Incompressible finite elements via hybridization

Jay Gopalakrishnan

University of Florida

July 2005

Collaborator: Bernardo Cockburn

University of Minnesota

Thanks: NSF

Stokes equations

$$
\begin{aligned}
-\boldsymbol{\Delta u} + \mathbf{grad}\, p &= \boldsymbol{f}, && \text{on } \Omega,\\
\operatorname{div} \boldsymbol{u} &= 0, && \text{on } \Omega,\\
\boldsymbol{u} &= \mathbf{0}, && \text{on } \partial\Omega.
\end{aligned}
$$

Since $-\boldsymbol{\Delta u} = \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{grad}\operatorname{div}\boldsymbol{u}$, the Stokes equations can be rewritten using vorticity $\boldsymbol{\omega}$:

$$
\begin{aligned}
\boldsymbol{\omega} - \mathbf{curl}\,\boldsymbol{u} &= 0, && \text{on } \Omega,\\
\mathbf{curl}\,\boldsymbol{\omega} + \mathbf{grad}\, p &= \boldsymbol{f}, && \text{on } \Omega,\\
\operatorname{div} \boldsymbol{u} &= 0, && \text{on } \Omega.
\end{aligned}
$$

Velocity-vorticity formulation

$$\boldsymbol{\omega} - \mathbf{curl}\,\boldsymbol{u} = 0 \implies (\boldsymbol{\omega}, \boldsymbol{\tau})_\Omega - (\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{\tau})_\Omega = 0$$

$$\mathbf{curl}\,\boldsymbol{\omega} + \mathbf{grad}\,p = \boldsymbol{f} \implies (\boldsymbol{v}, \mathbf{curl}\,\boldsymbol{\omega})_\Omega = (\boldsymbol{f}, \boldsymbol{v})_\Omega$$

$$\boldsymbol{\omega}, \boldsymbol{\tau} \in \mathcal{W} \quad := H(\mathbf{curl}\,)$$

$$\boldsymbol{u}, \boldsymbol{v} \in \mathring{\mathcal{V}} \quad := \{\boldsymbol{v} \in H(\mathrm{div}) : \ \mathrm{div}\,\boldsymbol{v} = 0, \quad \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega=0}\}.$$

Find $\boldsymbol{\omega} \in \mathcal{W}$ and $\boldsymbol{u} \in \mathring{\mathcal{V}}$ satisfying

$$(\boldsymbol{\omega}, \boldsymbol{\tau})_\Omega - (\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{\tau})_\Omega = 0, \qquad \forall \boldsymbol{\tau} \in \mathcal{W},$$

$$(\boldsymbol{v}, \mathbf{curl}\,\boldsymbol{\omega})_\Omega = (\boldsymbol{f}, \boldsymbol{v})_\Omega, \quad \forall \boldsymbol{v} \in \mathring{\mathcal{V}}.$$

Velocity-vorticity formulation

$$\boldsymbol{\omega} - \mathbf{curl}\,\boldsymbol{u} = 0 \implies (\boldsymbol{\omega}, \boldsymbol{\tau})_\Omega - (\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{\tau})_\Omega = 0$$

$$\mathbf{curl}\,\boldsymbol{\omega} + \mathbf{grad}\,p = \boldsymbol{f} \implies (\boldsymbol{v}, \mathbf{curl}\,\boldsymbol{\omega})_\Omega = (\boldsymbol{f}, \boldsymbol{v})_\Omega$$

$\boldsymbol{\omega}_h \in \mathcal{W}_h$:= Nédélec subspace of $H(\mathbf{curl}\,)$

$\boldsymbol{u}_h \in \mathring{\mathcal{V}}_h$:= divergence free subspace of Raviart-Thomas space

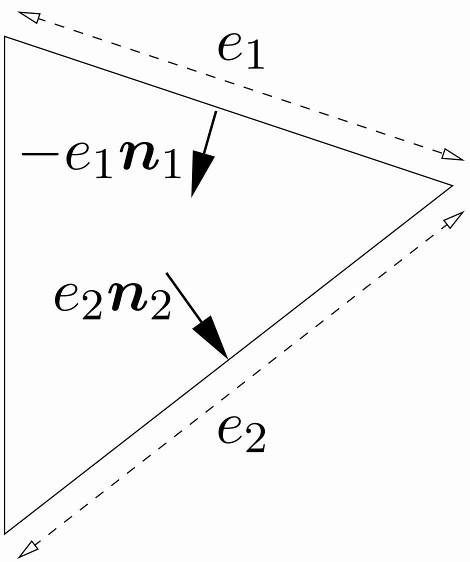
Find $\boldsymbol{\omega}_h \in \mathcal{W}_h$ and $\boldsymbol{u}_h \in \mathring{\mathcal{V}}_h$ satisfying

$$(\boldsymbol{\omega}_h, \boldsymbol{\tau})_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\,\boldsymbol{\tau})_\Omega = 0, \quad \forall \boldsymbol{\tau} \in \mathcal{W}_h,$$

$$(\boldsymbol{v}, \mathbf{curl}\,\boldsymbol{\omega}_h)_\Omega = (\boldsymbol{f}, \boldsymbol{v})_\Omega, \quad \forall \boldsymbol{v} \in \mathring{\mathcal{V}}_h.$$

Difficulties

Can one construct a basis for the divergence free space $\mathcal{V}_h$?

