $(\lambda_m \lambda_n \boldsymbol{\nabla} \lambda_l - \lambda_n \lambda_l \boldsymbol{\nabla} \lambda_m) \cdot \langle \lambda_l, \lambda_m, \lambda_n \rangle$

$(\lambda_n \lambda_l \boldsymbol{\nabla} \lambda_m - \lambda_l \lambda_m \boldsymbol{\nabla} \lambda_n) \cdot \langle \lambda_l, \lambda_m, \lambda_n \rangle$

Interior : $(\lambda_1 \lambda_2 \lambda_3 \boldsymbol{\nabla} \lambda_4 - \lambda_2 \lambda_3 \lambda_4 \boldsymbol{\nabla} \lambda_1)$

$(\lambda_2 \lambda_3 \lambda_4 \boldsymbol{\nabla} \lambda_1 - \lambda_3 \lambda_4 \lambda_1 \boldsymbol{\nabla} \lambda_2)$

$(\lambda_3 \lambda_4 \lambda_1 \boldsymbol{\nabla} \lambda_2 - \lambda_4 \lambda_3 \lambda_2 \boldsymbol{\nabla} \lambda_1)$

The hierarchy

Eg., $\boxed{k = 4}$

Edge(l,m) : $(\lambda_l \nabla \lambda_m - \lambda_m \nabla \lambda_l) \cdot \langle \lambda_l, \lambda_m \rangle \cdot \langle$edge bubbles$\rangle$

Face(l,m,n) : $(\lambda_m \lambda_n \nabla \lambda_l - \lambda_n \lambda_l \nabla \lambda_m) \cdot \langle \lambda_l, \lambda_m, \lambda_n \rangle \cdot \langle$side shape fn$\rangle$

$(\lambda_n \lambda_l \nabla \lambda_m - \lambda_l \lambda_m \nabla \lambda_n) \cdot \langle \lambda_l, \lambda_m, \lambda_n \rangle \cdot \langle$side shape fn$\rangle$

Interior : $(\lambda_1 \lambda_2 \lambda_3 \nabla \lambda_4 - \lambda_2 \lambda_3 \lambda_4 \nabla \lambda_1) \cdot \langle \lambda_1, \lambda_2, \lambda_3, \lambda_4 \rangle$

$(\lambda_2 \lambda_3 \lambda_4 \nabla \lambda_1 - \lambda_3 \lambda_4 \lambda_1 \nabla \lambda_2) \cdot \langle \lambda_1, \lambda_2, \lambda_3, \lambda_4 \rangle$

$(\lambda_3 \lambda_4 \lambda_1 \nabla \lambda_2 - \lambda_4 \lambda_3 \lambda_2 \nabla \lambda_1) \cdot \langle \lambda_1, \lambda_2, \lambda_3, \lambda_4 \rangle$

. . . and so on for higher k.

Conclusion

- We gave some characterizations of the Nédélec space $\boldsymbol{R}_k$.
- This helped us identify an elementary basis for the highest degree part $\boldsymbol{S}_k$.
- We established an isomorphism between $\boldsymbol{S}_k$ in $N+1$ dimensions and $\boldsymbol{R}_k$ in N dimensions.
- Using the isomorphism, we got a basis in affine coordinates for $\boldsymbol{R}_k$ of any order.
- Our basis can be used to develop shape functions that are hierarchical in the degree.
- While Nédélec-type spaces offer an alternative to the Nédélec space, whether they have advantages over the Nédélec space remains unclear.