The behavior of multigrid applied to some PDEs with complex coefficients

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Complex coefficient PDE's

Consider the simple boundary value problem

BVP

$$\begin{cases} -\nabla \cdot \alpha(x)\, \nabla u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

but let $\alpha : \Omega \longmapsto \mathbb{C}$ be a *complex* valued coefficient.

- Many practically important problems (especially in electromagnetics) have complex coefficients.
- PML is an example, but it is more complicated (complex tensor).
- Must understand the simple problem BVP first.
- Many standard results for real valued problems do *not* carry over.

Jay & Joe's experiments

Does multigrid work for complex coefficient PDE's?

Jay and Joe coded the V-cycle with point Gauss-Seidel smoothing:

Numerical Example A

$$\alpha(x_1, x_2) = 1 + \hat{\imath} K \sin(\pi(2x_2 - 1)/2),$$

(Here $\hat{\imath}$ = imaginary unit.)

and Ω = unit square, meshed uniformly.

$h_{\text{coarse}} = 1/4$ / h_{fine}	$K = 1$	$K = 20$	$K = 100$
1/8	7	20	$\star$
1/16	7	15	$\star$
1/32	7	13	$\star$
1/64	7	12	$\star$
1/128	7	12	$\star$

Number of V-cycle iterations (to reduce error by a factor of 10^{-5}).

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Jay and Joe's codes gave the same results.

Number of V-cycle iterations (to reduce error by a factor of 10^{-5}).

Jay & Joe's experiments

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Does multigrid work for complex coefficient PDE's?

Jay and Joe coded the V-cycle with point Gauss-Seidel smoothing:

Numerical Example B

$$\alpha = (1 - r)^2 + r^4 \exp(4\hat{\imath}\theta)$$

(Here $(r, \theta) =$ polar coordinates.)

and $\Omega =$ unit square, meshed uniformly.

$h_{\text{coarse}} = 1/4$ / h_{fine}	Joe's code diverges	Jay's code
1/16	$\star$	19
1/32	$\star$	19
1/64	$\star$	20
1/128	$\star$	20

Number of V-cycle iterations.

What is going on?!

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What is going on?

Q: Hmm ... somebody has a bug?

A: *No, no bugs!!* (More later....)

Existence of solutions

WeakForm
$$\underbrace{\int_\Omega \alpha \nabla u \cdot \nabla \bar v}_{\text{sesquilinear } a(u,v)} = \int_\Omega f\, \bar v, \qquad \forall v \in H_0^1(\Omega).$$

Theorem (Existence assuming uniqueness)

Assume that

$$\textit{if} \quad a(v,w) = 0 \quad \forall w \in H_0^1(\Omega), \qquad \textit{then } v = 0.$$

Then there exists a u in $H_0^1(\Omega)$ satisfying the WeakForm *above.*

Proof uses a perturbation argument using compactness, employing ideas due to Peetre & Tartar.

Examples

- The uniqueness assumption is not easy to verify for general complex α.
- But there are many complex coefficients for which it is obvious:

Example: Uniformly positive real part

$$\exists c_0 > 0 : \qquad c_0 \le \mathrm{Re}(\alpha(x)) \qquad \forall x \in \Omega.$$

More general example of essentially coercive coefficients

If there is a complex number β_0 and a $c_0 > 0$ satisfying

$$c_0 \le \mathrm{Re}(\beta_0 \alpha(x)) \qquad \forall x \in \Omega,$$

then uniqueness follows, because the above implies a coercivity inequality of the form

$$c_0 |w|^2_{H^1(\Omega)} \le |a(w, \bar{\beta}_0 w)| \quad \text{for all } w \in H^1_0(\Omega).$$

The discrete case

$V_h =$ standard continuous p.w. linear finite element subspace of $H_0^1(\Omega)$.

FEM
$$a(u_h, v_h) = \int_\Omega f\, \bar{v}_h, \qquad \forall v_h \in V_h.$$

Basic questions:

Is this method solvable?