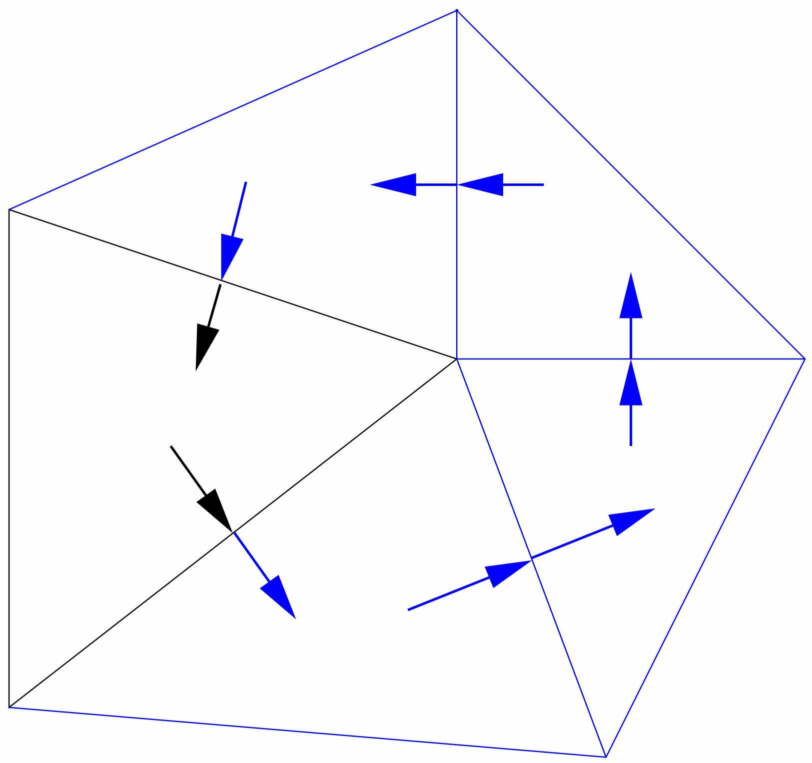


There is a constant function $\boldsymbol{\phi}$ on every triangle such that

$$\boldsymbol{\phi} \cdot \boldsymbol{n}_1 e_1 = -1,$$
$$\boldsymbol{\phi} \cdot \boldsymbol{n}_2 e_2 = 1,$$
$$\boldsymbol{\phi} \cdot \boldsymbol{n}_3 e_3 = 0.$$

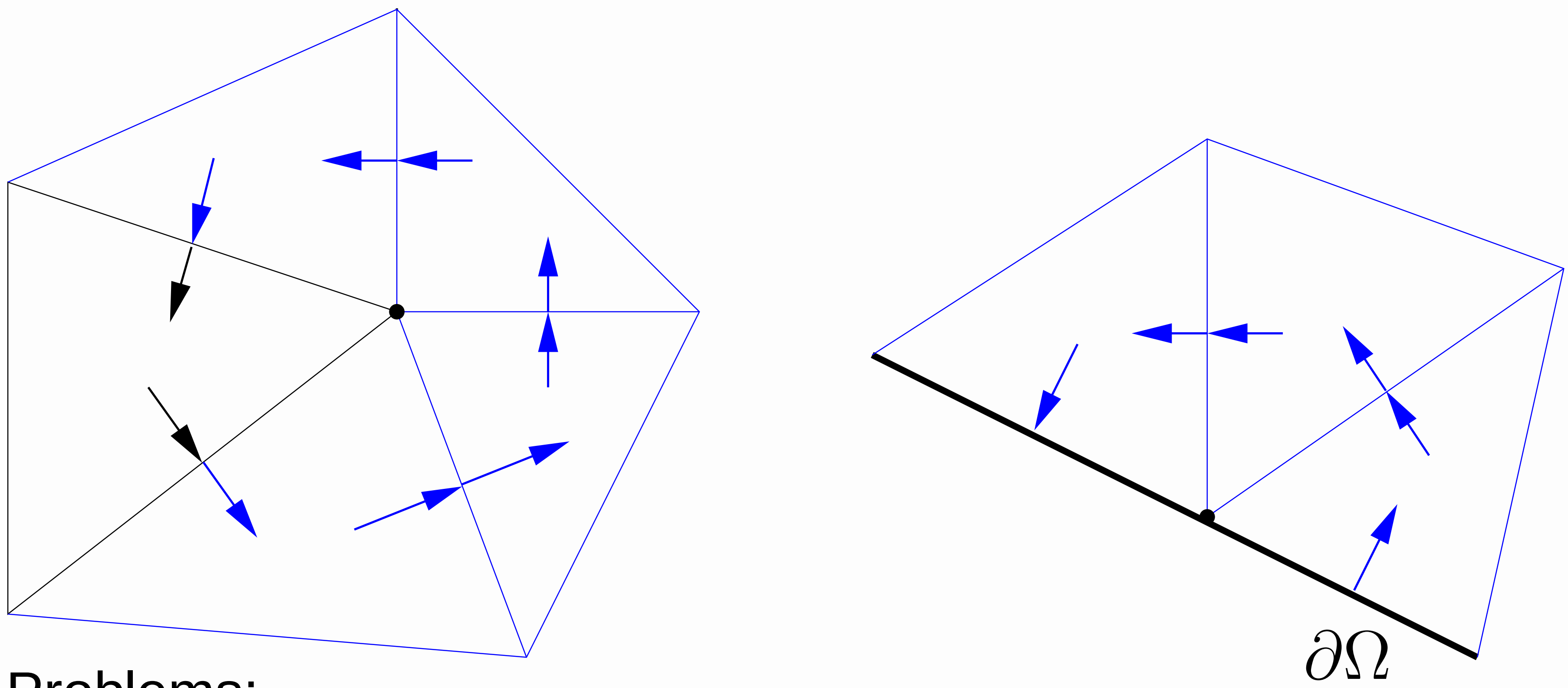
Difficulties

Can one construct a basis for the divergence free space $\mathcal{V}_h$*?*



Difficulties

Can one construct a basis for the divergence free space $\mathcal{V}_h$?



Problems:

- These functions are not linearly independent.
- They do not span $\mathcal{V}_h$ when Ω is not simply connected.

Why incompressible elements?

- Pressure disappears from the formulation.
- Can impose physically relevant (but as yet non-standard) boundary conditions:

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \Gamma_1,$$

$$\left.\begin{aligned} \boldsymbol{n} \times \boldsymbol{\omega} &= \boldsymbol{r} \\ \boldsymbol{u} \cdot \boldsymbol{n} &= g_n \end{aligned}\right\} \qquad \text{on } \Gamma_2,$$

$$\left.\begin{aligned} p &= s \\ \boldsymbol{u}_{\mathsf{T}} &= \boldsymbol{g}_{\mathsf{T}} \end{aligned}\right\} \qquad \text{on } \Gamma_3.$$

- When fluid velocity is coupled with convection problems, some schemes are more stable if the numerical velocity is exactly divergence free.

Background

- Stream function approach: Write divergence free finite element functions as curl of functions in the Nédélec space: [Girault & Raviart, 1986]

$$\boldsymbol{u}_h = \mathbf{curl}\, \boldsymbol{\psi}_h.$$

- Our approach: Since the inter-element continuity constraints of div-free spaces makes it hard to work with, remove the constraint from the spaces and impose it as an equation of the method, i.e., *hybridize*.

First hybridization

Mixed method: Find $\boldsymbol{\omega}_h \in \mathcal{W}_h$ and $\boldsymbol{u}_h \in \mathring{\mathcal{V}}_h$ satisfying

$$\begin{aligned} (\boldsymbol{\omega}_h, \boldsymbol{\tau})_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\,\boldsymbol{\tau})_\Omega &= 0, && \forall \boldsymbol{\tau} \in \mathcal{W}_h, \\ (\boldsymbol{v}, \mathbf{curl}\,\boldsymbol{\omega}_h)_\Omega &= (\boldsymbol{f}, \boldsymbol{v})_\Omega, && \forall \boldsymbol{v} \in \mathring{\mathcal{V}}_h. \end{aligned}$$

First hybridization: Break $H(\mathrm{div})$-continuity of $\boldsymbol{u}_h$ and reimpose it as an equation of the method. (Now $\boldsymbol{u}_h, \boldsymbol{v} \in V_h$.)

$$\begin{aligned} (\boldsymbol{\omega}_h, \boldsymbol{\tau})_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\,\boldsymbol{\tau})_\Omega &= 0, && \forall \boldsymbol{\tau} \in \mathcal{W}_h, \\ (\boldsymbol{v}, \mathbf{curl}\,\boldsymbol{\omega}_h)_\Omega + ([\![\boldsymbol{v} \cdot \boldsymbol{n}]\!], p_h)_\mathcal{E} &= (\boldsymbol{f}, \boldsymbol{v})_\Omega, && \forall \boldsymbol{v} \in V_h, \\ ([\![\boldsymbol{u}_h \cdot \boldsymbol{n}]\!], q)_\mathcal{E} &= 0, && \forall q \in P_h. \end{aligned}$$

Notations:

$V_h = \{\boldsymbol{v} : \boldsymbol{v}|_K \in P_k(K)^3,\ \mathrm{div}(\boldsymbol{v}|_K) = 0,\ \forall$ elements $K\}$.

$\mathcal{E} =$ union of all mesh faces.

First hybridization

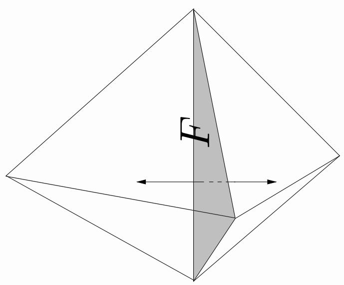
Mixed method: Find $\boldsymbol{\omega}_h \in \mathcal{W}_h$ and $\boldsymbol{u}_h \in \mathring{\mathcal{V}}_h$ satisfying

$$
\begin{aligned}
(\boldsymbol{\omega}_h, \boldsymbol{\tau})_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\, \boldsymbol{\tau})_\Omega &= 0, && \forall \boldsymbol{\tau} \in \mathcal{W}_h,\\
(\boldsymbol{v}, \mathbf{curl}\, \boldsymbol{\omega}_h)_\Omega &= (\boldsymbol{f}, \boldsymbol{v})_\Omega, && \forall \boldsymbol{v} \in \mathring{\mathcal{V}}_h.
\end{aligned}
$$

First hybridization: Break $H(\mathrm{div})$-continuity of $\boldsymbol{u}_h$ and reimpose it as an equation of the method. (Now $\boldsymbol{u}_h, \boldsymbol{v} \in V_h$.)