Is the FEM solution any good?

The discrete case

$V_h =$ standard continuous p.w. linear finite element subspace of $H_0^1(\Omega)$.

FEM $\qquad a(u_h, v_h) = \int_\Omega f\,\bar{v}_h, \qquad \forall v_h \in V_h.$

Assume:

1. *Uniqueness:* Let the uniqueness assumption for WeakForm hold.
2. *Ellipticity:* $\exists \alpha_0 > 0$ such that $\alpha_0 \le |\alpha(x)|$ for all x in Ω.
3. *Smoothness:* The coefficient $\alpha : \Omega \mapsto \mathbb{C}$ is $C^2(\overline{\Omega})$.

Theorem (Stability and Approximation)

$\exists\, h_0 > 0$ such that $\forall\, h \le h_0$, there is a unique solution u_h to FEM *and*

$$\|u_h\|_{H^1(\Omega)} \le C\, \|u\|_{H^1(\Omega)},$$

$$\|u - u_h\|_{H^1(\Omega)} \le C \inf_{w_h \in V_h} \|u - w_h\|_{H^1(\Omega)}.$$

Proof uses a "discrete" version of the Peetre-Tartar argument and the Schatz duality argument.

Multigrid algorithm

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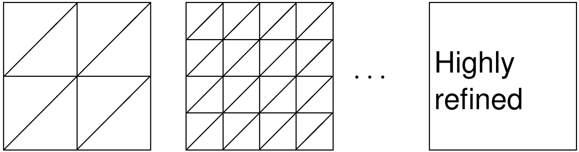
Illustrative multigrid setting



$k = 1$ Mesh 1 (Coarsest mesh)

$k = 2$ Mesh 2

...

$k = J$ Mesh J (Finest mesh)

Want to solve $A_J u = b$ at the finest level.

V-cycle: Set $\mathrm{MG}_{k_0}(v, w) = A_{k_0}^{-1} w$. Let $k > k_0$ and $v, w \in V_k$. Assuming that $\mathrm{MG}_{k-1}(\cdot, \cdot)$ has been defined, we define $\mathrm{MG}_k(v, w)$ as follows:

1. Set $x = v + R_k(w - A_k v)$. (Pre-smoothing)
2. Set $y = x + \mathrm{MG}_{k-1}(0, Q_{k-1}(w - A_k x))$. (Coarse-grid correction)
3. Define $\mathrm{MG}_k(v, w) = y + R_k'(w - A_k y)$. (Post-smoothing)

Multigrid perturbation

- Convergence of V-cycle for many non-symmetric and indefinite applications have been proven: [Bank, 1981], [Mandel,1986], [Bramble, Kwak & Pasciak, 1994].
- Main technique of analysis is a perturbation argument:

$$\text{Compare} \begin{Bmatrix} \text{MG for non-symmetric} \\ \text{or indefinite problem} \end{Bmatrix} \text{ with } \begin{Bmatrix} \text{MG for a nearby} \\ \text{SPD problem} \end{Bmatrix}.$$

- Previous papers handled "lower order" perturbative terms, e.g.:

$$\text{Compare} \quad (-\nabla\cdot A\nabla u + \underbrace{\gamma\cdot\nabla u + \eta u}_{\text{lower order}}) \quad \text{with} \quad (-\nabla\cdot A\nabla u).$$

- But we have a perturbation in the *highest order* term:

$$\text{Compare} \quad (-\nabla\cdot\alpha\nabla u) \quad \text{with} \quad (-\Delta u).$$

Convergence of multigrid

Although the perturbation is in the highest order term, we are able to prove a convergence theorem:

Theorem (Comparison of multigrid operators)

$\exists C > 0,\ H > 0,\ s > 0$ *such that whenever the coarsest meshsize in the algorithm, h_{k_0}, is less than H,*

$$\|\mathcal{E} - \hat{\mathcal{E}}\|_{H^1(\Omega)} \leq C\, h_{k_0}^{s/2}.$$

- Here $\begin{cases} \mathcal{E} = \text{Error reducer of the complex MG for } (\nabla \cdot \alpha \nabla) \\ \hat{\mathcal{E}} = \text{Error reducer of the standard MG for Laplacian } (-\Delta). \end{cases}$
- C and H are independent of the number of refinement levels.