$$
\begin{aligned}
(\boldsymbol{\omega}_h, \boldsymbol{\tau})_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\, \boldsymbol{\tau})_\Omega &= 0, && \forall \boldsymbol{\tau} \in \mathcal{W}_h,\\
(\boldsymbol{v}, \mathbf{curl}\, \boldsymbol{\omega}_h)_\Omega + ([\![\boldsymbol{v} \cdot \boldsymbol{n}]\!], p_h)_{\mathcal{E}} &= (\boldsymbol{f}, \boldsymbol{v})_\Omega, && \forall \boldsymbol{v} \in V_h,\\
([\![\boldsymbol{u}_h \cdot \boldsymbol{n}]\!], q)_{\mathcal{E}} &= 0, && \forall q \in P_h.
\end{aligned}
$$

$$
[\![\boldsymbol{v} \cdot \boldsymbol{n}]\!]\big|_F = \begin{cases} \text{jump of } \boldsymbol{v} \cdot \boldsymbol{n} \text{ across } F, & \text{for interior faces } F,\\ \boldsymbol{v} \cdot \boldsymbol{n}|_F, & \text{for boundary faces } F. \end{cases}
$$



First hybridization

Mixed method: Find $\boldsymbol{\omega}_h \in \mathcal{W}_h$ and $\boldsymbol{u}_h \in \mathring{\mathcal{V}}_h$ satisfying

$$
\begin{aligned}
(\boldsymbol{\omega}_h, \boldsymbol{\tau})_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\,\boldsymbol{\tau})_\Omega &= 0, && \forall \boldsymbol{\tau} \in \mathcal{W}_h,\\
(\boldsymbol{v}, \mathbf{curl}\,\boldsymbol{\omega}_h)_\Omega &= (\boldsymbol{f}, \boldsymbol{v})_\Omega, && \forall \boldsymbol{v} \in \mathring{\mathcal{V}}_h.
\end{aligned}
$$

First hybridization: Break $H(\mathrm{div})$-continuity of $\boldsymbol{u}_h$ and reimpose it as an equation of the method. (Now $\boldsymbol{u}_h, \boldsymbol{v} \in V_h$.)

$$
\begin{aligned}
(\boldsymbol{\omega}_h, \boldsymbol{\tau})_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\,\boldsymbol{\tau})_\Omega &= 0, && \forall \boldsymbol{\tau} \in \mathcal{W}_h,\\
(\boldsymbol{v}, \mathbf{curl}\,\boldsymbol{\omega}_h)_\Omega + ([\![\boldsymbol{v} \cdot \boldsymbol{n}]\!], p_h)_\mathcal{E} &= (\boldsymbol{f}, \boldsymbol{v})_\Omega, && \forall \boldsymbol{v} \in V_h,\\
([\![\boldsymbol{u}_h \cdot \boldsymbol{n}]\!], q)_\mathcal{E} &= 0, && \forall q \in P_h.
\end{aligned}
$$

The *Lagrange multiplier* p_h approximates pressure on mesh faces and lies in

$$P_h = \{q : \quad q = [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!] \text{ for some } \boldsymbol{v} \in V_h\}.$$

First hybridization

Mixed method: Find $\boldsymbol{\omega}_h \in \mathcal{W}_h$ and $\boldsymbol{u}_h \in \mathring{\mathcal{V}}_h$ satisfying

$$\begin{aligned} (\boldsymbol{\omega}_h, \boldsymbol{\tau})_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\, \boldsymbol{\tau})_\Omega &= 0, && \forall \boldsymbol{\tau} \in \mathcal{W}_h, \\ (\boldsymbol{v}, \mathbf{curl}\, \boldsymbol{\omega}_h)_\Omega &= (\boldsymbol{f}, \boldsymbol{v})_\Omega, && \forall \boldsymbol{v} \in \mathring{\mathcal{V}}_h. \end{aligned}$$

First hybridization: Break $H(\mathrm{div})$-continuity of $\boldsymbol{u}_h$ and reimpose it as an equation of the method. (Now $\boldsymbol{u}_h, \boldsymbol{v} \in V_h$.)

$$\begin{aligned} (\boldsymbol{\omega}_h, \boldsymbol{\tau})_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\, \boldsymbol{\tau})_\Omega &= 0, && \forall \boldsymbol{\tau} \in \mathcal{W}_h, \\ (\boldsymbol{v}, \mathbf{curl}\, \boldsymbol{\omega}_h)_\Omega + ([\![\boldsymbol{v} \cdot \boldsymbol{n}]\!], p_h)_\mathcal{E} &= (\boldsymbol{f}, \boldsymbol{v})_\Omega, && \forall \boldsymbol{v} \in V_h, \\ ([\![\boldsymbol{u}_h \cdot \boldsymbol{n}]\!], q)_\mathcal{E} &= 0, && \forall q \in P_h. \end{aligned}$$

Advantage: Need only implement div-free polynomials within an element.

Disadvantage: Increased degrees of freedom (cannot eliminate any of the variables easily). The next hybridization will remove this disadvantage...

Second hybridization

Method after first hybridization: Find $\boldsymbol{\omega}_h \in \mathcal{W}_h$, $\boldsymbol{u}_h \in V_h$, and $p_h \in P_h$:

$$\begin{aligned} (\boldsymbol{\omega}_h, \boldsymbol{\tau})_\Omega \quad - (\boldsymbol{u}_h, \mathbf{curl}\,\boldsymbol{\tau})_\Omega &= 0, & \forall \boldsymbol{\tau} \in \mathcal{W}_h, \\ (\boldsymbol{v}, \mathbf{curl}\,\boldsymbol{\omega}_h)_\Omega + (p_h, [\![\boldsymbol{v}\cdot\boldsymbol{n}]\!])_\mathcal{E} &= (\boldsymbol{f}, \boldsymbol{v})_\Omega, & \forall \boldsymbol{v} \in V_h, \\ (q, [\![\boldsymbol{u}_h\cdot\boldsymbol{n}]\!])_\mathcal{E} &= 0, & \forall q \in P_h. \end{aligned}$$

Second hybridization: Break $H(\mathbf{curl}\,)$-continuity of $\boldsymbol{\omega}_h$.

$$\begin{aligned} (\boldsymbol{\omega}_h, \boldsymbol{\tau})_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\,\boldsymbol{\tau})_\Omega - (\boldsymbol{\lambda}_h, [\![\boldsymbol{n}\times\boldsymbol{\tau}]\!])_\mathcal{E} &= 0, \\ (\boldsymbol{v}, \mathbf{curl}\,\boldsymbol{\omega}_h)_\Omega + \quad (p_h, [\![\boldsymbol{v}\cdot\boldsymbol{n}]\!])_\mathcal{E} &= (\boldsymbol{f}, \boldsymbol{v})_\Omega, \\ (q, [\![\boldsymbol{u}_h\cdot\boldsymbol{n}]\!])_\mathcal{E} &= 0, \\ (\boldsymbol{\mu}_h, [\![\boldsymbol{n}\times\boldsymbol{\omega}_h]\!])_\mathcal{E} &= 0. \end{aligned}$$

Now $\boldsymbol{\omega}_h$ is sought in

$$W_h = \{\boldsymbol{w} : \quad \boldsymbol{w}|_K \in \text{Nédélec space of degree } k{+}1 \text{ on } K, \quad \forall \text{ elements } K\}$$

and $\mathbf{curl}$ is applied element by element.

Second hybridization

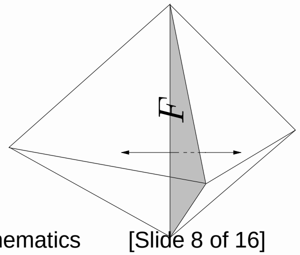
Method after first hybridization: Find $\boldsymbol{\omega}_h \in \mathcal{W}_h$, $\boldsymbol{u}_h \in V_h$, and $p_h \in P_h$:

$$
\begin{aligned}
(\boldsymbol{\omega}_h, \boldsymbol{\tau})_\Omega \quad - (\boldsymbol{u}_h, \mathbf{curl}\, \boldsymbol{\tau})_\Omega &= 0, && \forall \boldsymbol{\tau} \in \mathcal{W}_h,\\
(\boldsymbol{v}, \mathbf{curl}\, \boldsymbol{\omega}_h)_\Omega + (p_h, [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!])_{\mathcal{E}} &= (\boldsymbol{f}, \boldsymbol{v})_\Omega, && \forall \boldsymbol{v} \in V_h,\\
(q, [\![\boldsymbol{u}_h \cdot \boldsymbol{n}]\!])_{\mathcal{E}} &= 0, && \forall q \in P_h.
\end{aligned}
$$

Second hybridization: Break $H(\mathbf{curl}\,)$-continuity of $\boldsymbol{\omega}_h$.

$$
\begin{aligned}
(\boldsymbol{\omega}_h, \boldsymbol{\tau})_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\, \boldsymbol{\tau})_\Omega - (\boldsymbol{\lambda}_h, [\![\boldsymbol{n} \times \boldsymbol{\tau}]\!])_{\mathcal{E}} &= 0,\\
(\boldsymbol{v}, \mathbf{curl}\, \boldsymbol{\omega}_h)_\Omega + \quad (p_h, [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!])_{\mathcal{E}} &= (\boldsymbol{f}, \boldsymbol{v})_\Omega,\\
(q, [\![\boldsymbol{u}_h \cdot \boldsymbol{n}]\!])_{\mathcal{E}} &= 0,\\
(\boldsymbol{\mu}_h, [\![\boldsymbol{n} \times \boldsymbol{\omega}_h]\!])_{\mathcal{E}} &= 0.
\end{aligned}
$$

$$
[\![\boldsymbol{n} \times \boldsymbol{w}]\!]\big|_F = \begin{cases} \text{jump of } \boldsymbol{n} \times \boldsymbol{w} \text{ across } F, & \text{for interior faces } F,\\ \mathbf{0}, & \text{for boundary faces } F. \end{cases}
$$



Second hybridization

Method after first hybridization: Find $\boldsymbol{\omega}_h \in \mathcal{W}_h$, $\boldsymbol{u}_h \in V_h$, and $p_h \in P_h$:

$$\begin{aligned}
(\boldsymbol{\omega}_h, \boldsymbol{\tau})_\Omega \quad - (\boldsymbol{u}_h, \mathbf{curl}\, \boldsymbol{\tau})_\Omega &= 0, && \forall \boldsymbol{\tau} \in \mathcal{W}_h,\\
(\boldsymbol{v}, \mathbf{curl}\, \boldsymbol{\omega}_h)_\Omega + (p_h, [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!])_\mathcal{E} &= (\boldsymbol{f}, \boldsymbol{v})_\Omega, && \forall \boldsymbol{v} \in V_h,\\
(q, [\![\boldsymbol{u}_h \cdot \boldsymbol{n}]\!])_\mathcal{E} &= 0, && \forall q \in P_h.
\end{aligned}$$

Second hybridization: Break $H(\mathbf{curl}\,)$-continuity of $\boldsymbol{\omega}_h$.