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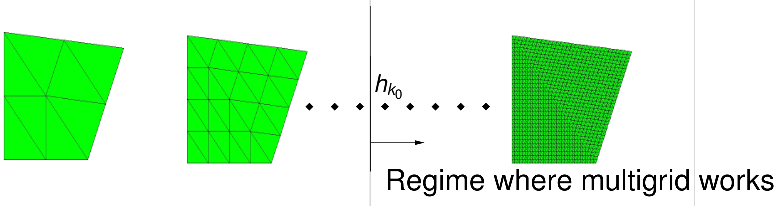
$$\|\mathcal{E} - \hat{\mathcal{E}}\|_{H^1(\Omega)} \le C\, h_{k_0}^{s/2}.$$

- This implies that the MG for complex coefficient converges if the coarse meshsize is sufficiently small.



Return to Numerical Example A

In accordance with the theorem, we see the complex MG iteration counts approaching that of MG for Laplacian $(-\Delta)$ as the *coarse* mesh is made finer:

Numerical Example A

$$\alpha(x, y) = 1 + \hat{\imath} K \sin(\pi(2y - 1)/2),$$

and $\Omega =$ unit square, meshed uniformly.

$h_{\text{fine}} = 1/256$ h_{coarse}	V-cycles MG for $(-\Delta)$	V-cycles $(K = 1)$	V-cycles $(K = 20)$	V-cycles $(K = 100)$
1/4	7	7	11	$\star$
1/8	7	7	9	$\star$
1/16	7	7	8	16
1/32	7	7	7	10
1/64	7	7	7	8
1/128	7	7	7	7

Return to Numerical Example B

Why did one code diverge, while the other converged?

Numerical Example B

$$\alpha = (1 - r)^2 + r^4 \exp(4\hat{\imath}\theta)$$

and $\Omega =$ unit square, meshed uniformly.

$h_{\text{coarse}} = 1/4$ / h_{fine}	Joe's code diverges	Jay's code
1/16	$\star$	19
1/32	$\star$	19
1/64	$\star$	20
1/128	$\star$	20

Within the MG, Gauss-Seidel depends on node ordering.

- Jay's code is in Red-Black node ordering.
- Joe's code is in Lexicographical node ordering.

Return to Numerical Example B

What happens if the coarse mesh is made finer?

Numerical Example B

$$\alpha = (1 - r)^2 + r^4 \exp(4\hat{\imath}\theta)$$

and $\Omega =$ unit square, meshed uniformly.

	Joe's code				
h_{fine}	$h_{\text{coarse}} = \frac{1}{4}$	$\cdots$	$h_{\text{coarse}} = \frac{1}{32}$	$h_{\text{coarse}} = \frac{1}{64}$	$h_{\text{coarse}} = \frac{1}{128}$
1/64	$\star$	$\cdots$	$\star$		
1/128	$\star$	$\cdots$	$\star$	43	
1/256	$\star$	$\cdots$	$\star$	7	7
1/512	$\star$	$\cdots$	$\star$	7	7

- Practically required coarse meshsize can depend on node ordering!

Conclusion

- We showed that multigrid for smooth elliptic complex coefficients converges at a mesh independent rate *if the coarse meshsize is sufficiently small.*

- This is similar to multigrid results for wave problems, where the folklore is that the "coarse grid must be small enough to resolve the wave".
- In contrast, for general complex coefficients, we have no idea how "small" the coarse meshsize needs to be.

- Our numerical experiments show that Gauss-Seidel smoother can be extremely sensitive to certain node orderings in the complex coefficient case.