$$\begin{aligned}
(\boldsymbol{\omega}_h, \boldsymbol{\tau})_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\, \boldsymbol{\tau})_\Omega - (\boldsymbol{\lambda}_h, [\![\boldsymbol{n} \times \boldsymbol{\tau}]\!])_\mathcal{E} &= 0,\\
(\boldsymbol{v}, \mathbf{curl}\, \boldsymbol{\omega}_h)_\Omega + \quad (p_h, [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!])_\mathcal{E} &= (\boldsymbol{f}, \boldsymbol{v})_\Omega,\\
(q, [\![\boldsymbol{u}_h \cdot \boldsymbol{n}]\!])_\mathcal{E} &= 0,\\
(\boldsymbol{\mu}_h, [\![\boldsymbol{n} \times \boldsymbol{\omega}_h]\!])_\mathcal{E} &= 0.
\end{aligned}$$

The second *Lagrange multiplier* $\boldsymbol{\lambda}_h$ approximates the tangential component of velocity on mesh faces and lies in

$$M_h = \{\boldsymbol{\mu} : \quad \boldsymbol{\mu} = [\![\boldsymbol{n} \times \boldsymbol{v}]\!] \text{ for some } \boldsymbol{v} \in W_h\}.$$

Hybridized method

Method after both hybridizations: Find $(\boldsymbol{\omega}_h, \boldsymbol{u}_h, p_h, \boldsymbol{\lambda}_h)$ in $W_h \times V_h \times P_h \times M_h$ satisfying

$$
\begin{aligned}
(\boldsymbol{\omega}_h, \boldsymbol{\tau})_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\,\boldsymbol{\tau})_\Omega - (\boldsymbol{\lambda}_h, [\![\boldsymbol{n} \times \boldsymbol{\tau}]\!])_{\mathcal{E}} &= 0, \\
(\boldsymbol{v}, \mathbf{curl}\,\boldsymbol{\omega}_h)_\Omega + \quad (p_h, [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!])_{\mathcal{E}} &= (\boldsymbol{f}, \boldsymbol{v})_\Omega, \\
(q, [\![\boldsymbol{u}_h \cdot \boldsymbol{n}]\!])_{\mathcal{E}} &= 0, \\
(\boldsymbol{\mu}, [\![\boldsymbol{n} \times \boldsymbol{\omega}_h]\!])_{\mathcal{E}} &= 0,
\end{aligned}
$$

for all test functions $(\boldsymbol{\tau}, \boldsymbol{v}, q, \boldsymbol{\mu})$ in $W_h \times V_h \times P_h \times M_h$.

PROPOSITION. *There is a unique solution to this hybridized method. Moreover, its velocity and vorticity approximations coincide with that of the mixed method.*

(*Problem*: Method has too many unknowns . . .)

Lagrange multipliers

THEOREM. *The Lagrange multipliers* $(\boldsymbol{\lambda}_h, p_h) \in M_h \times P_h$ *of the hybridized mixed method form the unique solution of*

$$\begin{aligned} a(\boldsymbol{\lambda}_h, \boldsymbol{\mu}) + b(\boldsymbol{\mu}, p_h) &= \ell_1(\boldsymbol{\mu}), && \forall \boldsymbol{\mu} \in M_h \text{ and} \\ b(\boldsymbol{\lambda}_h, q) - c(p_h, q) &= \ell_2(q), && \forall q \in P_h. \end{aligned}$$

Moreover, $\boldsymbol{\omega}_h$ *and* $\boldsymbol{u}_h$ *can be computed locally (element by element) once* $\boldsymbol{\lambda}_h$ *and* p_h *are determined from the above system.*

Thus we have a tangential velocity–pressure discretization on the mesh faces for the Stokes problem. (This discretization also yields interior velocity and vorticity locally).

Definitions of $a, b, c \ldots \longrightarrow$

Lifting maps

Define the result of two local lifting maps

$$\begin{aligned} M_h &\mapsto \quad W_h \times V_h & \qquad P_h &\mapsto \quad W_h \times V_h, \\ \boldsymbol{\lambda} &\mapsto (\, \boldsymbol{w}(\boldsymbol{\lambda}),\ \boldsymbol{u}(\boldsymbol{\lambda})\,) & p &\mapsto (\, \mathrm{w}(p),\ \mathbf{u}(p)\,), \end{aligned}$$

element by element, as the solutions of the equations

$$\begin{cases} (\boldsymbol{w}(\boldsymbol{\lambda}), \boldsymbol{\tau})_K - (\boldsymbol{u}(\boldsymbol{\lambda}), \mathbf{curl}\,\boldsymbol{\tau})_K = (\boldsymbol{\lambda}, \boldsymbol{n} \times \boldsymbol{\tau})_{\partial K}, \\ \qquad (\boldsymbol{v}, \mathbf{curl}\ \boldsymbol{w}(\boldsymbol{\lambda}))_K = 0, \end{cases}$$

$$\begin{cases} (\mathrm{w}(p), \boldsymbol{\tau})_K - (\mathbf{u}(p), \mathbf{curl}\,\boldsymbol{\tau})_K = 0, \\ \qquad (\boldsymbol{v}, \mathbf{curl}\ \mathrm{w}(p))_K = -(p, \boldsymbol{n} \cdot \boldsymbol{v})_{\partial K}, \end{cases}$$

for all $\boldsymbol{\tau} \in W_h$ and $\boldsymbol{v} \in V_h$. (Here K is any mesh element.)

Bilinear forms

The forms are defined using the lifting maps:

$$a(\boldsymbol{\lambda},\boldsymbol{\mu}) = (\boldsymbol{w}(\boldsymbol{\lambda}),\boldsymbol{w}(\boldsymbol{\mu}))_\Omega,$$
$$c(p,q) = (\mathrm{w}(p),\mathrm{w}(q))_\Omega,$$
$$b(\boldsymbol{\mu},p) = -(\boldsymbol{u}(\boldsymbol{\mu}),\mathbf{curl}\ \mathrm{w}(p))_\Omega$$

$$\ell_1(\boldsymbol{\mu}) = (\boldsymbol{f},\boldsymbol{u}(\boldsymbol{\mu}))_\Omega$$
$$\ell_2(q) = (\boldsymbol{f},\mathfrak{u}(q))_\Omega.$$

Thus the forms can be computed locally.

In fact, assembly of matrices can proceed using standard finite element techniques by computing local element matrices once local bases for the Lagrange multiplier spaces are developed.

Bases for multiplier spaces

Recall the definitions of the multiplier spaces:

$$P_h = \{q : \quad q = [\![\boldsymbol{n} \cdot \boldsymbol{v}]\!] \text{ for some } \boldsymbol{v} \in V_h\}.$$

$$M_h = \{\boldsymbol{\mu} : \quad \boldsymbol{\mu} = [\![\boldsymbol{n} \times \boldsymbol{v}]\!] \text{ for some } \boldsymbol{v} \in W_h\}.$$

THEOREM. *The space P_h is characterized as follows:*

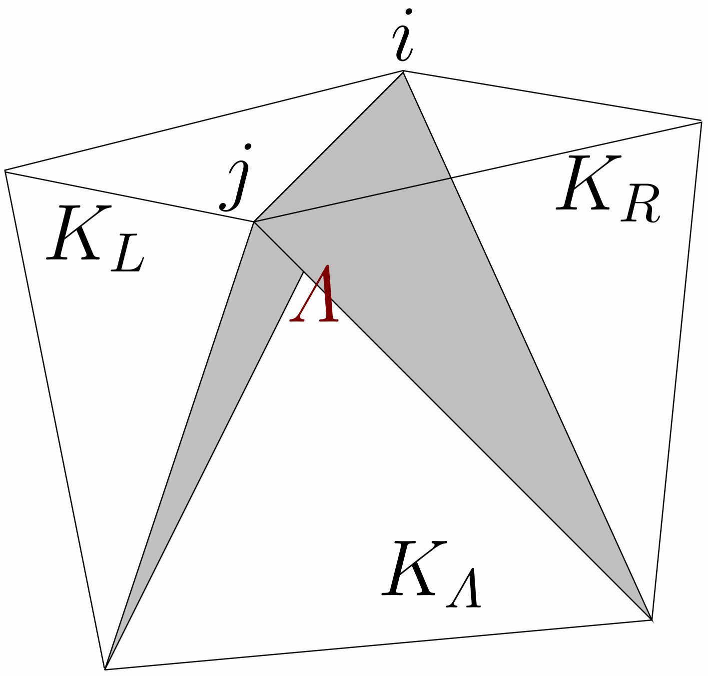
$$P_h = \left\{ p : p|_F \in P_k(F) \textit{ for all faces } F \textit{ and } \int_{\mathcal{E}} p \, \mathrm{d}s = 0 \right\}.$$

Thus, in computations, we can represent functions of P_h by a basis for the polynomial spaces $P_k(F)$ on each mesh face F.

To represent functions of M_h, we need a local basis for M_h. . .

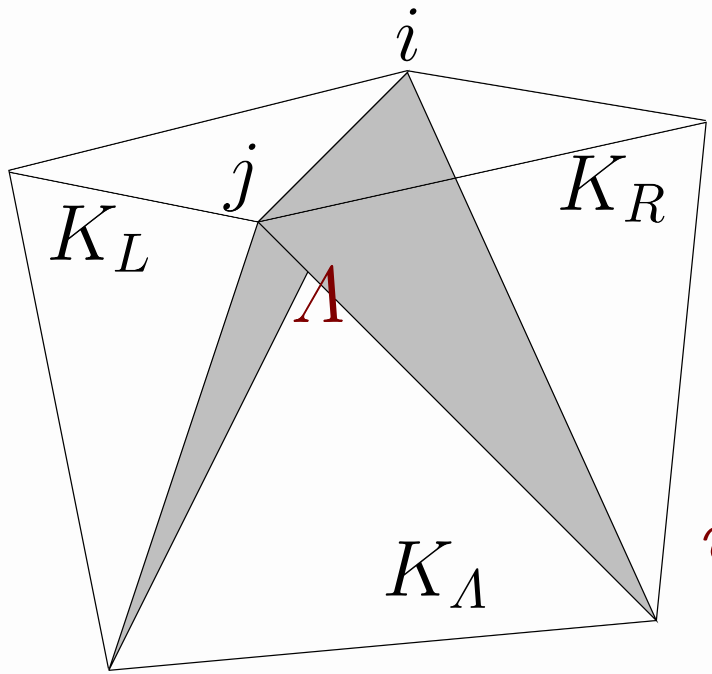
Wedge basis

We define a basis for M_h in the lowest order case using "wedges" of a mesh. A wedge Λ is the union of the two faces of a tetrahedron that share an edge.



Wedge basis

We define a basis for M_h in the lowest order case using "wedges" of a mesh. A wedge Λ is the union of the two faces of a tetrahedron that share an edge.



$$\phi_\Lambda = \begin{cases} \beta_i \,\boldsymbol{\nabla}\, \beta_j - \beta_j \,\boldsymbol{\nabla}\, \beta_i, & \text{on } K_\Lambda, \\ 0, & \text{on all other elements.} \end{cases}$$

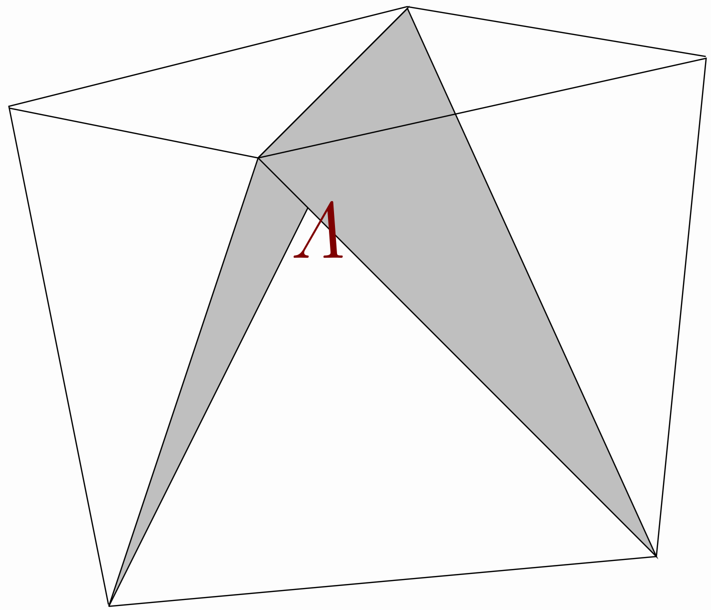
$$(\beta_i = \text{ barycentric coordinates on } K_\Lambda)$$

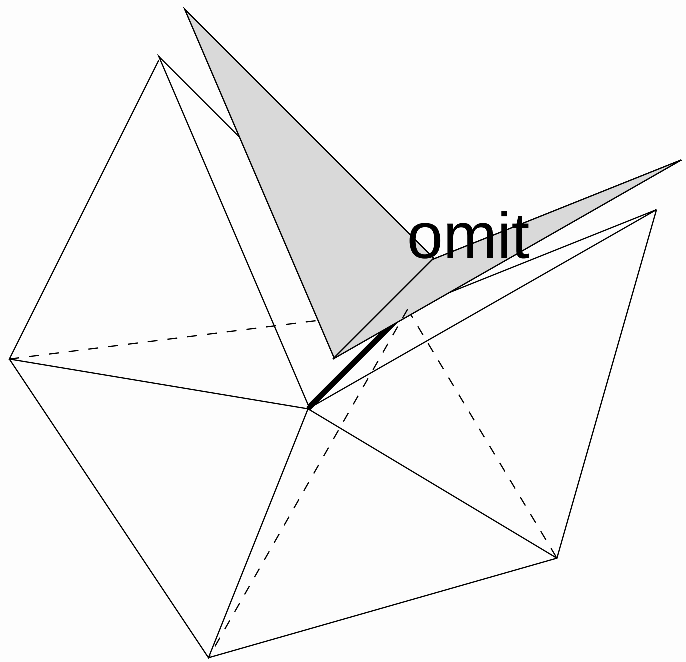
$$\psi_\Lambda = [\![\boldsymbol{n} \times \phi_\Lambda]\!].$$

The basis is constructed using ψ_Λ. But not all of them are linearly independent.

Wedge basis

We define a basis for M_h in the lowest order case using "wedges" of a mesh. A wedge Λ is the union of the two faces of a tetrahedron that share an edge.





THEOREM. *Collect the mesh wedges, omitting one wedge per edge, into a set* Υ. *Then the set* $\mathcal{B} = \{\psi_\Lambda : \Lambda \in \Upsilon\}$ *is a basis for* M_h.

Implementation

The Lagrange multiplier system gives the matrix equation

$$\begin{bmatrix} \mathsf{A} & \mathsf{B}^t \\ \mathsf{B} & -\mathsf{C} \end{bmatrix} \begin{bmatrix} \Lambda \\ \mathsf{P} \end{bmatrix} = \begin{bmatrix} \mathsf{L}_1 \\ \mathsf{L}_2 \end{bmatrix}$$

where Λ and P are the vectors of coefficients of $\boldsymbol{\lambda}_h$ and p_h in their local basis expansions. We can solve this equation by solving two *symmetric positive defi nite* systems:

- Solve for tangential velocity using the Schur complement

$$(\mathsf{B}^t\mathsf{C}^{-1}\mathsf{B} + \mathsf{A})\,\Lambda = \mathsf{L}_1 + \mathsf{B}^t\mathsf{C}^{-1}\mathsf{L}_2.$$

- Solve for pressure next:

$$\mathsf{C}\,\mathsf{P} = \mathsf{L}_2 - \mathsf{B}\Lambda.$$

Conclusion

- Our method for Stokes flow gives
  - exactly divergence free numerical velocity,
  - $H(\mathbf{curl}\,)$–conforming vorticity in 3D (or $H^1(\Omega)$–conforming vorticity in 2D), and
  - pressure approximations (generally discontinuous).
- The above approximations are obtained *locally* after solving one global “tangential velocity-pressure” system.
- This system is relatively small since it only couples unknowns *on mesh faces* (good for high order elements).
- The method has no topology dependence.
- We have shown that it is possible to hybridize methods that have edge (or vertex) degrees of freedom.